Алгебра и анализ
Том 9 (1997), вып. 3

St. Petersburg Math. J.
Vol. 9 (1998), No. 3

SMOOTHNESS OF A QUASICONFORMAL MAPPING AT A POINT

E. DYN'KIN

Dedicated to the memory of G. M. Goluzin

Abstract. Let f be a K-quasiconformal mapping on the plane with complex dilation μ. Let $p > 2K$, and let $\omega(r)$ be the mean value of $|\mu|$ of degree p over the disk $\{|z| < r\}$. If

$$\int_0^1 \frac{\omega(r)}{r^m}\, dr < +\infty$$

for a natural number m, then there exists a polynomial P of degree m such that

$$|f(z) - P(z)| \leq C_1 |z|^{m+1} + C_2 \Omega(|z|), \quad |z| < 1,$$

where

$$\Omega(\delta) = \delta^m \int_0^\delta \frac{\omega(r)}{r^m}\, dr + \delta^{m+1} \int_\delta^1 \frac{\omega(r)}{r^{m+1}}\, dr.$$

This is an intermediate result between the classical Teichmüller–Wittich–Belinskiĭ theorem and a recent result of Nikolaev and Shefel'.

In this paper we study the smoothness of plane quasiconformal mappings that are asymptotically conformal at one point. This is a natural supplement to the paper [3], where quasiconformal mappings asymptotically conformal on a circle were treated.

A plane quasiconformal mapping f (see [4]) is a homeomorphic solution of the Beltrami equation

$$f_{\bar z} = \mu\, f_z,$$

where the complex dilation μ is of class $L^\infty(\mathbb{C})$ and $\|\mu\|_{L^\infty} = \varkappa < 1$. If μ vanishes (in some sense) at a point, then f must be more or less "smooth" there.

In what follows we use also the "big" quasiconformality coefficient

$$K = \frac{1+\varkappa}{1-\varkappa}.$$

As usual, we put $z = x + iy$, $\zeta = \xi + i\eta$; C and c (with or without indices) will denote various constants; $\Delta(r)$ will stands for the disk $\{z : |z| < r\}$.

For any $p > 0$, the *mean value of* μ *of degree* p *over* $\Delta(r)$ is defined by the formula

$$\omega(r) = \omega_p(r) = \left(\frac{1}{\pi r^2} \int_{\Delta(r)} |\mu(z)|^p\, dx\, dy\right)^{1/p}.$$

1991 *Mathematics Subject Classification.* Primary 30C35, 30C45, 30C62.
Key words and phrases. Quasiconformal mappings, conformality at a point.
This research was supported by the Fund for Promotion of Research at the Technion.

Theorem. *If $p > 2K$ and*

$$\int_0^1 \frac{\omega(r)}{r^m}\, dr < +\infty$$

for some natural number m, then there exists an analytic polynomial P_m of degree m such that

$$|f(z) - P_m(z)| \le C_1|z|^{m+1} + C_2\Omega(|z|), \quad |z| < 1, \tag{1}$$

where

$$\Omega(\delta) = \delta^m \int_0^\delta \frac{\omega(r)}{r^m}\, dr + \delta^{m+1}\int_\delta^1 \frac{\omega(r)}{r^{m+1}}\, dr.$$

Remarks. 1. The right-hand side of (1) is always $o(|z|^m)$ as $|z| \to 0$.

2. If $\lim_{|z|\to 0}\mu(z) = 0$, then the theorem is true under the only assumption that $p > 2$.

3. The constants C_1 and C_2 depend on K only provided that f is normalized in a standard way (for instance, $f(0) = 0$, $f(\infty) = \infty$, and $f(1) = 1$).

Comment. The famous Teichmüller–Wittich–Belinskiĭ theorem [4] asserts that f has complex derivative at 0 if

$$\int_0^1 \frac{\omega_1(r)}{r}\, dr < +\infty.$$

Unfortunately, nobody has yet succeeded in deducing a finer estimate like (1) from this result. On the other hand, Nikolaev and Shefel′ [5] proved that

$$f(z) = P_m(z) + O(|z|^{m+\alpha}), \quad |z| \to 0,$$

if $\mu(z) = O(|z|^{m-1+\alpha})$, $0 < \alpha < 1$. The proof in [5] is based on the Belinskiĭ stability theorem (see [2]) and does not work if μ is small only in the mean.

Our theorem is an intermediate result; it relates the problem under consideration to the general approach suggested in [3].

In [5], Nikolaev and Shefel′ also proved an analog of their result for spatial quasiconformal mappings. It is unknown whether a spatial analog of our theorem is true.

Proof of the theorem. We may assume that $f(0) = 0$.

Step 1. The boundedness of distortion. The image $f(\Delta(r))$ of the disk $\Delta(r)$ is a quasidisk. We define the radius $R(r)$ of this quasidisk by the formula

$$R = \left(\int_{\Delta(r)} |f_z|^2\, dx\, dy\right)^{1/2}.$$

It is well known (see [4]) that

$$c_0 R < \min_{|z|=r}|f(z)| \le \max_{|z|=r}|f(z)| < C_0 R,$$

where $c_0 > 0$ and $C_0 < +\infty$ depend on K only. If $r_1 < r_2$ satisfy $C_0R(r_1) < c_0R(r_2)$, then the quasiannulus $Q = f(\Delta(r_2) \setminus \Delta(r_1))$ contains the annulus $\Delta(c_0R(r_2)) \setminus \Delta(C_0R(r_1))$ and is contained in the annulus $\Delta(C_0R(r_2)) \setminus \Delta(c_0R(r_1))$. Therefore, the modulus $M(Q)$ of the quasiannulus Q (see [4]) differs from

$$\log\frac{R(r_2)}{R(r_1)}$$

at most by a constant. On the other hand, inequality (6.9) in [4, Chapter 5, Subsection 6.3] implies that

$$\left|M(Q') - \log\frac{r_2}{r_1}\right| \le C\int_{r_1<|z|<r_2}\frac{|\mu(z)|}{|z|^2}\,dx\,dy.$$

This yields

$$\left|\log\frac{R(r_2)}{R(r_1)} - \log\frac{r_2}{r_1}\right| \le C_1 + C\int_{r_1<|z|<r_2}\frac{|\mu(z)|}{|z|^2}\,dx\,dy.$$

In particular, for any $r<1$ we have

$$|\log R(r) - \log r| \le C_1 + C\int_{r<|z|<1}\frac{|\mu(z)|}{|z|^2}\,dx\,dy \le C_1 + C\int_r^1\frac{\omega(t)}{t}\,dt.$$

Under the assumptions of the theorem, the integral

$$\int_0^1\frac{\omega(t)}{t}\,dt$$

converges, and $|\log R(r) - \log r|$ is uniformly bounded. Therefore,

$$R(r)\asymp r, \quad 0<r<1,$$

with a constant depending on K only.

Step 2. The construction of P_m. By the Cauchy–Green formula,

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)}{\zeta-z}\,d\zeta - \frac{1}{\pi}\int_{|\zeta|<1}\frac{f_{\bar\zeta}(\zeta)}{\zeta-z}\,d\xi d\eta, \quad |z|<1.$$

Since the first term is analytic in the unit disk, it can be represented in the form $P_m(z)+O(|z|^{m+1})$, where P_m is the corresponding Taylor polynomial of degree m. As for the second term, we have

$$\begin{aligned}\int_{|\zeta|<1}\frac{f_{\bar\zeta}(\zeta)}{\zeta-z}\,d\xi d\eta &= \sum_{k=0}^{m} z^k\int_{|\zeta|<1}\frac{f_{\bar\zeta}(\zeta)}{\zeta^{k+1}}\,d\xi d\eta + z^{m+1}\int_{|\zeta|<1}\frac{f_{\bar\zeta}(\zeta)}{\zeta^{m+1}(\zeta-z)}\,d\xi d\eta\\ &= \widetilde{P}_m(z) + S(z),\end{aligned}$$

where $\widetilde{P}_m$ is another polynomial of degree m. Now we must check the convergence of the coefficient integrals

$$\int_{|\zeta|<1}\frac{|f_{\bar\zeta}(\zeta)|}{|\zeta|^{k+1}}\,d\xi d\eta, \quad k=1,2,\dots,m, \tag{2}$$

and estimate the remainder term S.

Step 3. The convergence of the coefficient integrals. The integrals (2) do converge, because

$$\begin{aligned}\int_{|\zeta|<1}\frac{|f_{\bar\zeta}(\zeta)|}{|\zeta|^{k+1}}\,d\xi d\eta &\le \sum_{j=0}^{\infty}\frac{1}{(2^{-j})^{k+1}}\int_{|\zeta|<2^{-j+1}}|f_{\bar\zeta}|\\ &\le \sum_{j=0}^{\infty}\frac{1}{(2^{-j})^{k+1}}\left(\int_{|\zeta|<2^{-j+1}}|\mu(\zeta)|^2\right)^{1/2}\left(\int_{|\zeta|<2^{-j+1}}|f_\zeta|^2\right)^{1/2}\\ &\le C\sum_{j=0}^{\infty}\frac{1}{(2^{-j})^{k}}R(2^{-j+1})\omega(2^{-j+1})\\ &\le C\int_0^1\frac{\omega(r)}{r^k}\,dr<+\infty,\quad k=1,2,\dots,m.\end{aligned}$$

Step 4. The remainder estimate. We put $\delta=|z|$. Clearly,

$$|S(z)|\le C\delta^m\int_{|\zeta|<\delta/2}\frac{|f_{\bar\zeta}|}{|\zeta|^{m+1}}+C\delta^{m+1}\int_{|\zeta|>\delta/2}\frac{|f_{\bar\zeta}|}{|\zeta|^{m+2}}+C\int_{|\zeta-z|<\delta/2}\frac{|f_{\bar\zeta}|}{|\zeta-z|}.$$

The first two terms do not exceed $C\Omega(\delta)$. Indeed, for instance,

$$\begin{aligned}\delta^{m+1}\int_{|\zeta|>\delta/2}\frac{|f_{\bar\zeta}|}{|\zeta|^{m+2}} &\le \delta^{m+1}\sum_{j=0}^{\infty}\frac{1}{(2^{j-1}\delta)^{m+2}}\int_{|\zeta|<2^j\delta}|f_{\bar\zeta}|\\ &\le \delta^{m+1}\sum_{j=0}^{\infty}\frac{1}{(2^{j-1}\delta)^{m+2}}\left(\int_{|\zeta|<2^j\delta}|\mu|^2\right)^{1/2}\left(\int_{|\zeta|<2^j\delta}|f_\zeta|^2\right)^{1/2}\\ &\le C\delta^{m+1}\sum_{j=0}^{\infty}\frac{1}{(2^{j-1}\delta)^{m+1}}R(2^j\delta)\omega(2^j\delta)\\ &\le C\delta^{m+1}\int_\delta^1\frac{\omega(r)}{r^{m+1}}\,dr.\end{aligned}$$

In order to estimate the third term, we need a more powerful tool. In [1], Astala proved the following inverse Hölder inequality for quasiconformal mappings:

$$\left(\frac{1}{r^2}\int_{\Delta(r)}|f_z|^{2q}\,dx\,dy\right)^{\frac{1}{2q}}\le C\frac{1}{r^2}\int_{\Delta(r)}|f_z|^2\,dx\,dy\le C\frac{R(r)^2}{r^2}$$

for all q satisfying

$$q<\frac{K}{K-1}.$$

By the assumptions of our theorem, we have $p>2K$. So, choosing $q<\frac{K}{K-1}$ and fixing $\sigma<2$ such that

$$\frac{1}{p}+\frac{1}{2q}+\frac{1}{\sigma}=1,$$

we can apply the Hölder inequality with the above three exponents to obtain

$$\begin{aligned}\int_{|\zeta-z|<\delta/2}\frac{|\mu|\,|f_\zeta|}{|\zeta-z|} &\le \left(\int|\mu|^p\right)^{1/p}\left(\int|f_\zeta|^{2q}\right)^{\frac{1}{2q}}\left(\int\frac{1}{|\zeta-z|^\sigma}\right)^{1/\sigma}\\ &\le C\delta^2\omega(\delta)\frac{R(\delta)^2}{\delta^2}\,\frac{1}{\delta}\le C\delta\omega(\delta)\le C\Omega(\delta),\end{aligned}$$

by the Astala inequality. □

Remark. The standard localization procedure (see, e.g., [3]) shows that it suffices to assume that $|\mu| \le \varkappa$ only in a small neighborhood of the origin. The theorem remains true for these local values of $\varkappa$ and K.

In particular, if $\lim_{|z|\to 0} \mu(z) = 0$, then we can take any $K > 1$ and, therefore, any $p > 2$. In this case, the inverse Hölder inequality we need follows from an old result of Bojarski (see [4]) and does not require referring to Astala's work.

Acknowledgement. I am deeply grateful to J. M. Anderson for pointing out the paper [5] to me and for many discussions of the subject.

References

1. K. Astala, *Area distortion of quasiconformal mappings*, Acta Math. **173** (1994), 37–60.
2. P. P. Belinskiĭ, *General properties of quasiconformal mappings*, "Nauka", Sibirsk. Otdel., Novosibirsk, 1974. (Russian)
3. E. Dyn′kin, *Estimates for asymptotically conformal mappings*, Ann. Acad. Sci. Fenn. Ser. A1 Math. (to appear).
4. O. Lehto and K. Virtanen, *Quasiconformal mappings in the plane*, Grundlehren Math. Wiss., Bd. 126, Springer-Verlag, New York–Heidelberg, 1973.
5. I. G. Nikolaev and S. Z. Shefel′, *Differential properties of mappings that are conformal at a point*, Sibirsk. Mat. Zh. **27** (1986), no. 1, 132–142; English transl., Siberian Math. J. **27** (1986), no. 1, 106–114.

Department of Mathematics, Technion, Haifa 32000, Israel
E-mail address: `dynkin@techunix.technion.ac.il`

Received 24/DEC/96

Translated by THE AUTHOR