ON THE PACKING DIMENSION OF THE JULIA SET AND THE ESCAPING SET OF AN ENTIRE FUNCTION

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Abstract. Let f be a transcendental entire function. We give conditions which imply that the Julia set and the escaping set of f have packing dimension 2. For example, this holds if there exists a positive constant c less than 1 such that the minimum modulus $L(r, f)$ and the maximum modulus $M(r, f)$ satisfy $\log L(r, f) \leq c \log M(r, f)$ for large r. The conditions are also satisfied if $\log M(2r, f) \geq d \log M(r, f)$ for some constant d greater than 1 and all large r.

1. Introduction and results

The Fatou set $F(f)$ of an entire function f is defined as the set of all $z \in \mathbb{C}$ where the iterates f^n of f form a normal family. The Julia set is the complement of $F(f)$ and denoted by $J(f)$. The escaping set $I(f)$ is the set of all $z \in \mathbb{C}$ for which $f^n(z) \to \infty$ as $n \to \infty$. We note that $J(f) = \partial I(f)$ by a result of Eremenko [14]. For an introduction to the iteration theory of transcendental entire functions we refer to [5].

Considerable attention has been paid to the dimensions of Julia sets of entire functions; see [36] for a survey, as well as [3, 4, 8, 9, 10, 27, 28, 34] for some recent results not covered there. Many results in this area are concerned with the Eremenko-Lyubich class B consisting of all transcendental entire functions for which the set of critical and finite asymptotic values is bounded. By a result of Eremenko and Lyubich [15, Theorem 1] we have $I(f) \subset J(f)$ for $f \in B$. For a function in the Eremenko-Lyubich class, a lower bound for the dimension of the Julia set can thus be obtained from such a bound for the escaping set. This played a key role already in McMullen's seminal paper [24], and it has been used in many subsequent papers.

We denote the Hausdorff dimension, packing dimension and upper box dimension of a subset A of the complex plane $\mathbb{C}$ by $\dim_{\mathrm{H}} A$, $\dim_{\mathrm{P}} A$ and $\overline{\dim}_{\mathrm{B}} A$, respectively, noting that the upper box dimension is defined only for bounded sets A. We refer to the book by Falconer [16] for the definitions and a thorough treatment of these concepts. Here we only note that we always have [16, p. 48]

$$\dim_{\mathrm{H}} A \leq \dim_{\mathrm{P}} A \leq \overline{\dim}_{\mathrm{B}} A.$$

The exceptional set $E(f)$ of a transcendental entire function f consists of all points in $\mathbb{C}$ with finite backward orbit. It is an immediate consequence of Picard's theorem

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that $E(f)$ contains at most one point. The following result is one part of a theorem of Rippon and Stallard [30, Theorem 1.2].

Theorem A. *Let f be a transcendental entire function, A a backward invariant subset of $J(f)$ and U a bounded open subset of $\mathbb{C}$ whose closure does not intersect $E(f)$. Then $\overline{\dim}_{\mathrm{B}}(U \cap A) = \dim_{\mathrm{P}} A$. In particular,*

$$\overline{\dim}_{\mathrm{B}}(U \cap J(f)) = \dim_{\mathrm{P}} J(f).$$

For functions in the Eremenko-Lyubich class they obtained the following result [30, Theorem 1.1].

Theorem B. *Let $f \in B$. Then $\dim_{\mathrm{P}} J(f) = 2$.*

It follows from Theorem A that Theorem B is equivalent to the result that $\overline{\dim}_{\mathrm{B}}(U \cap J(f)) = 2$ for some bounded open set U satisfying $\overline{U} \cap E(f) = \emptyset$. In order to show this, Rippon and Stallard actually proved that $\overline{\dim}_{\mathrm{B}}(U \cap I(f)) = 2$ for such a set U and then used the result of Eremenko and Lyubich quoted above.

The maximum modulus and minimum modulus of an entire function f are defined by

$$M(r,f) := \max_{|z|=r} |f(z)| \quad \text{and} \quad L(r,f) := \min_{|z|=r} |f(z)|,$$

and the lower order of f is given by

$$\lambda(f) := \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

The $\cos\pi\lambda$-theorem (see [19, Chapter 5, Theorem 3.4] or [21, Section 6.2]) says that if $\lambda(f) < 1$, then

$$\limsup_{r\to\infty} \frac{\log L(r,f)}{\log M(r,f)} \geq \cos(\pi\lambda(f)). \tag{1.1}$$

It follows that $\lambda(f) \geq 1/2$ if $L(r,f)$ is bounded. In fact, the boundedness of $L(r,f)$ even implies (see [19, Chapter 5, Theorem 1.3] or [21, Theorem 6.4]) that

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{\sqrt{r}} > 0. \tag{1.2}$$

The main tool used by Eremenko and Lyubich to prove that $I(f) \subset J(f)$ for $f \in B$ is a logarithmic change of variable. This method shows in particular that if $f \in B$, then f is bounded on a curve tending to ∞; see [15, p. 993]. Hence $L(r,f)$ is bounded and thus $\lambda(f) \geq 1/2$ for $f \in B$ by (1.1). More precisely, we even have (1.2) for $f \in B$. (The observation that (1.2) holds for functions in B seems to have been made first in [7, Proof of Corollary 2] and [23, p. 1788]; see also [30, Lemma 3.5].)

It follows from a result of Baker [2, Corollary to Theorem 3.1] that all components of $F(f)$ are simply connected if f is bounded on a curve tending to ∞. In particular, if $f \in B$, then $F(f)$ has no multiply connected components [15, Proposition 3].

In view of these results the following theorem can be considered as a generalization of Theorem B.

Theorem 1.1. *Let f be a transcendental entire function satisfying*

$$\liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log\log r} = \infty. \tag{1.3}$$

If $F(f)$ has no multiply connected component, then

$$\dim_{\mathrm{P}}(I(f) \cap J(f)) = 2. \tag{1.4}$$

Since multiply connected components of $F(f)$ are contained in $I(f)$, we see that $I(f)$ has interior points if $F(f)$ has such a component. We conclude that $\dim_P I(f) = 2$ for all entire functions satisfying (1.3).

Since (1.2) implies (1.3), the following result is an immediate consequence of Theorem 1.1 and the results stated before it.

Corollary 1.2. *Let f be a transcendental entire function which is bounded on a curve tending to ∞. Then* $\dim_{\mathrm{P}}(I(f) \cap J(f)) = 2$.

More generally, we have the following result.

Corollary 1.3. *Let f be a transcendental entire function and suppose that*

$$\limsup_{r\to\infty} \frac{\log L(r,f)}{\log M(r,f)} < 1. \tag{1.5}$$

Then $\dim_{\mathrm{P}}(I(f) \cap J(f)) = 2$.

To deduce Corollary 1.3 from Theorem 1.1 we note that it follows from (1.5) and the $\cos \pi\lambda$-theorem (1.1) that $\lambda(f) > 0$, which in turn implies that (1.3) holds.

To see that (1.5) rules out the possibility of multiply connected components of $F(f)$ we note that Zheng [37, Corollary 1], as a corollary to the main result of his paper, proved that if $F(f)$ has a multiply connected component, then

$$\limsup_{r\to\infty} \frac{\log L(r,f)}{\log M(r,f)} > 0. \tag{1.6}$$

A slight extension of his argument shows that his main result actually yields that

$$\limsup_{r\to\infty} \frac{\log L(r,f)}{\log M(r,f)} = 1 \tag{1.7}$$

if $F(f)$ has a multiply connected component; see Proposition 4.1. Thus the hypotheses of Theorem 1.1 are satisfied if (1.5) holds. Actually the condition (1.5) can be further relaxed; cf. Remark 4.2.

Zheng [37, Corollary 5] also showed that the Fatou set of a transcendental entire function f has no multiply connected component if

$$\log M(2r,f) \geq d \log M(r,f) \tag{1.8}$$

for some $d > 1$ and all large r. It is easy to see that (1.8) implies that $\lambda(f) > 0$ and hence that (1.3) holds. Thus we obtain the following corollary to Theorem 1.1.

Corollary 1.4. *Let f be a transcendental entire function satisfying* (1.8). *Then* $\dim_{\mathrm{P}}(I(f) \cap J(f)) = 2$.

We mention that in Theorem B and in Theorem 1.1, as well as in the corollaries to Theorem 1.1, the packing dimension cannot be replaced by the Hausdorff dimension. In fact, it is shown in [28, Corollary 1.4] that there exists a function $f \in B$ for which $\dim_{\mathrm{H}} I(f) = 1$ and in [35] that for every $\varepsilon > 0$ there exists a function $f \in B$ such that $\dim_{\mathrm{H}} J(f) < 1 + \varepsilon$, and the functions considered in [28, 35] satisfy (1.8) as well. On the other hand, for every transcendental entire function f the Hausdorff dimension of $I(f) \cap J(f)$ is at least 1, since this set contains continua; see [29, Theorem 5] and [33, Theorem 1.3].

Concerning the proof of Theorem 1.1, we note that in view of Theorem A the conclusion of Theorem 1.1 is equivalent to the statement that

$$\overline{\dim}_{\mathrm{B}}(U \cap I(f) \cap J(f)) = 2 \tag{1.9}$$

for some bounded open set U satisfying $\overline{U} \cap E(f) = \emptyset$, which in turn is equivalent to (1.9) holding for all bounded open sets U. We shall show in our proof that (1.9) holds for some bounded open set U whose closure does not intersect $E(f)$.

The main tools used in the proof are certain estimates of the logarithmic derivative and a version of the Ahlfors islands theorem. A similar technique was used in [8] where it was shown that under a suitable regularity condition on the growth of f we even have $\dim_{\mathrm{H}}(I(f) \cap J(f)) = 2$.

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2. Preliminary lemmas

We use the standard terminology of Nevanlinna theory and, in particular, denote by $T(r, f)$ the Nevanlinna characteristic of a meromorphic function f; see [19, 20]. First we note that using the well-known inequality

$$T(r, f) \leq \log^+ M(r, f) \leq \frac{R+r}{R-r} T(R, f) \quad \text{for } 0 < r < R \tag{2.1}$$

we easily see that the growth condition (1.3) is equivalent to

$$\liminf_{r \to \infty} \frac{\log T(r, f)}{\log \log r} = \infty. \tag{2.2}$$

We shall need a number of lemmas and begin with the following estimate of the logarithmic derivative [19, p. 88].

Lemma 2.1. *Let f be an entire function satisfying $f(0) = 1$. Then*

$$\left| \frac{f'(z)}{f(z)} \right| \leq \frac{4s}{(s - |z|)^2} T(s, f) + \sum_{|z_j| \leq s} \frac{2}{|z - z_j|},$$

for $s > |z|$, where (z_j) is the sequence of zeros of f.

In order to estimate the sum on the right hand side we shall use the following result due to Fuchs and Macintyre [18]. Here and in the following we denote by $D(a, r)$ the open disk and by $\overline{D}(a, r)$ the closed disk of radius r around a point a.

Lemma 2.2. *Let $z_1, z_2, \ldots, z_n \in \mathbb{C}$ and let $H > 0$.*

(i) *There exist $l \in \{1, 2, \ldots, n\}$, $u_1, u_2, \ldots, u_l \in \mathbb{C}$ and $s_1, s_2, \ldots, s_l > 0$ satisfying*

$$\sum_{k=1}^{l} s_k^2 \leq 4H^2$$

such that

$$\sum_{k=1}^{n} \frac{1}{|z - z_k|} \leq \frac{2n}{H} \quad \text{for } z \notin \bigcup_{k=1}^{l} D(u_k, s_k).$$

(ii) *There exist* $m \in \{1, 2, \ldots, n\}$, $v_1, v_2, \ldots, v_m \in \mathbb{C}$ *and* $t_1, t_2, \ldots, t_m > 0$ *satisfying*

$$\sum_{k=1}^{m} t_k \le 2H$$

such that

$$\sum_{k=1}^{n} \frac{1}{|z - z_k|} \le \frac{n(1 + \log n)}{H} \quad \text{for } z \notin \bigcup_{k=1}^{m} D(v_k, t_k).$$

Actually Fuchs and Macintyre write An/H instead of $2n/H$ in part (i), with a constant A, but their argument shows that one can take $A = 2$. We note that the term $\log n$ in (ii) cannot be omitted; cf. [1].

We will also require the following version of the Borel-Nevanlinna growth lemma; see [19, Chapter 3, Theorem 1.2] or [12, Section 3.3]. Here a measurable subset E of $(0, \infty)$ is said to be of finite logarithmic measure if

$$\int_E \frac{dt}{t} < \infty.$$

Lemma 2.3. *Let* $F : [r_0, \infty) \to [t_0, \infty)$ *and* $\varphi : [t_0, \infty) \to (0, \infty)$ *be non-decreasing functions, with* $r_0, t_0 > 0$. *Suppose that*

$$\int_{t_0}^{\infty} \frac{dt}{\varphi(t)} < \infty.$$

Then there exists a set $E \subset [r_0, \infty)$ *of finite logarithmic measure such that*

$$F\left(r\left(1 + \frac{1}{\varphi(F(r))}\right)\right) \le F(r) + 1 \quad \text{for } r \notin E.$$

We shall also need a result from the Ahlfors theory of covering surfaces; cf. [20, Theorem 6.2] or [6]. Here $\mathbb{D} := D(0, 1)$ is the unit disk.

Lemma 2.4. *Let* D_1, D_2, D_3 *be Jordan domains with pairwise disjoint closures. Then there exists* $\mu > 0$ *such that if* $h : \mathbb{D} \to \mathbb{C}$ *is a holomorphic function satisfying*

$$\frac{|h'(0)|}{1 + |h(0)|^2} \ge \mu,$$

then $\mathbb{D}$ *has a subdomain which is mapped bijectively onto one of the domains* D_ν *by* h.

Finally, we shall repeatedly use the following result known as the Koebe distortion theorem and the Koebe one quarter theorem.

Lemma 2.5. *Let* $g : D(a, r) \to \mathbb{C}$ *be univalent,* $0 < \lambda < 1$ *and* $z \in \overline{D}(a, \lambda r)$. *Then*

$$\frac{1}{(1+\lambda)^2} |g'(a)| \le \frac{|g(z) - g(a)|}{|z - a|} \le \frac{1}{(1-\lambda)^2} |g'(a)|$$

and

$$\frac{1-\lambda}{(1+\lambda)^3} |g'(a)| \le |g'(z)| \le \frac{1+\lambda}{(1-\lambda)^3} |g'(a)|.$$

Moreover,

$$g(D(a, r)) \supset D\left(g(a), \frac{1}{4}|g'(a)| r\right).$$

Usually Koebe's theorems are stated only for the special case that $a = 0$, $r = 1$, $g(0) = 0$ and $g'(0) = 1$, but the above result easily follows from this special case.

3. Proof of Theorem 1.1

Without loss of generality we may assume that $f(0) = 1$. We denote by $n(r,0)$ the number of zeros of f in the closed disk of radius r around 0 and put

$$N(r,0) := \int_0^r \frac{n(t,0)}{t} dt.$$

By Nevanlinna's first fundamental theorem we have

$$T(s,f) \geq N(s,0) \geq \int_r^s \frac{n(t,0)}{t} dt \geq n(r,0) \int_r^s \frac{dt}{t} = n(r,0) \log\left(\frac{s}{r}\right) \tag{3.1}$$

for $s > r > 0$. Applying Lemma 2.3 with $\varphi(x) = x^2/6$ and $F(r) = \log T(r,f)$ we obtain a set E of finite logarithmic measure such that

$$T\left(r\left(1 + \frac{6}{[\log T(r,f)]^2}\right), f\right) \leq eT(r,f) \quad \text{for } r \notin E. \tag{3.2}$$

For $m \in \mathbb{N}$ we shall use the abbreviation

$$R_m(r) := r\left(1 + \frac{m}{[\log T(r,f)]^2}\right)$$

so that (3.2) takes the form

$$T\left(R_6(r), f\right) \leq eT(r,f) \quad \text{for } r \notin E. \tag{3.3}$$

Using (2.1) we find that if $r \notin E$ is sufficiently large, then

$$\log M(r,f) \leq \frac{R_6(r) + r}{R_6(r) - r} T(R_6(r), f) \leq T(r,f)[\log T(r,f)]^2. \tag{3.4}$$

For measurable $X \subset \mathbb{R}$ and $Y \subset \mathbb{C}$ we denote by $\operatorname{length} X$ the 1-dimensional Lebesgue measure of X and by $\operatorname{area} Y$ the 2-dimensional Lebesgue measure of Y.

Lemma 3.1. *For sufficiently large $r \notin E$ there exists a closed subset F_r of $[R_1(r), R_3(r)]$ with*

$$\operatorname{length} F_r \geq \frac{r}{[\log T(r,f)]^2} \tag{3.5}$$

such that

$$\left|\frac{f'(z)}{f(z)}\right| \leq \frac{T(r,f)[\log T(r,f)]^7}{r} \quad \text{for } |z| \in F_r. \tag{3.6}$$

Moreover, for $\eta > 0$ there exist an integer $l \in \{1, 2, \ldots, n\left(R_5(r), 0\right)\}$, points $u_1, u_2, \ldots, u_l \in \mathbb{C}$ and $s_1, s_2, \ldots, s_l > 0$ satisfying

$$\sum_{k=1}^{l} s_k^2 \leq \frac{r^2}{T(r,f)^\eta} \tag{3.7}$$

such that

$$\left|\frac{f'(z)}{f(z)}\right| \leq \frac{T(r,f)^{1+\eta}}{r} \quad \text{if } r \leq |z| \leq R_4(r) \text{ and } z \notin \bigcup_{k=1}^{l} D(u_k, s_k). \tag{3.8}$$

Proof. It follows from Lemma 2.1 that if $z_1, z_2, \ldots$ are the zeros of f, then

$$\left|\frac{f'(z)}{f(z)}\right| \leq \frac{4R_5(r)}{(R_5(r)-|z|)^2}T(R_5(r),f) + \sum_{|z_j|\leq R_5(r)} \frac{2}{|z-z_j|} \tag{3.9}$$

for $|z| < R_5(r)$. Using (3.3) and noting that $R_5(r) \leq 2r$ for large r we see that if $|z| \leq R_4(r)$, then

$$\frac{4R_5(r)}{(R_5(r)-|z|)^2}T(R_5(r),f) \leq 8e\frac{T(r,f)}{r}[\log T(r,f)]^4 \tag{3.10}$$

for large $r \notin E$. To estimate the sum on the right hand side of (3.9) we apply Lemma 2.2, part (ii), with

$$H = \frac{r}{8[\log T(r,f)]^2} \tag{3.11}$$

and conclude that

$$\sum_{|z_j|\leq R_5(r)} \frac{1}{|z-z_j|} \leq \frac{n(R_5(r),0)(1+\log n(R_5(r),0))}{H} \tag{3.12}$$

outside a union of disks whose sum of radii is at most $2H$. Let P be the set of all $s > 0$ such that $\{z \in \mathbb{C} : |z| = s\}$ intersects the union of these disks. Then length $P \leq 4H$. Thus $F_r := [R_1(r), R_3(r)] \setminus P$ satisfies (3.5), and (3.12) holds for $|z| \in F_r$.

From (3.1) and (3.3) we can deduce that

$$\begin{aligned} n(R_5(r),0) &\leq \frac{T(R_6(r),f)}{\log(R_6(r)/R_5(r))} \\ &\leq 2T(R_6(r),f)[\log T(r,f)]^2 \leq 2eT(r,f)[\log T(r,f)]^2 \end{aligned} \tag{3.13}$$

for large $r \notin E$. Combining this with (3.11) and (3.12) we obtain

$$\begin{aligned} &\sum_{|z_j|\leq R_5(r)} \frac{1}{|z-z_j|} \\ &\leq \frac{16eT(r,f)[\log T(r,f)]^4\left(1+\log\left(2eT(r,f)[\log T(r,f)]^2\right)\right)}{r} \\ &\leq \frac{T(r,f)[\log T(r,f)]^6}{r} \end{aligned} \tag{3.14}$$

if $|z| \in F_r$ and r is large. Now (3.6) follows from (3.9), (3.10) and (3.14).

In order to prove (3.8) we use part (i) of Lemma 2.2 with

$$H = \frac{r}{2T(r,f)^{\eta/2}}. \tag{3.15}$$

This yields l disks $D(u_k, s_k)$ satisfying (3.7) such that

$$\sum_{|z_j|\leq R_5(r)} \frac{1}{|z-z_j|} \leq \frac{2n(R_5(r),0)}{H} \quad \text{for } z \notin \bigcup_{k=1}^{l} D(u_k,s_k), \tag{3.16}$$

with $l \leq n(R_5(r),0)$. Combining (3.13), (3.15) and (3.16) we find that

$$\sum_{|z_j|\leq R_5(r)} \frac{1}{|z-z_j|} \leq \frac{8eT(r,f)[\log T(r,f)]^2 T(r,f)^{\eta/2}}{r}$$

if $z \notin \bigcup_{k=1}^{l} D(u_k, s_k)$. This, together with (3.9) and (3.10), implies (3.8). $\square$

Lemma 3.2. *Let F_r be as in Lemma 3.1 and let $\delta > 0$. If r is sufficiently large, then for each $s \in F_r$ there exists a closed subset J_s of $[0, 2\pi]$ with*

$$\text{length } J_s \geq \frac{1}{T(r,f)^\delta} \tag{3.17}$$

such that if $\theta \in J_s$, then

$$|f(se^{i\theta})| \geq \sqrt{M(r,f)} \tag{3.18}$$

and

$$\left| \frac{f'(se^{i\theta})}{f(se^{i\theta})} \right| \geq \frac{T(r,f)^{1-\delta}}{r}. \tag{3.19}$$

Proof. First we consider the case that

$$\log L(s,f) = \min_{|z|=s} \log |f(z)| \leq \frac{1}{2} \log M(s,f). \tag{3.20}$$

Then there exists z_1 and z_2 with $|z_1| = |z_2| = s$ satisfying

$$\log |f(z_1)| = \frac{1}{2} \log M(s,f) \quad \text{and} \quad \log |f(z_2)| = \log M(s,f)$$

while

$$\frac{1}{2} \log M(s,f) \leq \log |f(z)| \leq \log M(s,f)$$

on one of the two arcs between z_1 and z_2. With $z_1 = se^{i\theta_1}$ and $z_2 = se^{i\theta_2}$ we may assume without loss of generality that $\theta_1 < \theta_2$ and

$$\frac{1}{2} \log M(s,f) \leq \log |f(se^{i\theta})| \leq \log M(s,f)$$

for $\theta_1 \leq \theta \leq \theta_2$. Let J_s be the set of all $\theta \in [\theta_1, \theta_2]$ for which (3.19) holds. Since (3.18) holds for all $\theta \in [\theta_1, \theta_2]$ and thus in particular for $\theta \in J_s$, we only have to estimate the length of J_s. By the choice of z_1 and z_2 we have

$$\begin{aligned} \frac{1}{2} \log M(s,f) &= \log |f(z_2)| - \log |f(z_1)| = \operatorname{Re} \left(\int_{z_1}^{z_2} \frac{f'(z)}{f(z)} dz \right) \\ &\leq \int_{z_1}^{z_2} \left| \frac{f'(z)}{f(z)} \right| |dz| = s \int_{\theta_1}^{\theta_2} \left| \frac{f'(se^{i\theta})}{f(se^{i\theta})} \right| d\theta. \end{aligned} \tag{3.21}$$

Now

$$\begin{aligned} s \int_{\theta_1}^{\theta_2} \left| \frac{f'(se^{i\theta})}{f(se^{i\theta})} \right| d\theta &= s \int_{J_s} \left| \frac{f'(se^{i\theta})}{f(se^{i\theta})} \right| d\theta + s \int_{[\theta_1,\theta_2] \setminus J_s} \left| \frac{f'(se^{i\theta})}{f(se^{i\theta})} \right| d\theta \\ &\leq s \frac{T(r,f)[\log T(r,f)]^7}{r} \text{length } J_s + 2\pi s \frac{T(r,f)^{1-\delta}}{r} \\ &\leq 2T(r,f)[\log T(r,f)]^7 \text{length } J_s + 4\pi \, T(r,f)^{1-\delta}, \end{aligned} \tag{3.22}$$

where we used $s \leq R_3(r) \leq 2r$ in the last inequality. Since $T(r,f) \leq \log M(r,f)$ we have

$$4\pi \, T(r,f)^{1-\delta} \leq \frac{1}{4} \log M(r,f)$$

for large r and this, together with (3.21) and (3.22), yields

$$\frac{1}{4} \log M(r,f) \leq 2T(r,f)[\log T(r,f)]^7 \text{length } J_s.$$

We deduce that

$$\operatorname{length} J_s \geq \frac{1}{8[\log T(r,f)]^7}$$

and thus obtain (3.17).

Now we consider the case that (3.20) does not hold. Then (3.18) holds for all $\theta \in [0, 2\pi]$. Let J_s be the subset of all $\theta \in [0, 2\pi]$ for which (3.19) holds. By the argument principle, we have

$$n(s,0) = \frac{1}{2\pi i}\int_{|z|=s} \frac{f'(z)}{f(z)} dz \leq \frac{s}{2\pi}\int_0^{2\pi} \left|\frac{f'(se^{i\theta})}{f(se^{i\theta})}\right| d\theta.$$

The same argument as in (3.22) now yields

$$2\pi\, n(s,0) \leq 2T(r,f)[\log T(r,f)]^7 \operatorname{length} J_s + 4\pi\, T(r,f)^{1-\delta}. \tag{3.23}$$

Since we are assuming that (3.20) does not hold, we have $|f(re^{i\theta})| \geq 1$ for $0 \leq \theta \leq 2\pi$ and thus $m(r, 1/f) = 0$, where $m(r,\cdot)$ denotes the Nevanlinna proximity function. Nevanlinna's first fundamental theorem, together with the assumption that $f(0) = 1$, yields

$$T(r,f) = N(r,0) \leq N(1,0) + n(r,0)\log r$$

and thus

$$2\pi\, n(s,0) \geq 2\pi\, n(r,0) \geq \frac{T(r,f)}{\log r} \geq 2T(r,f)^{1-\delta/2} \tag{3.24}$$

for large r by (2.2). Combining (3.23) and (3.24) we obtain

$$T(r,f)^{1-\delta/2} \leq 2T(r,f)[\log T(r,f)]^7 \operatorname{length} J_s,$$

from which (3.17) easily follows. $\square$

Remark 3.1. In the proof of Lemma 3.2, the hypothesis (1.3) is used only in (3.24) and thus only in the case that (3.20) does not hold. The hypothesis (1.3) is not used in Lemma 3.1.

For small $\delta > 0$ and large $r \notin E$ we consider the set

$$A(r) := \Bigg\{ z \in \mathbb{C} : R_1(r) \leq |z| \leq R_3(r), \\ |f(z)| \geq \sqrt{M(r,f)},\ \left|\frac{f'(z)}{f(z)}\right| \geq \frac{T(r,f)^{1-\delta}}{r} \Bigg\}.$$

With the notation of Lemmas 3.1 and 3.2 we have

$$A(r) \supset \left\{ se^{i\theta} : s \in F_r,\ \theta \in J_s \right\}$$

and thus we can deduce from these lemmas that

$$\begin{aligned} \operatorname{area} A(r) &\geq \operatorname{length}(F_r) \inf_{s\in F_r} (s \operatorname{length}(J_s)) \\ &\geq \frac{r^2}{[\log T(r,f)]^2 T(r,f)^{\delta}} \geq \frac{2r^2}{T(r,f)^{2\delta}} \end{aligned} \tag{3.25}$$

for large r. We put

$$\rho(r) := \frac{r}{T(r,f)^{1-2\delta}}$$

and

$$B(r) := A(r) \setminus \bigcup_{j=1}^{l} D\left(u_j, s_j + \rho(r)\right),$$

where $l, u_1, \ldots, u_l, s_1, \ldots, s_l$ are chosen according to Lemma 3.1, taking $\eta := 3\delta$ there.

We use the notation $\operatorname{ann}(r, R) := \{z \in \mathbb{C} : r < |z| < R\}$ for an annulus with radii r and R. Given a constant $M > 1$ we have $M\rho(r) \le r/[\log T(r, f)]^2$ for large r and thus

$$D(b, M\rho(r)) \subset \operatorname{ann}(r, R_4(r)) \quad \text{for } b \in A(r). \tag{3.26}$$

Hence

$$D(b, \rho(r)) \subset \operatorname{ann}(r, R_4(r)) \setminus \bigcup_{j=1}^{l} D(u_j, s_j)$$

and thus

$$\left|\frac{f'(z)}{f(z)}\right| \le \frac{T(r, f)^{1+3\delta}}{r} \quad \text{if } b \in B(r) \text{ and } z \in D(b, \rho(r)) \tag{3.27}$$

by (3.8). In particular, f has no zeros in $D(b, \rho(r))$.

We want to show that the area of $B(r)$ is not much smaller than that of $A(r)$. In order to do so we note that (3.7), (3.13) and the definition of $\rho(r)$ yield

$$\begin{aligned}
\operatorname{area}\left(\bigcup_{j=1}^{l} D\left(u_j, s_j + \rho(r)\right)\right) &= \pi \sum_{j=1}^{l} \left(s_j + \rho(r)\right)^2 \\
&\le 2\pi \sum_{j=1}^{l} \left(s_j^2 + \rho(r)^2\right) \\
&\le 2\pi \frac{r^2}{T(r, f)^{3\delta}} + 2\pi\rho(r)^2 n(R_5(r), 0) \\
&\le 2\pi \frac{r^2}{T(r, f)^{3\delta}} + 4\pi e \frac{r^2[\log T(r, f)]^2}{T(r, f)^{1-4\delta}} \\
&\le \frac{r^2}{T(r, f)^{2\delta}}
\end{aligned}$$

for large r, provided $\delta < 1/6$. Combining this with (3.25) we obtain

$$\operatorname{area} B(r) \ge \operatorname{area} A(r) - \operatorname{area}\left(\bigcup_{j=1}^{l} D\left(u_j, s_j + \rho(r)\right)\right) \ge \frac{r^2}{T(r, f)^{2\delta}} \tag{3.28}$$

for large r.

Now let $m(r)$ be the maximal number of pairwise disjoint disks of radius $\rho(r)$ whose centers are in $B(r)$. Then $B(r)$ is contained in a union of $m(r)$ disks of radius $2\rho(r)$ and thus

$$\operatorname{area} B(r) \le 4\pi m(r)\rho(r)^2 = 4\pi m(r) \frac{r^2}{T(r, f)^{2-4\delta}}.$$

for large r. Together with (3.28) we obtain

$$m(r) \ge \frac{1}{4\pi} T(r, f)^{2-6\delta} \ge T(r, f)^{2-7\delta}. \tag{3.29}$$

Recall that if $b \in B(r)$, for some large $r \notin E$, then f has no zeros in $D(b, \rho(r))$. Thus we can define a branch ϕ of the logarithm of f in $D(b, \rho(r))$; that is, there exists a holomorphic function $\phi : D(b, \rho(r)) \to \mathbb{C}$ such that $\exp \phi(z) = f(z)$ for $z \in D(b, \rho(r))$. Of course, the other branches of the logarithm of f are then given by $z \mapsto \phi(z) + 2\pi i n$ where $n \in \mathbb{Z}$.

For $a \in \mathbb{R}$ we will consider the domain

$$Q(a) := \{z \in \mathbb{C} : |\operatorname{Re} z - a| < 1, |\operatorname{Im} z| < 2\pi\}$$

Lemma 3.3. *Let $b \in B(r)$, with $r \notin E$ sufficiently large. Then there exists a branch ϕ of the logarithm of f defined in $D(b, \rho(r))$ and a subdomain U of $D(b, \rho(r))$ such that ϕ maps U bijectively onto $Q(\log |f(b)|)$.*

Proof. Let ϕ_0 be a fixed branch of the logarithm of f defined in $D(b, \rho(r))$. Define $h : \mathbb{D} \to \mathbb{C}$, $h(z) = \phi_0(b + \rho(r)z) - \phi_0(b)$. Then $h(0) = 0$ and

$$|h'(0)| = \rho(r)|\phi_0'(b)| = \rho(r) \left| \frac{f'(b)}{f(b)} \right| \geq \rho(r) \frac{T(r, f)^{1-\delta}}{r} = T(r, f)^{\delta}$$

by the definition of $\rho(r)$ and $A(r)$ and since $B(r) \subset A(r)$. For $k \in \{1, 2, 3\}$ we put $D_k := \{z \in \mathbb{C} : |\operatorname{Re} z| < 1, |\operatorname{Im} z - 12k\pi| < 4\pi\}$. Lemma 2.4 now implies that if $r \notin E$ is large enough, then there exists a subdomain V of $\mathbb{D}$ and $k \in \{1, 2, 3\}$ such that h maps V bijectively onto D_k. This implies that $D(b, \rho(r))$ contains a subdomain W which is mapped bijectively onto

$$\{z \in \mathbb{C} : |\operatorname{Re} z - \log |f(b)|| < 1, |\operatorname{Im} z - \operatorname{Im} \phi_0(b) - 12\pi k| < 4\pi\}$$

by ϕ_0. Choosing $n \in \mathbb{Z}$ such that $|2\pi n - \operatorname{Im} \phi_0(b) - 12k\pi| \leq \pi$ we find that there exists a domain $U \subset W \subset D(b, \rho(r))$ which is mapped bijectively onto

$$\{z \in \mathbb{C} : |\operatorname{Re} z - \log |f(b)|| < 1, |\operatorname{Im} z - 2\pi n| < 2\pi\}$$

by ϕ_0. The branch ϕ of the logarithm given by $\phi(z) = \phi_0(z) - 2\pi i n$ now has the required property. $\square$

We note that if U is as in Lemma 3.3, then

$$f(U) = \exp Q(\log |f(b)|) = \operatorname{ann} \left(\frac{1}{e} |f(b)|, e|f(b)| \right). \tag{3.30}$$

Now we fix some large $r_0 \notin E$, put $\rho_0 := \rho(r_0)$ and choose $b_0 \in B(r_0)$. Lemma 3.3 yields the existence of a branch ϕ_0 of the logarithm of f and a domain $U_0 \subset D(b_0, \rho_0)$ which is mapped bijectively onto $Q(\log |f(b_0)|)$ by ϕ_0. Since $|f(b_0)| \geq \sqrt{M(r_0, f)}$ we also have $|f(b_0)| \geq 2r_0$ and the interval $[|f(b_0)|, 2|f(b_0)|]$ contains a point r_1 which does not belong to E, provided r_0 is chosen large enough. Note that $r_1 \geq 2r_0$. We now choose $b_1 \in B(r_1)$ and with $\rho_1 := \rho(r_1)$ we find a domain $U_1 \subset D(b_1, \rho_1)$ which is mapped bijectively onto $Q(\log |f(b_1)|)$ by a branch ϕ_1 of the logarithm of f.

Inductively we thus obtain sequences (r_k), (ρ_k), (b_k), (U_k) and (ϕ_k) satisfying

$$r_k \in [\,|f(b_{k-1})|, 2|f(b_{k-1})|\,] \setminus E \subset [2r_{k-1}, \infty) \setminus E,$$

$\rho_k = \rho(r_k)$ and $b_k \in B(r_k)$ such that U_k is a subdomain of $D(b_k, \rho_k)$ and ϕ_k is a branch of the logarithm of f with the property that $\phi_k : U_k \to Q(\log |f(b_k)|)$ is bijective.

For large r_0 we have $R_4(r_k) \leq 5r_k/4 \leq 5|f(b_k)|/2$ for all k and thus

$$\begin{aligned} D(b_k, M\rho_k) &\subset \operatorname{ann}(r_k, R_4(r_k)) \\ &\subset \operatorname{ann}\left(|f(b_{k-1})|, \frac{5}{2}|f(b_{k-1})|\right) \subset \exp Q(\log|f(b_{k-1})|) \end{aligned} \tag{3.31}$$

by (3.26) and (3.30). Hence there exists a branch L of the logarithm which maps $D(b_k, M\rho_k)$ into $Q(\log|f(b_{k-1})|)$. Then $\psi_k := \phi_{k-1}^{-1} \circ L$ is a branch of the inverse function of f which maps $D(b_k, M\rho_k)$ into U_{k-1}.

We put

$$V_k := (\psi_1 \circ \psi_2 \circ \ldots \circ \psi_k)\left(\overline{D}(b_k, \rho_k)\right).$$

Then V_k is compact and $V_{k+1} \subset V_k$. Thus $\bigcap_{k=1}^{\infty} V_k \neq \emptyset$. We will show that this intersection contains only one point.

In order to do so we note that since $\psi_k : D(b_k, M\rho_k) \to U_{k-1}$ is univalent and $U_{k-1} \subset D(b_{k-1}, \rho_{k-1}) \subset D(\psi_k(b_k), 2\rho_{k-1})$, Koebe's one quarter theorem implies that $2\rho_{k-1} \geq M\rho_k|\psi_k'(b_k)|/4$ and thus $|\psi_k'(b_k)| \leq 8\rho_{k-1}/(M\rho_k)$. The Koebe distortion theorem, applied with $\lambda = 1/M$, now yields

$$\begin{aligned} \sup_{z \in \overline{D}(b_k, \rho_k)} |\psi_k'(z)| &\leq \frac{1+\lambda}{(1-\lambda)^3}|\psi_k'(b_k)| \\ &= \frac{M^2(M+1)}{(M-1)^3}|\psi_k'(b_k)| \leq \frac{8M(M+1)}{(M-1)^3}\frac{\rho_{k-1}}{\rho_k}. \end{aligned}$$

Choosing $M = 20$ we obtain

$$\sup_{z \in \overline{D}(b_k, \rho_k)} |\psi_k'(z)| \leq \frac{1}{2}\frac{\rho_{k-1}}{\rho_k}.$$

If $K \subset \overline{D}(b_k, \rho_k)$ is compact, we thus have

$$\operatorname{diam} \psi_k(K) \leq \frac{1}{2}\frac{\rho_{k-1}}{\rho_k} \operatorname{diam} K,$$

where $\operatorname{diam} K$ denotes the diameter of K. Hence

$$\frac{\operatorname{diam} \psi_k(K)}{\rho_{k-1}} \leq \frac{1}{2}\frac{\operatorname{diam} K}{\rho_k}.$$

Inductively we obtain

$$\frac{\operatorname{diam} V_k}{\rho_0} = \frac{\operatorname{diam}(\psi_1 \circ \psi_2 \circ \ldots \circ \psi_k)\left(\overline{D}(b_k, \rho_k)\right)}{\rho_0} \leq \frac{1}{2^k}\frac{\operatorname{diam} \overline{D}(b_k, \rho_k)}{\rho_k} = \frac{1}{2^{k-1}}$$

and thus

$$\lim_{k \to \infty} \operatorname{diam} V_k = 0 \tag{3.32}$$

so that

$$\bigcap_{k=1}^{\infty} V_k = \{z_0\}$$

for some z_0.

It follows from the definition of V_k and (3.26) that

$$f^k(V_k) = \overline{D}(b_k, \rho_k) \subset \operatorname{ann}(r_k, R_4(r_k))$$

and hence that $z_0 \in I(f)$. Moreover,

$$\begin{aligned} f^{k+1}(V_k) &= f\left(\overline{D}(b_k,\rho_k)\right) \supset f(U_k) \\ &= \operatorname{ann}\left(\frac{1}{e}|f(b_k)|, e|f(b_k)|\right) \supset \operatorname{ann}\left(\frac{1}{e}r_{k+1}, \frac{e}{2}r_{k+1}\right). \end{aligned}$$

As $F(f)$ does not have multiply connected components,

$$\operatorname{ann}\left(\frac{1}{e}r_{k+1}, \frac{e}{2}r_{k+1}\right) \cap J(f) \neq \emptyset$$

for large k. Since $J(f)$ is completely invariant, we conclude that V_k intersects $J(f)$, and since $J(f)$ is closed, this yields that $z_0 \in I(f) \cap J(f)$.

In order to estimate the upper box dimension of $I(f)\cap J(f)$, we note that in the above construction of the sequences (r_k), (ρ_k), (b_k), (U_k) and (ϕ_k) we have $m(r_k)$ choices for the point $b_k \in B(r_k)$ such that the disks of radius ρ_k around these points are pairwise disjoint, with $m(r_k)$ satisfying (3.29). We fix $k \in \mathbb{N}$, choose r_j, ρ_j, b_j, U_j and ϕ_j for $0 \leq j \leq k-1$ as before and denote by b_k^ν choices of b_k with the above property, with $1 \leq \nu \leq m_k := m(r_k)$

In other words, we take $r_k \in [|f(b_{k-1})|, 2|f(b_{k-1})|] \setminus E$ and $\rho_k := \rho(r_k)$ as before and choose $b_k^\nu \in B(r_k)$, where $1 \leq \nu \leq m_k$, such that

$$D(b_k^i, \rho_k) \cap D(b_k^j, \rho_k) = \emptyset \quad \text{for } i \neq j. \tag{3.33}$$

Then for $1 \leq \nu \leq m_k$ there exists a branch $\psi_k^\nu : D(b_k^\nu, 2\rho_k) \to U_{k-1}$ of the inverse function of f that is of the form $\psi_k^\nu = \phi_{k-1}^{-1} \circ L_\nu$ for some branch L_ν of the logarithm which maps $D(b_k^\nu, 2\rho_k)$ into $Q(\log|f(b_{k-1})|)$. For $a \in \mathbb{R}$ we put

$$P(a) := \left\{ z \in \mathbb{C} :\ 0 \leq \operatorname{Re} z - a \leq \log\frac{5}{2},\ |\operatorname{Im} z| \leq \frac{3}{2}\pi \right\}.$$

In view of (3.31) we may actually assume that L_ν maps $D(b_k^\nu, 2\rho_k)$ into the compact subset $P(\log|f(b_{k-1})|)$ of $Q(\log|f(b_{k-1})|)$. Since ϕ_{k-1}^{-1} is univalent in $Q(\log|f(b_{k-1})|)$ we thus conclude that there exists an absolute constant $\alpha > 1$ such that

$$\frac{1}{\alpha} \leq \frac{|(\phi_{k-1}^{-1})'(\zeta)|}{|(\phi_{k-1}^{-1})'(z)|} \leq \alpha \quad \text{for } \zeta, z \in P(\log|f(b_{k-1})|). \tag{3.34}$$

An explicit upper bound for α could be determined from the Koebe distortion theorem, but we do not need such an estimate. Put

$$\Lambda_k^\nu := \psi_1 \circ \psi_2 \circ \ldots \circ \psi_{k-1} \circ \psi_k^\nu = \psi_1 \circ \psi_2 \circ \ldots \circ \psi_{k-1} \circ \phi_{k-1}^{-1} \circ L_\nu.$$

Since $\psi_1 \circ \psi_2 \circ \ldots \circ \psi_{k-1}$ is univalent in $D(b_{k-1}, 2\rho_{k-1})$, we deduce from (3.34) and the Koebe distortion theorem that there exists $\beta > 1$ such that

$$\frac{1}{\beta} \leq \frac{|(\Lambda_k^\nu)'(b_k^\nu)|}{|(\Lambda_k^1)'(b_k^1)|} \leq \beta \quad \text{for } 1 \leq \nu \leq m_k. \tag{3.35}$$

We put

$$V_k^\nu := \Lambda_k^\nu\left(\overline{D}(b_k^\nu, \rho_k)\right) \quad \text{and} \quad v_k^\nu := \Lambda_k^\nu(b_k^\nu).$$

As above we see that each V_k^ν contains a point of $I(f) \cap J(f)$.

Since Λ_k^ν is univalent in $D(b_k^\nu, 2\rho_k)$ we deduce from the Koebe distortion theorem (with $\lambda = 1/2$) that

$$\overline{D}\left(v_k^\nu, \frac{4}{9}\sigma_k^\nu\right) \subset V_k^\nu \subset \overline{D}\left(v_k^\nu, 4\sigma_k^\nu\right) \tag{3.36}$$

where

$$\sigma_k^\nu := |(\Lambda_k^\nu)'(b_k^\nu)|\,\rho_k = \frac{\rho_k}{|(f^k)'(v_k^\nu)|}.$$

We put $\sigma_k := \sigma_k^1$. It follows from (3.32) and (3.36) that

$$\lim_{k\to\infty} \sigma_k = 0.$$

Using (3.35) and (3.36) we obtain

$$\overline{D}\left(v_k^\nu, \frac{4}{9\beta}\sigma_k\right) \subset V_k^\nu \subset \overline{D}\left(v_k^\nu, 4\beta\sigma_k\right) \tag{3.37}$$

Fix a square of sidelength σ_k centered at a point c and denote by N the cardinality of the set of all $\nu \in \{1,\dots,m_k\}$ for which V_k^ν intersects this square. It follows from (3.37) that if if V_k^ν intersects this square, then

$$V_k^\nu \subset D(c, (8\beta+1)\sigma_k).$$

On the other hand, (3.37) also says that V_k^ν contains a disk of radius $4\sigma_k/(9\beta)$. Since the V_k^ν have pairwise disjoint interior by (3.33), we obtain

$$N\pi\left(\frac{4\sigma_k}{9\beta}\right)^2 \leq \pi\left((8\beta+1)\sigma_k\right)^2.$$

Thus $N \leq N_0 := \lfloor 81\beta^2(8\beta+1)^2/16 \rfloor$.

We now put a grid of sidelength σ_k over U_0. Then each square of this grid can intersect at most N_0 of the m_k domains V_k^ν. Recalling that each of the domains V_k^ν contains a point of $I(f)\cap J(f)$ we see that at least m_k/N_0 squares of our grid intersect $I(f)\cap J(f)$. We conclude that

$$\overline{\dim}_{\mathrm{B}}(U_0\cap I(f)\cap J(f)) \geq \limsup_{k\to\infty} \frac{\log(m_k/N_0)}{-\log\sigma_k}. \tag{3.38}$$

By (3.29) we have

$$m_k = m(r_k) \geq T(r_k, f)^{2-7\delta}. \tag{3.39}$$

It remains to estimate σ_k. In order to do so we note that

$$(f^k)'(v_k^1) = \prod_{j=0}^{k-1} f'(f^j(v_k^1)). \tag{3.40}$$

Since

$$f^j(v_k^1) \subset U_j \subset D(b_j, \rho_j)$$

it follows from (3.27) that

$$\left|\frac{f'(f^j(v_k^1))}{f(f^j(v_k^1))}\right| \leq \frac{T(r_j, f)^{1+3\delta}}{r_j}.$$

This yields

$$\left|f'(f^j(v_k^1))\right| \leq \frac{T(r_j, f)^{1+3\delta}}{r_j}\left|f^{j+1}(v_k^1)\right| \leq e\frac{T(r_j, f)^{1+3\delta}}{r_j} r_{j+1}. \tag{3.41}$$

We deduce from (3.40) and (3.41) that

$$\left|(f^k)'(v_k^1)\right| \leq \prod_{j=0}^{k-1} e\frac{T(r_j, f)^{1+3\delta}}{r_j} r_{j+1} = e^k\frac{r_k}{r_0}\left(\prod_{j=0}^{k-1} T(r_j, f)\right)^{1+3\delta}$$

Using the definition of ρ_k and σ_k we thus have

$$\sigma_k \geq \frac{r_0}{e^k T(r_k, f)^{1-2\delta} \left(\prod_{j=0}^{k-1} T(r_j, f) \right)^{1+3\delta}} \tag{3.42}$$

By construction, we have

$$\log r_{j+1} \geq \log |f(b_j)| \geq \frac{1}{2} \log M(r_j, f)$$

for $0 \leq j \leq k-1$. Given a large positive number q, we deduce from (1.3) that if r_0 is sufficiently large, then

$$\log M(r_{j+1}, f) \geq (2 \log r_{j+1})^{q+1}$$

for $0 \leq j \leq k-1$. Combining the last two estimates with (3.4) we find that

$$T(r_j, f) \leq \log M(r_j, f) \leq 2 \log r_{j+1} \leq (\log M(r_{j+1}, f))^{1/(q+1)} \leq T(r_{j+1}, f)^{1/q}$$

for large r_0. We conclude that

$$\prod_{j=0}^{k-1} T(r_j, f) \leq T(r_k, f)^{\tau}, \tag{3.43}$$

where

$$\tau := \sum_{j=1}^{k} \left(\frac{1}{q} \right)^j \leq \sum_{j=1}^{\infty} \left(\frac{1}{q} \right)^j = \frac{1}{q-1}.$$

For large q we have $\tau(1+3\delta) \leq \delta$ and thus

$$\left(\prod_{j=0}^{k-1} T(r_j, f) \right)^{1+3\delta} \leq T(r_k, f)^{\delta}.$$

We can also deduce from (3.43) that

$$T(r_k, f)^{\delta} \geq e^k$$

if r_0 is chosen large enough. Combining the last two estimates with (3.42), and assuming that $r_0 \geq 1$, we conclude that $\sigma_k \geq 1/T(r_k, f)$.

Together with (3.38) and (3.39) we thus find that

$$\overline{\dim}_{\mathrm{B}}(U_0 \cap I(f) \cap J(f)) \geq \limsup_{k \to \infty} \frac{(2-7\delta) \log T(r_k, f) - \log N_0}{\log T(r_k, f)} = 2 - 7\delta.$$

Since $\delta > 0$ was arbitrary, we obtain (1.9) for $U = U_0$. We may assume that the closure of U_0 does not intersect the exceptional set. As mentioned in the introduction, Theorem 1.1 now follows.

Remark 3.2. Theorem B has been extended to meromorphic functions with finitely many poles [31] and in fact to meromorphic functions with a logarithmic tract [10, Theorem 1.4]. It is conceivable that our result admits similar extensions.

4. The minimum modulus of entire functions with multiply connected Fatou components

Zheng [37] proved that if the Fatou set of a transcendental entire function f has a multiply connected component U, then there exist sequences (r_k) and (R_k) satisfying $\lim_{k\to\infty} r_k = \lim_{k\to\infty} R_k/r_k = \infty$ such that $\operatorname{ann}(r_k, R_k) \subset f^k(U)$ for large k. It is shown in [11] that one can actually take $R_k = r_k^c$ for some $c > 1$ and this is then used to show that if $F(f)$ has a multiply connected component, then there exists $C > 0$ such that

$$\log L(r,f) \geq \left(1 - \frac{C}{\log r}\right) \log M(r,f) \tag{4.1}$$

on some unbounded sequence of r-values.

Using Zheng's result [37] instead of [11] yields the following proposition referred to in the introduction. We include its short proof for completeness.

Proposition 4.1. *Let f be a transcendental entire function for which $F(f)$ has a multiply connected component. Then* (1.7) *holds.*

Proof. Zheng [37, Corollary 1] used hyperbolic geometry to prove that the hypothesis of the proposition implies (1.6). We will use the same idea and denote the hyperbolic distance of two points a, b in a hyperbolic domain V by $\lambda_V(a,b)$; see, e.g., [25, Section 2.2] for the properties of the hyperbolic metric that are used.

Let U be a multiply connected component of $F(f)$ and let (r_k) and (R_k) be as in Zheng's result mentioned above. Put $s_k := \sqrt{R_k r_k}$ and $U_k := f^k(U)$. Choose $|a_k| = |b_k| = s_k$ such that $|f(a_k)| = L(s_k, f)$ and $|f(b_k)| = M(s_k, f)$. Since $R_k/r_k \to \infty$ we easily see that $\lambda_{U_k}(a_k, b_k) \leq \lambda_{\operatorname{ann}(r_k,R_k)}(a_k, b_k) \to 0$. Thus $\lambda_{U_{k+1}}(f(a_k), f(b_k)) \to 0$. Since $0, 1 \notin U_{k+1}$ for large k we obtain

$$\lambda_{\mathbb{C}\setminus\{0,1\}}(f(a_k), f(b_k)) \to 0 \tag{4.2}$$

as $k \to \infty$. As the density $\rho_{\mathbb{C}\setminus\{0,1\}}(z)$ of the hyperbolic metric in $\mathbb{C} \setminus \{0,1\}$ satisfies $\rho_{\mathbb{C}\setminus\{0,1\}}(z) \geq c/(|z| \log |z|)$ for some $c > 0$ and large $|z|$, we obtain

$$\lambda_{\mathbb{C}\setminus\{0,1\}}(f(a_k), f(b_k)) \geq c \int_{|f(a_k)|}^{|f(b_k)|} \frac{dt}{t \log t} = c \log \frac{\log M(s_k, f)}{\log L(s_k, f)}. \tag{4.3}$$

Now (1.7) follows from (4.2) and (4.3). ☐

Remark 4.1. An alternative way to deduce Proposition 4.1 from Zheng's result is via Harnack's inequality [26, p. 14]. This method was used by Hinkkanen [22, Lemma 2] and Rippon and Stallard [32, Lemma 5], and it is also used in [11].

With the notation as in the above proof, put $t_k := \log s_k = \log \sqrt{R_k/r_k}$ and define

$$u_k : \{z \in \mathbb{C} : |\operatorname{Re} z| < t_k\} \to \mathbb{R}, \quad u_k(z) = \log |f(s_k e^z)|.$$

We may assume that $|f(z)| > 1$ for $z \in \operatorname{ann}(r_k, R_k) \subset f^k(U)$ so that u_k is a positive harmonic function. Choose $y_1, y_2 \in \mathbb{R}$ with $|y_1 - y_2| \leq \pi$ such that $u(iy_1) = \log L(s_k, f)$ and $u(iy_2) = \log L(s_k, f)$. By Harnack's inequality we have

$$u(iy_2) \leq \frac{t_k + \pi}{t_k - \pi} u(iy_1) = (1 + o(1)) u(iy_1)$$

as $k \to \infty$, and (1.7) follows.

Remark 4.2. It follows from a result of Fenton ([17], see also [13]) that if

$$p := \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log\log r} < \infty \tag{4.4}$$

and $\varepsilon > 0$, then

$$\log L(r,f) \geq \log M(r,f) - (\log r)^{p-2+\varepsilon} \tag{4.5}$$

on some sequence of r-values tending to ∞. Hence

$$\log L(r,f) \geq \log M(r,f) - \frac{\log M(r,f)}{(\log r)^{2-2\varepsilon}} \tag{4.6}$$

on such a sequence. Choosing $\varepsilon < 1/2$ we see that if

$$\lim_{r\to\infty} \left(1 - \frac{\log L(r,f)}{\log M(r,f)}\right) \log r = \infty, \tag{4.7}$$

then (4.6) and hence (4.4) cannot hold and thus (1.3) holds. The result of [11] quoted before Proposition 4.1 shows that if (4.7) holds, then $F(f)$ has no multiply connected components. Thus we obtain the following consequence of Theorem 1.1.

Corollary 4.1. *Let f be a transcendental entire function satisfying* (4.7). *Then* $\dim_P(I(f) \cap J(f)) = 2$.

Finally we mention that it was actually shown in [11] and [17] that (4.1) and (4.5) hold on sets or r-values of a certain size. This could be used to further strengthen the statement of Corollary 4.1.

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