Periodic Points of Entire Functions: Proof of a Conjecture of Baker

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Let f be an entire transcendental function and denote the n-th iterate of f by f_n. Our main result is that if $n \geq 2$, then there are infinitely many fixpoints of f_n which are not fixpoints of f_k for any k satisfying $1 \leq k < n$. This had been conjectured by I. N. Baker in 1967. Actually, we prove that there are even infinitely many repelling fixpoints with this property. We also give a new proof of a conjecture of F. Gross from 1966 which says that if h and g are entire transcendental functions, then the composite function $h \circ g$ has infinitely many fixpoints. We show that $h \circ g$ has even infinitely many repelling fixpoints.

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1 Introduction and main results

Let f be an entire transcendental function. The iterates f_n are defined by $f_0(z) = z$ and $f_n(z) = f(f_{n-1}(z))$ for $n \geq 1$. We say that z_0 is a *periodic point* of f, if $f_n(z_0) = z_0$ for some positive integer n. In this case, n is called a *period* of z_0 and the smallest n with this property is called the *primitive period* of z_0. The periodic points of period 1 are the *fixpoints* of f. If z_0 is a periodic point of period n, then $f_n'(z_0)$ is called the *multiplier* of z_0 (with respect to n). A periodic point z_0 is said to be *attracting*, *indifferent*, or *repelling* according as the modulus of its multiplier is less than, equal to, or greater than 1. The multiplier of an indifferent periodic point z_0 is of the form $e^{2\pi i\alpha}$ for some real α. If α is rational, then z_0 is called *rationally indifferent*. Otherwise, z_0 is called *irrationally indifferent*.

The periodic points play an important role in iteration theory. For an introduction into iteration theory, we refer to the classical papers of Fatou [23] and Julia [32] for rational functions and Fatou [24] for entire functions. More recent presentations of the theory have been given e. g. by Blanchard [16], Brolin [17], and Lyubich [34] for rational functions and by Baker [9], Eremenko and Lyubich [22], and Gross [25, Chapter 8] for rational and entire functions.

The main objects studied in iteration theory are the *set of normality* or *Fatou set* of f, which is defined to be the set of all complex numbers where the family of iterates of f is normal, and the *Julia set* of f, which, by definition, is the complement of the Fatou set. It is easy to see that repelling periodic points are in the Julia set while attracting periodic points are in the Fatou set. It is also well-known that rationally indifferent periodic points are in the Julia set. For irrationally indifferent periodic points, it is generally not easy to decide whether they are in the Julia set or in the Fatou set. Both possibilities do occur.

The importance of the periodic points is also illustrated by a result of Fatou [24, p. 354] who proved that every point in the Julia set is a limit point of periodic points. This was strengthened by Baker [8] who proved that every point in the Julia set is a limit point of repelling periodic points, that is, the Julia set is the closure of the set of repelling periodic points.

In particular, since the Julia set is known to be non-empty, Fatou's and Baker's results imply that there are infinitely many periodic and in fact infinitely many repelling periodic points. The question arises whether anything can be said about the periods and primitive periods of these points. This question was already addressed by Fatou in 1926. Fatou [24, p. 345] proved that there exists at least one periodic point of period 2. He also noted that there are in fact infinitely many periodic points of period 2 and sketched the proof of this assertion. In 1948, this result was generalized by Rosenbloom [35] who proved that there are infinitely many periodic points of period n for any $n \geq 2$. Baker [29, Problem 2.20] (see also [6, p. 284] and [28, Appendix, p. 184]) conjectured in 1967 that if $n \geq 2$, then there exist infinitely many periodic points of primitive period n. Earlier, he had proved [6] that there exists at most one positive integer n (depending on f) for which this fails to be true. For other partial results concerning this conjecture we refer to [4, 5, 11]. One of the purposes of this paper is to prove that Baker's conjecture is correct. In fact, we shall prove the following more general result.

Theorem 1 *Let f be an entire transcendental function and $n \geq 2$. Then f has infinitely many repelling periodic points of primitive period n.*

We remark that entire functions need not have attracting periodic points. An example, already known to Fatou [24, p. 370] (see also [9, §8]), is given by $f(z) = e^z$. Also, if $n = 1$, then the conclusion of Theorem 1 need not hold, as shown by examples like $f(z) = e^z + z + a$ where $|a - 1| \leq 1$. For functions of lower order less than $\frac{1}{2}$ or of order $\frac{1}{2}$ and type 0, however, Whittington [38, p. 532], using a theorem of Kjellberg [33] on the minimum modulus, proved that there exist infinitely many fixpoints which are repelling or have multiplier 1. For rational functions (of degree not less than two), it was proved by Fatou [23, p. 168] and Julia [32, p. 85 and p. 243] that there exists at least one fixpoint which is repelling or has multiplier 1.

We note two consequences of Theorem 1.

Corollary 1 *If all fixpoints of an entire transcendental function F have different multipliers, then F is not the n-th iterate, $n > 1$, of any entire function f.*

Corollary 2 *An entire transcendental function F with at most finitely many repelling fixpoints is not the n-th iterate, $n > 1$, of any entire function f.*

These results were proved by Baker [4, p. 152] and Whittington [38, p. 533] under additional assumptions on the order of F or f. A considerable generalization of Corollary 2 will be stated and proved in §9.

This paper is organized as follows. In §2 and §3, we state some known results. They are used in §4 to prove a theorem about repelling fixpoints of composite entire functions. This result is stated as Theorem 2. In §5, we state some facts from iteration theory. These results, together with Theorem 2, are used in §6 to prove Theorem 1. Corollary 1 is also proved in §6. The proof of Corollary 2 is immediate and will be omitted. In §7, some remarks and open questions concerning periodic points of entire functions are given. In §8, we give a new proof of a conjecture of Gross (see [21, p. 542, Problem 32] and [25, Problem 5]) which says that any composite function of entire transcendental functions has infinitely many fixpoints. The first proof was given in [12]. This result is generalized in §9, where we prove that such functions have in fact infinitely many repelling fixpoints.

2 Some results from Wiman-Valiron theory

We denote the central index of an entire function g by $\nu(r,g)$. By F we denote an exceptional set of finite logarithmic measure, not necessarily the same at each occurrence. We quote some standard results from Wiman-Valiron theory (e. g. [30, 37]).

Lemma 1 *Let g be an entire transcendental function and let K and η be positive constants. If $|z_0| = r$, $|g(z_0)| \geq \eta M(r,g)$, and $|\tau| \leq K/\nu(r,g)$, then*

$$g(z_0 e^{\tau}) \sim g(z_0) e^{\nu(r,g)\tau} \qquad (r \notin F) \tag{1}$$

and

$$g'(z_0 e^{\tau}) \sim \frac{\nu(r,g)}{z_0 e^{\tau}} g(z_0 e^{\tau}) \qquad (r \notin F). \tag{2}$$

The following lemma is a consequence of Lemma 1 and Rouché's Theorem. For the details of the proof, we refer to [10, Lemma 2].

Lemma 2 *Let g be an entire transcendental function and let K, η, and ε be positive constants. If $|\sigma_1| < K$, $|g(z_0)| \geq \eta M(r,g)$, and $|z_0| = r \notin F$, then there exists τ_1 such that $|\nu(r,g)\tau_1 - \sigma_1| < \varepsilon$ and $g(z_0 e^{\tau_1}) = g(z_0) e^{\sigma_1}$. If $\varepsilon < 2\pi$ and if $r \notin F$ is large enough, then τ_1 is unique.*

Lemma 3 *Let g be an entire transcendental function and let C and η be positive constants. Let j be an integer and suppose that $r \notin F$ and that z_0 satisfies $|z_0| = r$ and $|g(z_0)| \geq \eta M(r,g)$. Then there exists an analytic function $\tau_j(z)$ defined for $|z - z_0| \leq Cr/\nu(r,g)$ which satisfies*

$$|\tau_j(z)\nu(r,g) - 2\pi ij| \to 0 \quad as \quad r \to \infty,$$

$$g(ze^{\tau_j(z)}) = g(z),$$

and

$$\frac{d}{dz}(ze^{\tau_j(z)}) \sim 1 \quad as \quad r \to \infty.$$

Lemma 3 was proved in [12, Lemma 3] for the case $j = 1$. The general case can be proved by the same method, as already pointed out in [11, Lemma 3].

Clunie [18, p. 76] has pointed out that one can use (1) to prove the following result.

Lemma 4 *If h and g are transcendental entire functions, then*

$$M(r, h \circ g) = M((1-o(1))M(r,g), h) \qquad (r \notin F)$$

and

$$M(r, h \circ g) = M(M((1-o(1))r, g), h).$$

3 Results of Ahlfors, Dufresnoy, and Hayman

One of the main tools that we shall use is the following result.

Lemma 5 *Let G_1, G_2, and G_3 be three simply connected domains whose closures are pairwise disjoint. If $f(z)$ is analytic in $|z - z_0| < R$ and fails to map any subregion of $|z - z_0| < R$ conformally onto one of the domains G_j $(j = 1, 2, 3)$, then*

$$R \leq \frac{2\mu(\log \mu + A)}{|f'(z_0)|},$$

where $\mu = \max\{1, |f(z_0)|\}$ and A is a constant depending only on the domains G_j.

With $2\mu(\log\mu+A)$ replaced by $A(1+|f(z_0)|^2)$, this result is due to Ahlfors [1, p. 9] and Dufresnoy [20, p. 224]. In this form, it also holds for meromorphic f, if we take five regions instead of three regions. It can be deduced from Ahlfors's theory of covering surfaces (see [2] or [28, Chapter 5]).

In the above form, the result is essentially due to Hayman [27]. It is a direct consequence of Theorems 6.8, 6.6, and 5.5 of his book [28].

4 Repelling fixpoints of composite functions

It was proved in [11] that certain composite meromorphic functions have infinitely many repelling fixpoints. If we restrict ourselves to entire functions, then we can obtain stronger results.

Theorem 2 *Let h and g be entire transcendental functions and let K and ε be positive constants, $K > 1$. Suppose that $h \circ g$ has only finitely many repelling fixpoints z' for which $|g(z')| \geq M(|z'|/2, g)$. If $r \notin F$ and if w_1 and w_2 are complex numbers satisfying*

$$\frac{1}{K}M(r,g) \leq |w_j| \leq KM(r,g) \quad (j=1,2), \tag{3}$$

then

$$\frac{1}{(1+\varepsilon)}\log|h(w_1)| \leq \log|h(w_2)| \leq (1+\varepsilon)\log|h(w_1)|. \tag{4}$$

Proof. Suppose that there exist K, ε, w_1, and w_2 satisfying the hypotheses of the theorem, but that (4) is not satisfied. Interchanging w_1 and w_2 if necessary, we may assume that

$$(1+\varepsilon)\log|h(w_2)| < \log|h(w_1)|.$$

We claim that there exists w_0 satisfying

$$\frac{1}{K}M(r,g) \leq |w_0| \leq KM(r,g), \tag{5}$$

$$|h(w_0)| \geq |w_0|, \tag{6}$$

and

$$\frac{|h'(w_0)|}{|h(w_0)|\log|h(w_0)|} \geq \frac{\delta}{|w_0|} \tag{7}$$

for some positive δ depending on K and ε, provided r is large enough. We prove this claim only for the case that $|w_1| \le |w_2|$. The case $|w_2| < |w_1|$ is similar. We may assume that $|h(w_1)| = M(|w_1|, h)$ and from (3) we can deduce that $|w_2| \le K^2|w_1|$. Hence there exists a curve $\gamma(t)$, $0 \le t \le 1$, of length at most $(K^2 - 1 + \pi)|w_1|$ such that $\gamma(0) = w_1$, $\gamma(1) = w_2$, and

$$\frac{1}{K}M(r,g) \le |w_1| \le |\gamma(t)| \le |w_2| \le KM(r,g).$$

Define $t_1 = \inf\{t : (1+\varepsilon)\log|h(\gamma(t))| \le \log|h(w_1)|\}$. Then $t_1 > 0$ and $\log|h(w_1)| = (1+\varepsilon)\log|h(\gamma(t_1))|$. For a suitable branch of the logarithm, we have

$$|\log h(\gamma(t_1))| \le \log|h(\gamma(t_1))| + \pi \le (1 + \frac{\varepsilon}{2})\log|h(\gamma(t_1))|,$$

if r is large enough. It follows that

$$\begin{aligned}
& |\log\log h(w_1) - \log\log h(\gamma(t_1))| \\
\ge\; & \log|\log h(w_1)| - \log|\log h(\gamma(t_1))| \\
\ge\; & \log\log|h(w_1)| - \log\log|h(\gamma(t_1))| - \log(1 + \frac{\varepsilon}{2}) \\
=\; & \log(1+\varepsilon) - \log(1 + \frac{\varepsilon}{2}).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\log(1+\varepsilon) - \log(1 + \frac{\varepsilon}{2}) &\le |\log\log h(w_1) - \log\log h(\gamma(t_1))| \\
&= |\int_0^{t_1} \frac{h'(\gamma(t))\gamma'(t)}{h(\gamma(t))\log h(\gamma(t))}dt| \\
&\le \max_{0\le t\le t_1} |\frac{h'(\gamma(t))}{h(\gamma(t))\log h(\gamma(t))}| \int_0^{t_1} |\gamma'(t)|dt \\
&\le \frac{|h'(\gamma(t_0))|}{|h(\gamma(t_0))|\log|h(\gamma(t_0))|}(K^2 - 1 + \pi)|w_1|,
\end{aligned}$$

if the maximum is attained for $t = t_0$. We define $w_0 = \gamma(t_0)$ and, since $|w_1| \le |w_0|$, we deduce that (7) holds with

$$\delta = \frac{\log(1+\varepsilon) - \log(1+\frac{\varepsilon}{2})}{K^2 - 1 + \pi},$$

provided r is large enough. Clearly, (5) and (6) also hold for large r.

Now we choose z_0 such that $|z_0| = r \notin F$ and $|g(z_0)| = M(r, g)$. Since $w_0 = e^{\sigma} g(z_0)$ for some σ satisfying $|\text{Re}\,\sigma| \leq \log K$ and $|\text{Im}\,\sigma| \leq \pi$, we deduce from Lemma 2 that there exists s satisfying

$$|s| \leq (1 + o(1)) \frac{\log K + \pi}{\nu(r, g)}$$

and $g(z_0 e^s) = w_0$. We define $u_0 = z_0 e^s$. Then $g(u_0) = w_0$,

$$|g'(u_0)| \sim |\frac{\nu(r, g)}{r} g(u_0)| = \frac{\nu(r, g)}{r} |w_0|$$

by (2), and

$$|u_0 - z_0| = r|e^s - 1| \leq (1 + o(1)) \frac{(\log K + \pi)r}{\nu(r, g)} \leq \frac{(\log K + 4)r}{\nu(r, g)}.$$

We define

$$f(z) = \frac{\nu(r, g)}{u_0} (h(g(z)) - u_0).$$

Then

$$|f(u_0)| \sim \frac{\nu(r, g)}{r} |h(w_0) - u_0| \sim \frac{\nu(r, g)}{r} |h(w_0)| \tag{8}$$

by (6). In particular, we have $|f(u_0)| > 1$ for large r by (5) and (6). It is easy to see that $\log \nu(r, g) = o(\log M(r, g))$ for $r \notin F$. It follows that

$$\log |f(u_0)| \sim \log |h(w_0)|. \tag{9}$$

Moreover,

$$|f'(u_0)| \sim \frac{\nu(r, g)}{r} |h'(g(u_0)) g'(u_0)| \sim \frac{\nu(r, g)^2}{r^2} |h'(w_0) w_0| \tag{10}$$

by (2). Combining the last three equations with (7) we find that

$$\begin{aligned} & \frac{2|f(u_0)|(\log |f(u_0)| + A)}{|f'(u_0)|} \\ = \; & (1 + o(1)) \frac{2|h(w_0)| \log |h(w_0)| r}{|w_0 h'(w_0)| \nu(r, g)} \\ \leq \; & \frac{3}{\delta} \frac{r}{\nu(r, g)}, \end{aligned} \tag{11}$$

if A is a constant and $r \notin F$.

We now choose an integer N satisfying $N > 5/\delta$ and apply Lemma 5 for $G_j = D(2\pi ijN, 5/\delta)$. Here and in the following $D(a,R)$ denotes the disk of radius R about a. It follows that there exist $j \in \{1,2,3\}$ and a domain G contained in $D(u_0, 3r/\delta\nu(r,g))$ such that f maps G conformally onto G_j, provided $r \notin F$. We choose τ_{-jN} according to Lemma 3 and define $\sigma(z) = z\exp(\tau_{-jN}(z))$. Moreover, we define $u_j = u_0(1+2\pi ijN/\nu(r,g))$. It is not difficult to see that $D(u_0, 3r/\delta\nu(r,g)) \subset \sigma(D(u_j, 4r/\delta\nu(r,g)))$ for $r \notin F$. Hence there exists H contained in $D(u_j, 4r/\delta\nu(r,g))$ such that $G = \sigma(H)$ and $f(\sigma(z))$ maps H conformally onto G_j. Hence $h(g(\sigma(z)))$ maps H conformally onto $D(u_j, 5|u_0|/\delta\nu(r,g))$. By Lemma 3, we have $h(g(z)) = h(g(\sigma(z)))$. It follows that $h(g(z))$ maps H conformally onto $D(u_j, 5|u_0|/\delta\nu(r,g))$. Since, for $r \notin F$ large enough, the closure of H is contained in $D(u_j, 5|u_0|/\delta\nu(r,g))$, we deduce as Baker in [8, p. 255] from Rouché's theorem and Schwarz's lemma that the inverse function of $h \circ g$ has an attracting fixpoint z' in H. Clearly, z' is a repelling fixpoint of $h \circ g$.

We note that

$$\begin{aligned} |z' - z_0| &\leq |z' - u_j| + |u_j - u_0| + |u_0 - z_0| \\ &\leq \frac{5|u_0|}{\delta\nu(r,g)} + \frac{2\pi jN|u_0|}{\nu(r,g)} + \frac{(\log K + 4)r}{\nu(r,g)} \\ &\leq \frac{Cr}{\nu(r,g)} \end{aligned}$$

if $C > 5/\delta + 2\pi jN + \log K + 4$ and $r \notin F$. This implies that z' is of the form $z' = z_0 e^t$ for some t satisfying $|t| \leq (1+o(1))C/\nu(r,g)$. Hence we have

$$|g(z')| \geq (1+o(1))e^{-C}M(r,g) \geq M(\frac{|z'|}{2}, g) \tag{12}$$

by Lemma 1, provided $r \notin F$ is large enough.

Altogether, since z_0 and hence z' can be chosen arbitrarily large, we obtain a contradiction to our hypothesis that $h \circ g$ has only finitely many repelling fixpoints z' which satisfy (12). This completes the proof of Theorem 2.

5 Results from iteration theory

In the proof of Theorem 1, we shall use some results about attracting and rationally indifferent periodic points. The results that we need about rationally indifferent periodic points are summarized in the following lemma. They are essentially due to Fatou [23, Chapters 2 and 4]. A more recent exposition can be found in the survey article of Lyubich [34, §1.10]. Although Fatou and Lyubich consider only rational functions, the results hold for entire functions as well. Fatou's results were used by Baker [7] who determined in which cases polynomials or rational functions may fail to have periodic points of some primitive period. Our presentation of Fatou's results follows that of Baker [7, Lemma 4] and Lyubich [34, §1.10].

Lemma 6 *Let f be an entire function and let z_0 be a periodic point of primitive period p. Suppose that z_0 is rationally indifferent and let t be the smallest positive integer such that $(f_p'(z_0))^t = 1$. Then f_{pt} has a Taylor series*

$$f_{pt}(z) = z_0 + (z - z_0) + \sum_{\nu=m+1}^{\infty} a_\nu (z - z_0)^\nu \quad (a_{m+1} \neq 0)$$

where m is of the form $m = kt$ for some positive integer k. Moreover, if we define $z_j = f_j(z_0)$ for $1 \le j \le p-1$, then for each j satisfying $0 \le j \le p-1$ there are exactly m components D_{ij} $(1 \le i \le m)$ of the Fatou set of f which contain z_j as a boundary point and have the property that $\lim_{\nu\to\infty} f_{\nu p}(z) = z_j$ for $z \in D_{ij}$. The domains D_{ij} are called Leau domains. The set of the $pm = pkt$ Leau domains D_{ij} falls into k disjoint subsets called cycles of Leau domains, each of pt domains, the domains of each subset being permuted cyclically by f. Each cycle of Leau domains contains at least one singularity of f_{-1}, the inverse function of f.

The Leau domains D_{ij} contain subdomains L_{ij} called Leau petals which have z_j as a boundary point and which are bounded by piecewise analytic curves. Let L be the union of all L_{ij} and assume that $\varepsilon > 0$. Then the L_{ij} can be chosen such that $f(L) \subset L$ and $L_{ij} \subset D(\varepsilon, z_j)$ for all i and j.

For attracting periodic points, the situation is simpler. It is described in the following lemma (cf. e. g. [23, Chapter 4] or [34, §1.8]).

Lemma 7 *Let f be an entire function and let z_0 be an attracting periodic point of primitive period p. Define $z_j = f_j(z_0)$ for $1 \le j \le p-1$, denote by D_j the component of the Fatou set that contains z_j, and define D to be the union of all D_j, $0 \le j \le p-1$. Then D contains at least one singularity of f_{-1}.*

We shall also need a result which says that the Leau domains and the domains D_j in Lemma 7 are bounded under certain hypotheses. This has been proved by Bhattacharyya [15, Theorem 1] if the order of f is less than $\frac{1}{2}$ or if f has order $\frac{1}{2}$ and type 0. We use Bhattacharyya's method to prove the following result.

Lemma 8 *Let f be an entire function and let z_0 be a rationally indifferent or attracting periodic point of primitive period p. Define $z_j = f_j(z_0)$ for $0 \leq j \leq p-1$. If z_0 is rationally indifferent, let D be the union of the Leau domains which have one z_j as a boundary point. If z_0 is attracting, let D be as in Lemma 7. Suppose that there exists a simple closed curve Γ_0 such that $\Gamma_0 \cap f(\Gamma_0) = \emptyset$ and $\Gamma_0 \subset f(\mathrm{int}(\Gamma_0))$ where $\mathrm{int}(\Gamma_0)$ denotes the interior of Γ_0. Furthermore suppose that $z_j \in \mathrm{int}(\Gamma_0)$ for $0 \leq j \leq p-1$. Then $D \subset \mathrm{int}(\Gamma_0)$.*

Proof. We give the proof only for the case that z_0 is rationally indifferent. The case that z_0 is attracting is similar. Let L be the union of the Leau petals of the z_j according to Lemma 6. We may assume that $f(L) \subset L$ and $L \subset \mathrm{int}(\Gamma_0)$. Suppose now that $D \not\subset \mathrm{int}(\Gamma_0)$, that is, $D \cap \Gamma_0 \neq \emptyset$. Then there is a curve γ contained in D which connects one of the Leau petals with Γ_0. Since γ is a compact subset of D, there exists j satisfying $0 \leq j \leq p-1$ such that $\lim_{\nu\to\infty} f_{\nu p}(z) = z_j$ uniformly for $z \in \gamma$. On the other hand, one easily sees that for each $\nu \geq 1$, there exists $z_\nu \in \gamma$ such that $f_\nu(z_\nu) \in \Gamma_0$. This is a contradiction and the lemma is proved.

6 Proof of Theorem 1

Suppose that f has only finitely many repelling periodic points of primitive period n. Define $l = \max\{p : p < n, p|n\}$ and $h = f_l$. Moreover, define $m = n - l$ and $g = f_m$. We shall prove that the hypotheses of Theorem 2 are satisfied. Suppose that z' is a repelling fixpoint of $h \circ g$, that is suppose that z' is a repelling periodic point of f of period n. By our assumption, there are only finitely many repelling periodic points of primitive period n. Hence, if $|z'|$ is large enough, then z' has primitive period j for some $j < n$. Clearly, we have $1 \leq j \leq l \leq m$. It follows that

$$|g(z')| = |f_m(z')| = |f_{m-j}(f_j(z'))| = |f_{m-j}(z')| \leq M(|z'|, f_{m-j}).$$

Also, we have

$$|z'| < \frac{1}{2}M(\frac{|z'|}{2}, f_j),$$

if $|z'|$ is large. Combining the last two inequalities with Lemma 4 we deduce that

$$|g(z')| < M(\frac{1}{2}M(\frac{|z'|}{2}, f_j), f_{m-j}) \leq M(\frac{|z'|}{2}, f_m) = M(\frac{|z'|}{2}, g),$$

provided $j < m$ and $|z'|$ is large. But if $j = m$, then we have $|g(z')| = |z'| < M(|z'|/2, g)$ for large $|z'|$. Altogether, we deduce that the hypotheses of Theorem 2 are satisfied. Similarly, we deduce that this also holds if g and h are interchanged.

It follows from Theorem 2 (with g and h interchanged) that there exists an unbounded sequence (t_ν) such that

$$\log|g(z)| \sim \log M(t_\nu, g) \quad \text{for} \quad |z| = t_\nu. \tag{13}$$

Now let δ be a positive constant, $\delta < \frac{1}{2}$. We choose s_ν such that $s_\nu \notin F$ and $(t_\nu)^{1-2\delta} \leq s_\nu \leq (t_\nu)^{1-\delta}$, which is possible if ν is large enough. From Hadamard's three circles theorem we can deduce that

$$\log M(r^\eta, g) \geq (1 - o(1))\eta \log M(r, g) \tag{14}$$

for any $\eta > 1$. It follows that

$$\log M(s_\nu, g) \leq (1 + o(1))(1 - \delta) \log M(t_\nu, g) < \log|g(z)|$$

for $|z| = t_\nu$, provided ν is large enough. A result of Baker [3, p. 129] (see also [4, p. 147]) implies that there exists a simple closed curve Γ which is contained in the annulus $s_\nu \leq |z| \leq t_\nu$ and surrounds the origin once and on which $|g(z)| = M(s_\nu, g)$. Since we have chosen $s_\nu \notin F$, we deduce from Theorem 2 that

$$\begin{aligned} \log|f_n(z)| &= \log|h(g(z))| \\ &= (1 + o(1)) \log M(M(s_\nu, g), h) \\ &\geq (1 + o(1)) \log M(s_\nu, f_n) \end{aligned} \tag{15}$$

for $z \in \Gamma$. Now we choose r_ν such that $r_\nu \notin F$ and $(s_\nu)^{1-2\delta} \leq r_\nu \leq (s_\nu)^{1-\delta}$. Then $\log|f_n(z)| > \log M(r_\nu, f_n)$ for $z \in \Gamma$ by (14) and (15), provided ν is large enough. The argument of Baker used above shows that there exists a simple closed curve Γ_0 which is contained in $\text{int}(\Gamma)$, the interior of Γ, and surrounds the origin once and on which $|f_n(z)| = M(r_\nu, f_n)$. Clearly, Γ_0 is contained in the annulus $r_\nu \leq |z| \leq t_\nu$ and, for any given positive ε, we can achieve $t_\nu \leq (r_\nu)^{1+\varepsilon}$ by a suitable choice of δ. Also, we can achieve by a suitable choice of r_ν that $f_k'(z) \neq 0$ for $z \in \Gamma_0$ and $1 \leq k \leq n$.

We define $G_0 = \text{int}(\Gamma_0)$, that is, Γ_0 is the boundary of G_0. For $1 \leq k \leq n$, we define $G_k = f_k(G_0)$ and denote the boundary of G_k by Γ_k. Clearly, Γ_n is the circle of radius $M(r_\nu, f_n)$ about the origin. Also, it is not difficult to see that if $1 \leq k \leq n-1$ and ν is large, then $\Gamma_k \subset \{z : M(r_\nu, f_k)/2 \leq |z| \leq M(t_\nu, f_k)\} \subset G_{k+1}$ and Γ_k is a simple closed curve which surrounds the origin once. However, if $z(t)$ runs through Γ_k once, then $f(z(t))$ runs several times through Γ_{k+1}, say p_{k+1} times. It follows from the argument principle that if $a \in G_k$, then the equation $f(z) = a$ has exactly p_k solutions in G_{k-1}, counted according to multiplicity. We deduce that if $a \in G_k$, then f_k takes the value a exactly P_k times in G_0, where $P_k = p_1 p_2 \cdots p_k$. It follows from Rouché's theorem that f_k has P_k fixpoints in G_0. Also, f' has exactly $p_{k+1}-1$ zeros in G_k, again counted according to multiplicity.

In many ways, the function f, regarded as a mapping from G_k onto G_{k+1}, behaves like a polynomial of degree p_{k+1}. In fact, it is a polynomial-like mapping of degree p_{k+1} in the sense of Douady and Hubbard [19]. In the following, however, we do not need the results of Douady and Hubbard about polynomial-like mappings, but the elementary properties noted above suffice for our purposes.

We note that p_k and hence P_k depend not only on k but also on ν. It is not difficult to see that $p_k \to \infty$ as $\nu \to \infty$. It follows that P_n, the number of periodic points of period n in G_0, is much larger than P_k for any $k < n$. This alone, however, does not prove the existence of periodic points of primitive period n, since we have to take into account the multiplicities of the periodic points.

Let $\overline{P}_n$ be the number of periodic points of period n in G_0, where multiplicities are ignored. The numbers of periodic points of primitive period n corresponding to P_n and $\overline{P}_n$ are denoted by N_n and $\overline{N}_n$. Then

$$\overline{P}_n - \overline{N}_n \leq \sum_{k<n, k|n} \overline{P}_k \leq \sum_{k<n, k|n} P_k.$$

It follows that

$$\overline{N}_n \geq \overline{P}_n - \sum_{k<n, k|n} P_k = P_n - \sum_{k<n, k|n} P_k - (P_n - \overline{P}_n).$$

If a periodic point z_0 of period n contributes to the term $P_n - \overline{P}_n$, then we have $f_n'(z_0) = 1$. It follows that if z_0 has primitive period p and if t is chosen according to Lemma 6, then $pt \leq n$. Lemma 6 implies that if the multiplicity of z_0 regarded as a zero of $f_n(z) - z$ equals $m+1$, then the Leau domains associated with z_0 contain at least $\frac{m}{t}$ singularities of f_{-1}. By (15), f has no asymptotic values. It follows (cf. [37, Appendix D]) that the only singularities of f_{-1} are points of the form $f(c)$ where $f'(c) = 0$. In fact, these values $f(c)$ are singularities of the inverse function of the restriction of f to the corresponding cycle of Leau domains, which, by Lemma 8, is contained in G_{p-1}. Hence this cycle of Leau domains contains a zero of f' and we deduce that there are at least $\frac{m}{t}$ zeros of f' in G_{p-1} which, under iteration of f_p, tend to $f_j(z_0)$ for some j satisfying $0 \leq j \leq p-1$. Also, the contribution of $\{f_j(z_0) : 0 \leq j \leq p-1\}$ to the term $P_n - \overline{P}_n$ equals pm. Altogether, since $G_{p-1} \subset G_{n-1}$ and $\frac{m}{t} \geq \frac{pm}{n}$ and since f' has $p_n - 1$ zeros in G_{n-1}, we find that $P_n - \overline{P}_n \leq n(p_n - 1)$. It follows that

$$\overline{N}_n \geq P_n - \sum_{k<n, k|n} P_k - n(p_n - 1).$$

Since $p_k \to \infty$ as $\nu \to \infty$, we deduce that $\overline{N}_n \to \infty$ as $\nu \to \infty$, and this proves the conjecture of Baker referred to in the introduction.

To prove Theorem 1, we write $\overline{N}_n = \overline{N}_{\text{att}} + \overline{N}_{\text{rat}} + \overline{N}_{\text{irr}} + \overline{N}_{\text{rep}}$, where $\overline{N}_{\text{att}}$, $\overline{N}_{\text{rat}}$, $\overline{N}_{\text{irr}}$, and $\overline{N}_{\text{rep}}$ denote the numbers of attracting, rationally indifferent, irrationally indifferent, and repelling periodic points of primitive period n in G_0.

First we note that any attractive or rationally indifferent cycle attracts one singularity of f_{-1} by Lemmas 6 and 7. Again, it also attracts a zero of f'. As above we deduce that $\overline{N}_{\text{att}} + \overline{N}_{\text{rat}} \leq n(p_n - 1)$.

To estimate $\overline{N}_{\text{irr}}$, let Q be a polynomial such that $Q(z_0) = 0$ and $Q'(z_0) = f'(z_0)$ for all periodic points z_0 of f which have period n and are contained in G_{n-1}. For $0 < \xi < 1$, we define $F(z) = f(z) - \xi Q(z)$. Obviously, every periodic point of f of period n contained in G_0 is also a periodic point of F of period n. Also, we have $F_n'(z_0) = (1-\xi)^n f_n'(z_0)$ for any periodic point z_0 of f

which is contained in G_0 and has period n, that is, indifferent periodic points of f of period n are attracting periodic points of F of period n. By Lemma 7, every attracting periodic point of F of period n contained in G_0 attracts a singularity of F_{-1}. As above, we deduce that this singularity corresponds to a zero of F' and, if ξ is small enough, then this zero of F' is contained in G_{n-1}. Also, F' and f' have the same number of zeros in G_{n-1} by Hurwitz's theorem, provided ξ is small enough. Since f' has exactly $p_n - 1$ zeros in G_{n-1}, we deduce that F has at most $n(p_n - 1)$ attracting periodic points of period n in G_0. It follows that $\overline{N}_{\text{irr}} \leq n(p_n - 1)$. Altogether, we have

$$\overline{N}_{\text{rep}} = \overline{N}_n - \overline{N}_{\text{att}} - \overline{N}_{\text{rat}} - \overline{N}_{\text{irr}} \geq P_n - \sum_{k<n, k|n} P_k - 3n(p_n - 1).$$

We deduce that $\overline{N}_{\text{rep}} \to \infty$ as $\nu \to \infty$, contrary to our assumption. This completes the proof of Theorem 1.

Proof of Corollary 1. It has been noted already by Baker [6, p. 284] that Corollary 1 follows from Theorem 1. We include the short proof only for completeness and follow the argument of Baker. Suppose that $F(z) = f_n(z)$ for some $n \geq 2$. Let z_0 be a periodic point of f of primitive period n and define $z_j = f_j(z_0)$ for $1 \leq j \leq n - 1$. Then $z_i \neq z_j$ but $F'(z_i) = f_n'(z_i) = f_n'(z_j) = F'(z_j)$ for $0 \leq i < j \leq n - 1$. This contradicts the hypothesis and the proof of Corollary 1 is complete.

7 Some remarks and open questions

Some interesting problems concern the number $n(r, 1/(f_n(z) - z))$ of periodic points of period n in $|z| \leq r$. This number has been estimated by Baker [4] for functions of order less than $\frac{1}{2}$ and in [13] for functions of finite order and positive lower order. In the general case, however, no lower bounds for $n(r, 1/(f_n(z) - z))$ seem to be known. Although the method used in this paper can be used to obtain some estimation for $n(r, 1/(f_n(z) - z))$, it does not seem to be suitable to give "good" results of this type. Baker [6, p. 284] has asked whether $N(r, 1/(f_n(z) - z))$ and $T(r, f_n)$ are always of the same order of magnitude. (For a definition of these terms of Nevanlinna theory, cf. e. g. [28, 31].)

One may also consider the number of repelling periodic points of primitive period n in $|z| \leq r$. Using the results of Baker [4], one can obtain some lower bound for this number for functions of order less than $\frac{1}{2}$. Baker [4, Lemma 1] proved, without any assumptions on the periodic points of f, that if the order of f is less than $\frac{1}{2}$, then an equation similar to (15) holds. There are only two differences between his equation and (15). Firstly, he does not obtain $(t_\nu)^{1-2\delta} \leq s_\nu$ for any given positive δ, but only $(t_\nu)^\sigma \leq s_\nu$ for some positive σ depending on n and the order of f. Secondly, his equation holds for all large s_ν, not only for some unbounded sequence of s_ν-values. Baker [4, Theorem 1] deduced that $N(r, 1/(f_n(z)-z)) \geq (1-o(1)) \log M(r^\sigma, f_n)$. He noted that this implies that $N(r, 1/(f_n(z)-z))$ is much larger than $N(r, 1/(f_p(z)-z))$ for $p < n$, but that this does not prove his conjecture for functions of order less than $\frac{1}{2}$, since the N-functions count according to multiplicity. However, we can estimate the multiplicities and the number of attracting and indifferent periodic points with the method used in the proof of Theorem 1. If we do so, then we obtain the following result.

Let f be an entire function of order less than $\frac{1}{2}$ and $n \geq 2$. Denote by $n(r)$ the number of repelling periodic points of primitive period n of f in $0 < |z| \leq r$ and define $N(r) = \int_0^r (n(t)/t)dt$. Then there exists a positive number σ depending only on n and the order of f such that $N(r) \geq \log M(r^\sigma, f_n)$ for all large r.

It does not seem unlikely that the above estimation holds for all entire functions, without any restriction on the order. Possibly, it even holds for any $\sigma < 1$.

For polynomials or rational functions, there is only a finite number of attracting and indifferent periodic points (see e. g. [23, 36]). For entire transcendental functions, there may be infinitely many attracting and indifferent periodic points. One may ask, however, whether there is an upper bound for their number in $|z| \leq r$ in terms of $\log M(r, f)$ or the Nevanlinna characteristic $T(r, f)$. (Of course, we always have the trivial estimation $N(r, 1/(f_n(z) - z)) \leq T(r, f_n) + O(\log r)$.)

8 A new proof of a conjecture of Gross

We state the result mentioned in the introduction.

Theorem 3 *Let h and g be entire transcendental functions. Then the com-*

posite function $h \circ g$ has infinitely many fixpoints.

This result had been conjectured by Gross in 1966 (see [21, p. 542, Problem 32] and [25, p. 247, Problem 5]). The first proof was given in [12]. For a discussion of the result and references to earlier partial results concerning Gross's conjecture we refer to [12]. We shall give a new proof of Theorem 3. Although some of the underlying ideas are similar, this proof is quite different from (and probably simpler than) the proof given in [12]. However, it fails to yield the more general result proved in [12] that $h(g(z)) = p(z)$ has infinitely many solutions for any nonconstant polynomial p.

Proof of Theorem 3. Suppose that $h \circ g$ has only finitely many fixpoints. As pointed by Gross and Yang ([26, p. 214, proof of Theorem 2], see also Lemma 9 in §9), this implies that $g \circ h$ has only finitely many fixpoints. As in the proof of Theorem 1 we can deduce from Theorem 2 that there exists a simple closed curve Γ which is contained in the annulus $s_\nu \leq |z| \leq t_\nu$ and surrounds the origin once and on which $\log |h(g(z))| = (1 + o(1)) \log M(M(s_\nu, g), h)$. Rouché's theorem implies that the number of fixpoints of $h \circ g$ in the interior of Γ equals the number of solutions of $h(g(z)) = a$ in the interior of Γ, provided ν is large enough. Here a is any fixed complex number. A suitable choice of a yields that $h \circ g$ has infinitely many fixpoints, contrary to our assumption. This completes the proof of Theorem 3.

Remark If we consider Theorem 2 only as an intermediate step in the proof of Theorem 3, then we can do without the word "repelling" in Theorem 2. This simplifies the proof. We do not need Lemma 5, but only a strong form of Landau's theorem [28, p. 169]:

If $h(z)$ is analytic for $|z - z_0| < R$ and fails to take the values zero and one there, then $|h'(z_0)|R \leq 2|h(z_0)|(|\log |h(z_0)|| + K)$ for some absolute constant K.

To prove the version of Theorem 2 where the word "repelling" is omitted, we proceed as in the proof of Theorem 2 and define w_0 and u in the same way. However, we define

$$f(z) = \frac{h(g(z)) - z}{ze^{\tau_1(z)} - z}.$$

Again, we find that (8), (9), and (10) hold. Hence (11) also holds and we deduce that f takes one of the values zero and one in $D(z_0, Cr/\nu(r, g))$, if the constant C is chosen large enough and $r \notin F$. Using Lemma 3 we find that if $C' > C + 2\pi$, then $h \circ g$ has a fixpoint z' in $D(z_0, C'r/\nu(r,g))$, provided $r \notin F$. Also, we see that z' satisfies (12), and this completes the proof of this version of Theorem 2.

We remark that this weaker version of Theorem 2 also suffices for the proof of Baker's conjecture referred to in the introduction. Theorem 2 in the form stated in §4 is needed, however, for the proof of Theorem 1. It will also be used in the next section.

9 A generalization of Theorem 3

In this section, we shall prove the following generalization of Theorem 3 and Corollary 2.

Theorem 4 *Let h and g be entire transcendental functions. Then $h \circ g$ has infinitely many repelling fixpoints.*

First we need the following lemma.

Lemma 9 *Let h and g be entire transcendental functions. Then $h \circ g$ has infinitely many repelling fixpoints if and only if $g \circ h$ does.*

This result without the word "repelling" is due to Gross and Yang [26, p. 214, proof of Theorem 2] and was already used in the proof of Theorem 3. The following proof follows the argument of Gross and Yang.

Proof of Lemma 9. Suppose that $h \circ g$ has infinitely many repelling fixpoints $z_0, z_1, z_2, \ldots$ and define $w_k = g(z_k)$ for all k. Then $(g \circ h)(w_k) = g(h(g(z_k))) = g(z_k) = w_k$ for all k. Moreover, if $w_j = w_k$, then $z_j = h(g(z_j)) = h(w_j) = h(w_k) = h(g(z_k)) = z_k$. It follows that $g \circ h$ has infinitely many fixpoints w_0, w_1, $w_2, \ldots$. Since $(g \circ h)'(w_k) = g'(z_k)h'(w_k) = (h \circ g)'(z_k)$ and since z_k is a repelling fixpoint of $h \circ g$, it follows that w_k is a repelling fixpoint of $g \circ h$ for all k. This completes the proof of Lemma 9.

Proof of Theorem 4. Suppose that $h \circ g$ has only finitely many repelling fixpoints. Using Lemma 9, we find as in the proof of Theorem 1 that there exists a simple closed curve Γ_0 which is contained in the annulus $r_\nu \le |z| \le (r_\nu)^{1+\varepsilon}$ and surrounds the origin once such that

$$|h(g(z))| = M(r_\nu, h \circ g) \quad (z \in \Gamma_0). \tag{16}$$

We define $G_0 = \mathrm{int}(\Gamma_0)$, $G_1 = g(G_0)$, $\Gamma_1 = \partial G_1$, and $G_2 = h(G_1)$. Clearly, G_2 is the disk around zero of radius $M(r_\nu, h \circ g)$. As in the proof of Theorem 1, we find that there exist positive integers p_1 and p_2 (depending on ν) with the following properties:

(i) For every value $a \in G_1$, the equation $g(z) = a$ has p_1 solutions in G_0 and g' has $p_1 - 1$ zeros in G_0, counted according to multiplicity.

(ii) For every value $b \in G_2$, the equation $h(z) = b$ has p_2 solutions in G_1 and h' has $p_2 - 1$ zeros in G_1, counted according to multiplicity.

(iii) $h \circ g$ has $p_1 p_2$ fixpoints in G_0, counted according to multiplicity as zeros of $h(g(z)) - z$.

(iv) $p_1 \to \infty$ and $p_2 \to \infty$ as $\nu \to \infty$.

We define $P = p_1 p_2$, that is, P is the number of fixpoints of $h \circ g$ in G_0, counted according to multiplicity. By $\overline{P}$ we denote the corresponding number where multiplicities are ignored.

Suppose that a fixpoint z_0 contributes to the term $P - \overline{P}$, that is, suppose that z_0 has multiplicity $m + 1$ for some positive integer m. As in the proof of Theorem 1, we deduce from Lemma 6 and (16) that there are m zeros $z_1, z_2, \ldots, z_m$ of $(h \circ g)'$ in G_0, which tend to z_0 under iteration of $h \circ g$. Moreover, $z_1, z_2, \ldots, z_m$ can be chosen such that $h(g(z_i)) \ne h(g(z_j))$ for $i \ne j$. This latter assertion follows from the fact that we may assume that each Leau domain corresponding to z_0 contains exactly one z_j.

We say that two zeros z_1 and z_2 of $(h \circ g)'$ in G_0 are equivalent, if $h(g(z_1)) = h(g(z_2))$. By N we denote the number of equivalence classes. The above observation can now be written in the form $P - \overline{P} \le N$. Similarly, we write $\overline{P}_{\mathrm{att}}$, $\overline{P}_{\mathrm{rat}}$, $\overline{P}_{\mathrm{irr}}$, and $\overline{P}_{\mathrm{rep}}$ for the number of attracting, rationally indifferent, irrationally indifferent, and repelling fixpoints of $h \circ g$ in G_0. As in the proof of Theorem 1, we can show that $\overline{P}_{\mathrm{att}} + \overline{P}_{\mathrm{rat}} \le N$ and, using again a small perturbation of $h \circ g$, we find that $\overline{P}_{\mathrm{irr}} \le N$. Altogether, we have

$$\overline{P}_{\mathrm{rep}} = P - (\overline{P}_{\mathrm{att}} + \overline{P}_{\mathrm{rat}}) - \overline{P}_{\mathrm{irr}} - (P - \overline{P}) \ge P - 3N.$$

On the other hand, since g' has $p_1 - 1$ zeros in G_0 and since h' has $p_2 - 1$ zeros in G_1, we have $N \leq (p_1 - 1) + (p_2 - 1) = p_1 + p_2 - 2$. Since $P = p_1 p_2$, it follows that $\overline{P}_{\text{rep}} \geq p_1 p_2 - 3p_1 - 3p_2 + 6$. We deduce that $\overline{P}_{\text{rep}} \to \infty$ as $\nu \to \infty$. This is a contradiction and the theorem is proved.

Concluding Remarks Some of the questions asked in §7 for iterated entire functions may also be asked for composite entire functions. For instance, one may ask for lower bounds for the number of fixpoints of $h \circ g$ in $|z| \leq r$. It seems likely to me that $N(r, 1/(h(g(z)) - z))$ and $T(r, h \circ g)$ are always of the same order of magnitude in some sense. In fact, I do not know an example where the Nevanlinna deficiency $\delta(0, h(g(z)) - z)$ is greater than zero. Similarly, one may ask for lower bounds for the number of repelling fixpoints of $h \circ g$ in $|z| \leq r$.

It was proved in [14] that $h \circ g$ has infinitely many fixpoints if h and g are transcendental meromorphic functions and if g has at least three poles. The proof shows that in this case one of the poles of g is a limit point of fixpoints of $h \circ g$. Choosing $h = f$ and $g = f_{n-1}$ we see that a transcendental meromorphic function f has infinitely many periodic points of primitive period n, if $n \geq 2$ and if f_{n-1} has at least three poles. This result remains valid if f_{n-1} has only one or two poles, as can be seen by suitable modifications of the method used in this paper.

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References

[1] L. V. Ahlfors, Sur les domaines dans lesquels une fonction méromorphe prend des valeurs appartenant à une région donnée, *Acta Soc. Sci. Fennicae (Nova Series A)* 2 (1933), no. 2, 1-17.

[2] L. V. Ahlfors, Zur Theorie der Überlagerungsflächen, *Acta Math.* 65 (1935), 157-194.

[3] I. N. Baker, Zusammensetzungen ganzer Funktionen, *Math. Z.* 69 (1958), 121-163.

[4] I. N. Baker, Fixpoints and iterates of entire functions, *Math. Z.* 71 (1959), 146-153.

[5] I. N. Baker, Some entire functions with fixpoints of every order, *J. Australian Math. Soc.* 1 (1959/60), 203-209.

[6] I. N. Baker, The existence of fixpoints of entire functions, *Math. Z.* 73 (1960), 280-284.

[7] I. N. Baker, Fixpoints of polynomials and rational functions, *J. London Math. Soc.* 39 (1964), 615-622.

[8] I. N. Baker, Repulsive fixpoints of entire functions, *Math. Z.* 104 (1968), 252-256.

[9] I. N. Baker, Iteration of entire functions: an introductory survey, in *Lectures on complex analysis*, World Scientific, Singapore, New Jersey, London, Hong Kong, 1987, pp. 1-17.

[10] W. Bergweiler, On the fix-points of composite functions, *Pacific J. Math.* 143 (1990), 1-8.

[11] W. Bergweiler, Fix-points of meromorphic functions and iterated entire functions, *J. Math. Anal. Appl.* 151 (1990), 261-274.

[12] W. Bergweiler, Proof of a conjecture of Gross concerning fix-points, *Math. Z.* 204 (1990), 381-390.

[13] W. Bergweiler, On the number of fix-points of iterated entire functions, *Arch. Math.* 55 (1990), 558-563.

[14] W. Bergweiler, On the existence of fixpoints of composite meromorphic functions, *Proc. Amer. Math. Soc.*, to appear.

[15] P. Bhattacharyya, On the domain of normality of an attractive fixpoint, *Trans. Amer. Math. Soc.* 153 (1971), 89-98.

[16] P. Blanchard, Complex analytic dynamics on the Riemann sphere, *Bull. Amer. Math. Soc.* 11 (1984), 85-141.

[17] H. Brolin, Invariant sets under iteration of rational functions, *Ark. Mat.* 6 (1967), 103-141.

[18] J. Clunie, The composition of entire and meromorphic functions, in *Mathematical essays dedicated to A. J. Macintyre*, Ohio University Press, Athens, Ohio, 1970, pp. 75-92.

[19] A. Douady and J. H. Hubbard, On the dynamics of polynomial-like mappings, *Ann. Sci. École Norm. Sup.* (4) 18 (1985), 287-343.

[20] J. Dufresnoy, Sur les domaines couvertes par les valeurs d'une fonction méromorphe ou algébroïde, *Ann. Sci. École Norm. Sup.* (3) 58 (1941), 179-259.

[21] L. Ehrenpreis, Problems, in *Entire functions and related parts of analysis*, American Mathematical Society, Providence, R. I., 1968, pp. 533-546.

[22] A. E. Eremenko and M. Yu. Lyubich, The dynamics of analytic transforms, *Leningrad Math. J.* 1 (1990), 563-634.

[23] P. Fatou, Sur les équations fonctionelles, *Bull. Soc. Math. France* 47 (1919), 161-271; 48 (1920), 33-94, 208-314.

[24] P. Fatou, Sur l'itération des fonctions transcendantes entières, *Acta Math.* 47 (1926), 337-360.

[25] F. Gross, *Factorization of meromorphic functions*, U. S. Government Printing Office, Washington, D. C., 1972.

[26] F. Gross and C. C. Yang, The fix-points and factorization of meromorphic functions, *Trans. Amer. Math. Soc.* 168 (1972), 211-219.

[27] W. K. Hayman, Uniformly normal families, in *Lectures on functions of a complex variable*, Michigan University Press, Ann Arbor, Michigan, 1955, pp. 199-212.

[28] W. K. Hayman, *Meromorphic functions*, Clarendon Press, Oxford 1964 (reprinted with Appendix 1975).

[29] W. K. Hayman, *Research problems in function theory*, Athlone Press, London 1967.

[30] W. K. Hayman, The local growth of power series: a survey of the Wiman-Valiron method, *Can. Math. Bull.* (3) 17 (1974), 317-358.

[31] G. Jank and L. Volkmann, *Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen*, Birkhäuser, Basel, Boston, Stuttgart, 1985.

[32] G. Julia, Sur l'itération des fonctions rationelles, *J. Math. Pures Appl.* (7) 4 (1918), 47-245.

[33] B. O. Kjellberg, On the minimum modulus of entire functions of lower order less than one, *Math. Scand.* 8 (1960), 189-197.

[34] M. Yu. Lyubich, The dynamics of rational transforms: the topological picture, *Russian Math. Surveys* 41:4 (1986), 43-117, translation from *Uspekhi Mat. Nauk* 41:4 (1986), 35-95.

[35] P. C. Rosenbloom, L'itération des fonctions entières, *C. R. Acad. Sci. Paris* 227 (1948), 382-383.

[36] M. Shishikura, On the quasi-conformal surgery of rational functions, *Ann. Sci. École Norm. Sup.* (4) 20 (1987), 1-29.

[37] G. Valiron, *Lectures on the general theory of integral functions*, Édouard Privat, Toulouse 1923.

[38] J. E. Whittington, On the fixpoints of entire functions, *Proc. London Math. Soc.* (3) 17 (1967), 530-546.