ESCAPE RATE AND HAUSDORFF MEASURE FOR ENTIRE FUNCTIONS

WALTER BERGWEILER AND JÖRN PETER

Abstract. The escaping set of an entire function is the set of points that tend to infinity under iteration. We consider subsets of the escaping set defined in terms of escape rates and obtain upper and lower bounds for the Hausdorff measure of these sets with respect to certain gauge functions.

1. Introduction and results

The *escaping set* $I(f)$ of a non-linear and non-constant entire function f consists of all points in the complex plane $\mathbb{C}$ which tend to ∞ under iteration of f. The *Julia set* $J(f)$ is the subset of $\mathbb{C}$ where the iterates fail to be normal. By a result of Eremenko [14] we have $J(f) = \partial I(f)$. For an introduction to the dynamics of entire functions we refer to [5].

Considerable attention has been paid to the Hausdorff dimension and Hausdorff measures of these sets; see [37] for a survey. We denote the Hausdorff dimension of a subset A of $\mathbb{C}$ by $\dim A$ and the Hausdorff measure of A with respect to a gauge function h by $H_h(A)$; see section 2 for these definitions and some discussion of them.

The first results on the Hausdorff dimension of Julia sets of transcendental entire functions are due to McMullen who proved [19, Theorem 1.2] that $\dim J(\lambda e^z) = 2$ for $\lambda \in \mathbb{C}$, $\lambda \neq 0$. He notes [19, p. 336] that $H_h(J(\lambda e^z)) = \infty$ if $h(t) = t^2 \log\log \ldots \log(1/t)$, for any number of logarithms. A refinement of this result is given in [21]. McMullen actually showed that $\dim I(\lambda e^z) = 2$ and then noted that $I(\lambda e^z) \subset J(\lambda e^z)$.

McMullen's results initiated a large body of research on Hausdorff dimension and measure of Julia sets and escaping sets; see the survey [37] as well as more recent results in [3, 4, 9, 22, 25, 28, 33]. Most of these results, exceptions being [8, 34], have been concerned with the *Eremenko-Lyubich class B* which consists of all transcendental entire functions f for which the set $\operatorname{sing}(f^{-1})$ of singularities of the inverse of f is bounded. We note that $\operatorname{sing}(f^{-1})$ coincides with the set of critical and finite asymptotic values of f. For $f(z) = \lambda e^z$ we have $\operatorname{sing}(f^{-1}) = \{0\}$ and thus $f \in B$. Eremenko and Lyubich [15, Theorem 1] proved that if $f \in B$, then $I(f) \subset J(f)$ and thus $J(f) = \overline{I(f)}$.

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Baker [2] showed that if f is transcendental entire, not necessarily in the Eremenko-Lyubich class, then $J(f)$ contains continua. Rippon and Stallard [29] proved that $I(f)$ also contains continua. In fact, they showed [31, Theorem 1.3] that $I(f) \cap J(f)$ contains continua. It follows that $\dim J(f) \geq 1$ and $\dim I(f) \geq 1$ for every transcendental entire function f. Bishop [11] has recently constructed an entire transcendental function f satisfying $\dim J(f) = 1$. On the other hand, Stallard ([35], see also [4] and [10, Theorem 1.5]) proved that if $f \in B$, then $\dim J(f) > 1$. Moreover, she showed [36] that for each $d \in (1, 2]$ there exists a function $f \in B$ such that $\dim J(f) = \dim I(f) = d$.

In contrast to Stallard's result that $\dim J(f) > 1$ for $f \in B$, Rempe and Stallard [28] showed that there exists a function $f \in B$ with $\dim I(f) = 1$. Thus $H_h(I(f)) = 0$ if $h(t) = t^{1+\varepsilon}$ with $\varepsilon > 0$.

In recent studies of the escaping set, several subsets of the escaping set defined in terms of escape rates have turned out to be important; see, e.g., [30, 31]. For example, the key idea in the proof in [29] that $I(f)$ has at least one unbounded component was to consider a subset $A(f)$ of $I(f)$ which, roughly speaking, consists of all points tending to ∞ at the fastest rate compatible with the growth of f. This set $A(f)$, now called the *fast escaping set*, was introduced in [7] to study a problem concerning permutability. It is shown in [29] that all components of $A(f)$ are unbounded continua. In contrast, the set $I(f) \setminus A(f)$ need not contain continua. In fact, if $f(z) = \lambda e^z$ with $0 < \lambda < 1/e$, then $I(f) \setminus A(f)$ is totally disconnected; cf. [12, 18, 27]. Theorem 1 below implies that $\dim(I(f) \setminus A(f)) \geq 1$ for $f \in B$.

For a sequence (p_n) of positive real numbers tending to ∞ we denote by $\operatorname{Esc}(f, (p_n))$ the set of all $z \in I(f)$ such that $|f^n(z)| \leq p_n$ for all large n. We prove that $H_h(\operatorname{Esc}(f, (p_n))) = \infty$ if $h(t)$ tends to 0 slower than the functions $t \mapsto t^{1+\varepsilon}$.

Theorem 1. *Let $f \in B$ and let h be a gauge function satisfying*

$$\lim_{t \to 0} \frac{\log h(t)}{\log t} = 1. \tag{1.1}$$

Let (p_n) be a sequence of positive real numbers tending to ∞. Then

$$H_h(\operatorname{Esc}(f, (p_n))) = \infty.$$

In particular, $H_h(I(f)) = \infty$ and $\dim \operatorname{Esc}(f, (p_n)) \geq 1$.

In contrast to Theorem 1, we show that the set of points which escape with a definitive speed (in particular the fast escaping set) can have zero H_h-measure for certain gauge functions satisfying (1.1).

To formulate the result, let again (p_n) be a real sequence tending to ∞ and let $\operatorname{Unb}(f, (p_n))$ be the set of all $z \in \mathbb{C}$ such that $|f^n(z)| > p_n$ for infinitely many n. Thus

$$\operatorname{Esc}(f, (p_n)) = I(f) \setminus \operatorname{Unb}(f, (p_n)).$$

Note that points in $\text{Unb}(f,(p_n))$ have unbounded orbits, but need not be in $I(f)$.

Theorem 2. *Let (p_n) be a real sequence tending to ∞ and let h be a gauge function satisfying*

$$\lim_{t\to 0}\frac{h(t)}{t}=0. \tag{1.2}$$

Then there exists $f\in B$ such that

$$H_h(\text{Unb}(f,(p_n)))=0.$$

Note that (1.2) cannot be weakened since $A(f)$ contains continua [29]. Similarly, (1.1) cannot be weakened by [28].

There exist gauge functions h satisfying both (1.1) and (1.2). Thus Theorem 1 and Theorem 2 imply that the set of slow escaping points can be larger (from a measure-theoretic point of view) than the fast escaping set. This is in contrast to the situation for the functions λe^z where the fast escaping set is larger [17].

We describe the idea of the proof of Theorem 1. An important ingredient is a result of Barański, Karpińska and Zdunik [4] saying that the hyperbolic dimension of the Julia set of a transcendental meromorphic function with a logarithmic tract is strictly greater than 1. In the proof the authors construct iterated function schemes consisting of inverse branches of the second iterate of F which map a certain square S into itself. Here F is a function obtained from f by a logarithmic change of variable (cf. section 3). As the result is stated in [4] not quite in the form we need, we formulate a version of it as Lemma 3.4 below; cf. the remark at the end of section 4 for the difference to [4]. For the convenience of the reader we also include the proof.

We will take a suitable sequence (S_k) of such squares tending to infinity and obtain a subset of $\text{Esc}(f,(p_n))$ by considering points in S_1 which stay in S_1 under many iterations of F^2, are then mapped to S_2, stay there for many iterations of F^2, and so forth.

In order to estimate the Hausdorff measure of the set obtained, we extend in section 2 a result of [16] concerning the Hausdorff dimension of the limit set of iterated function schemes. The definitions and basic properties of Hausdorff measures and the Hausdorff dimension as well as iterated function schemes are recalled at the beginning of that section. The results of sections 2 and 3 are then combined in section 4 to prove Theorem 1.

The function f constructed in Theorem 2 will be large inside a narrow strip and bounded outside the strip. In section 5 we will give preliminary results for the construction, in particular we discuss some results of Ahlfors [1] concerning conformal mappings of strips. The actual proof of Theorem 2 is then carried out in section 6.

2. Hausdorff measures

We recall the definition of Hausdorff measure and Hausdorff dimension; see the book by Falconer [16] for more details.

A *gauge function* (or *dimension function*) is an increasing, continuous function $h\colon [0,\eta) \to [0,\infty)$ which satisfies $h(0) = 0$, where $\eta > 0$. For $A \subset \mathbb{C}$ and $\delta > 0$ we call a sequence (A_j) of subsets of $\mathbb{C}$ a *δ-cover* of A if $\operatorname{diam} A_j < \delta$ for all $j \in \mathbb{N}$ and

$$A \subset \bigcup_{j=1}^{\infty} A_j.$$

Here $\operatorname{diam} A_j$ denotes the (Euclidean) diameter of A_j. For a gauge function h we put

$$H_h^\delta(A) = \inf\left\{\sum_{j=1}^{\infty} h(\operatorname{diam} A_j)\colon (A_j) \text{ is } \delta\text{-cover of } A\right\}$$

and call

$$H_h(A) = \lim_{\delta\to 0} H_h^\delta(A)$$

the *Hausdorff measure* of A with respect to the gauge function h. Note that $H_h^\delta(A)$ is a non-increasing function of δ so that the limit defining $H_h(A)$ exists. It is possible that $H_h^\delta(A) = \infty$, meaning that $\sum_{j=1}^{\infty} h(\operatorname{diam} A_j)$ diverges for all δ-covers (A_j). Similarly, we may have $H_h(A) = \infty$. Given gauge functions h_1, h_2 and $A \subset \mathbb{C}$, we have

$$H_{h_1}(A) \le H_{h_2}(A) \quad \text{if } h_1(t) \le h_2(t) \text{ for all small } t. \tag{2.1}$$

In the special case that $h(t) = t^s$ for some $s > 0$, we call $H_h(A)$ the *s-dimensional Hausdorff measure.* There exists $d \ge 0$ such that $H_{t^s}(A) = \infty$ for $0 < s < d$ and $H_{t^s}(A) = 0$ for $s > d$. This value d is called the *Hausdorff dimension* of A and denoted by $\dim A$.

The following result [24, Theorem 7.6.1] about the Hausdorff measure is known as the mass distribution principle. Here $D(x, r)$ denotes the open disk of radius r around a point $x \in \mathbb{C}$.

Lemma 2.1. *Let h be a gauge function and $A \subset \mathbb{C}$. If there exists a Borel probability measure μ supported on A such that*

$$\lim_{r\to 0} \frac{\mu(D(x,r))}{h(r)} = 0 \quad \textit{for all } x \in A,$$

then $H_h(A) = \infty$.

Let $D \subset \mathbb{C}$ be compact. A mapping $T\colon D \to D$ is called a *contraction* on D if there exists $c \in (0,1)$ such that $|T(z) - T(w)| \le c\,|z - w|$ for all $z, w \in D$. Clearly a contraction is continuous. Let $T_1, \dots, T_m$ be contractions on D. Then the family $T = \{T_1, \dots, T_m\}$ is called an *iterated function scheme on D.* It can be shown (see [16, Theorem 9.1]) that there exists a unique non-empty compact set $X \subset D$ which is invariant for the T_i, that is, which satisfies

$$X = \bigcup_{i=1}^{m} T_i(X).$$

Moreover, we have

$$X = \bigcap_{n=1}^{\infty} \bigcup_{i_1,\ldots,i_n} \left(T_{i_1} \circ \ldots \circ T_{i_n}\right)(F)$$

for every non-empty compact set $F \subset D$ satisfying $T_i(F) \subset F$ for all i. The set X is called *limit set* of the iterated function scheme T.

The following result [16, Proposition 9.7] gives a lower bound for the Hausdorff dimension of the limit set of an iterated function scheme; see [16, Proposition 9.6] for an analogous upper bound.

Lemma 2.2. *Let $D \subset \mathbb{C}$ be compact and $T_1, \ldots, T_m \colon D \to D$ be contractions. Suppose that $T_i(D) \cap T_j(D) = \emptyset$ for all $i \neq j$ and that for each i there exists $b_i \in (0,1)$ with*

$$|T_i(z) - T_i(w)| \geq b_i\,|z - w| \tag{2.2}$$

Define s by

$$\sum_{i=1}^{m} b_i^s = 1.$$

Then the limit set X of the iterated function scheme $T = \{T_1, \ldots, T_m\}$ satisfies $\dim X \geq s$*.*

We need a version of Lemma 2.2 for sequences of iterated function schemes. Let $(T_k)_{k\in\mathbb{N}}$ be such a sequence, with $T_k = \{T_{k,1}, \ldots, T_{k,m_k}\}$. Define the *limit set* X of (T_k) by

$$X = \bigcap_{k=1}^{\infty} \bigcup_{\substack{1\leq i_l \leq m_l,\\ 1\leq l\leq k}} (T_{1,i_1} \circ \ldots \circ T_{k,i_k})(D).$$

Lemma 2.3. *Let $D \subset \mathbb{C}$ be compact, (R_k) a sequence of iterated function schemes on D, with $R_k = \{R_{k,1}, \ldots, R_{k,m_k}\}$, and h a gauge function satisfying* (1.1)*. Let X_k be the limit set of R_k. Suppose that, for all $k \in \mathbb{N}$,*

(1) *$R_{k,i}(D) \cap R_{k,j}(D) = \emptyset$ for $i \neq j$;*
(2) *for $1 \leq j \leq m_k$ there exists $\tilde{b}_{k,j} \in (0,1)$ with*

$$\sum_{j=1}^{m_k} \tilde{b}_{k,j} > 1 \quad \textit{and} \quad |R_{k,j}(z) - R_{k,j}(w)| \geq \tilde{b}_{k,j}\,|z - w| \ \textit{ for } z, w \in D.$$

Then there exists an increasing sequence (n_i) of positive integers such that if (T_k) is a sequence of iterated function schemes with

$$T_k \in \{R_1, \dots, R_i\} \text{ for all } k \le n_i \tag{2.3}$$

and X is its limit set, then $H_h(X) = \infty$.

Remark. It is clear that the conclusion also holds for any increasing sequence (n_i') satisfying $n_i' \ge n_i$ for all i. We will use this fact later.

Proof. Let $h(t) = t^{1+\varepsilon(t)}$. By (1.1) we have $\varepsilon(t) \to 0$ as $t \to 0$. We may assume that ε is non-decreasing, since otherwise we can replace $\varepsilon(t)$ by $\max_{0\le s\le t} \varepsilon(s)$ and use (2.1). Similarly, we may assume that $\varepsilon(t) \to 0$ as $t \to 0$ so slowly that with $g(t) = t^{\varepsilon(t)}$ and hence $h(t) = tg(t)$ we have

$$g(t) \to 0 \quad \text{ as } t \to 0. \tag{2.4}$$

By condition (2), the solution $\tilde{s}_k$ of $\sum_j \tilde{b}_{k,j}^{\tilde{s}_k} = 1$ satisfies $\tilde{s}_k > 1$ for all k. Let an increasing sequence (n_i) be given. We will show that (n_i) has the required properties if it tends to infinity sufficiently fast. Let (T_k) be a sequence of iterated function schemes satisfying (2.3) and let X be its limit set. If $T_k = R_m$, we set $b_{k,j} = \tilde{b}_{m,j}$ and $s_k = \tilde{s}_m$.

By a standard argument, it is possible to construct a probability measure μ supported on X such that

$$\mu((T_{1,j_1} \circ \dots \circ T_{k,j_k})(D)) = b_{1,j_1}^{s_1} \cdot \dots \cdot b_{k,j_k}^{s_k}.$$

Note that since $\sum_j b_{k,j}^{s_k} = 1$ for all k, we have

$$\mu\left(\bigcup_{j_1,\dots,j_k} (T_{1,j_1} \circ \dots \circ T_{k,j_k})(D) \right) = 1.$$

Assuming that (n_i) tends to ∞ sufficiently fast, we will show that if $x \in X$ and $r > 0$ is sufficiently small, then there exist $k \in \mathbb{N}$ and $p_1, \dots, p_k$ such that

$$D(x,r) \cap X \subset (T_{1,p_1} \circ \dots \circ T_{k,p_k})(D) \tag{2.5}$$

and

$$\mu((T_{1,p_1} \circ \dots \circ T_{k,p_k})(D)) \le r^{1+2\varepsilon(r)}. \tag{2.6}$$

Let

$$d_k = \min_{l\le k} \min_{j\ne j'} \ \operatorname{dist}(T_{l,j}(D), T_{l,j'}(D)).$$

Rescaling if necessary, we may assume that $d_k < 1$ for all k. Further, define

$$\alpha_k = \min_{l\le k} \min_j b_{l,j}, \quad \beta_k = \max_{l\le k} \max_j b_{l,j}, \quad \gamma_k = \min_{l\le k} s_l, \quad \delta_k = \max_{l\le k} s_l.$$

It follows from these definitions that the quantities $\alpha_{n_i}, \beta_{n_i}, \gamma_{n_i}$ and δ_{n_i} depend only on $R_1, \dots, R_i$ and not on the choice of the sequence (n_i).

So we can choose (n_i) such that

$$\gamma_{n_{i+1}} - \frac{\delta_{n_{i+1}} \log(\alpha_{n_{i+1}} d_{n_{i+1}})}{\log(d_{n_i}(\beta_{n_i})^{n_i})} \geq 1 + 2\varepsilon(d_{n_i}(\beta_{n_i})^{n_i}). \tag{2.7}$$

To see that this condition can be satisfied we note that $\gamma_{n_{i+1}} > 1$, the denominator tends to $-\infty$ if $n_i \to \infty$ and the right side tends to 1 as $n_i \to \infty$. The latter statements follow since $(\beta_{n_i})^{n_i}$ decreases to 0 as $n_i \to \infty$ (note that $\beta_{n_i} < 1$).

Now let $r > 0$ be small and choose $k \in \mathbb{N}$ with

$$d_{k+1} \prod_{l=1}^{k+1} b_{l,p_l} \leq r \leq d_k \prod_{l=1}^{k} b_{l,p_l},$$

where $p_1, \dots, p_{k+1}$ are the uniquely determined positive integers such that $x \in (T_{1,p_1} \circ \dots \circ T_{k+1,p_{k+1}})(D)$.

If $(q_1, \dots, q_k) \neq (p_1, \dots, p_k)$ and $y = (T_{1,q_1} \circ \dots \circ T_{k,q_k})(w)$ for some $w \in D$, we obtain with $k_0 = \min\{l \in \{1, \dots, k\} \colon q_l \neq p_l\}$ that

$$|y - x| \geq b_{1,p_1} \cdot \ldots \cdot b_{k_0-1,p_{k_0-1}} \cdot d_{k_0} \geq d_k \prod_{l=1}^{k} b_{l,p_l} \geq r,$$

so (2.5) holds.

Let i be such that $n_i \leq k < n_{i+1}$. By the choice of k we have $r \leq d_{n_i}(\beta_{n_i})^{n_i}$. Since ε is a non-decreasing function, (2.7) implies that

$$\gamma_{n_{i+1}} - \frac{\delta_{n_{i+1}} \log(\alpha_{n_{i+1}} d_{n_{i+1}})}{\log r} \geq 1 + 2\varepsilon(r).$$

Multiplying this by $\log r$ and taking exponentials, it follows that

$$r^{\gamma_{n_{i+1}}} \frac{1}{(\alpha_{n_{i+1}} d_{n_{i+1}})^{\delta_{n_{i+1}}}} \leq r^{1+2\varepsilon(r)}.$$

By the definition of the quantities above, we have

$$r^{\gamma_{k+1}} \leq r^{\gamma_{n_{i+1}}}, \quad d_{k+1}^{-\gamma_{k+1}} \leq d_{n_{i+1}}^{-\delta_{n_{i+1}}} \quad \text{and} \quad b_{k+1,p_{k+1}}^{-s_{k+1}} \leq \alpha_{n_{i+1}}^{-\delta_{n_{i+1}}}.$$

This implies that

$$\left(\frac{r}{d_{k+1}}\right)^{\gamma_{k+1}} \cdot b_{k+1,p_{k+1}}^{-s_{k+1}} \leq r^{1+2\varepsilon(r)}$$

so that

$$\begin{aligned}
\mu((T_{1,p_1} \circ \dots \circ T_{k,p_k})(D)) &= \prod_{l=1}^{k} b_{l,p_l}^{s_l} = b_{k+1,p_{k+1}}^{-s_{k+1}} \prod_{l=1}^{k+1} b_{l,p_l}^{s_l} \\
&\leq b_{k+1,p_{k+1}}^{-s_{k+1}} \left(\prod_{l=1}^{k+1} b_{l,p_l}\right)^{\gamma_{k+1}} \leq b_{k+1,p_{k+1}}^{-s_{k+1}} \left(\frac{r}{d_{k+1}}\right)^{\gamma_{k+1}} \leq r^{1+2\varepsilon(r)},
\end{aligned}$$

which is (2.6).

Using (2.5) and (2.6) we obtain $\mu(D(x,r)) \leq r^{1+2\varepsilon(r)}$ and thus

$$\frac{\mu(D(x,r))}{h(r)} = \frac{\mu(D(x,r))}{r^{1+\varepsilon(r)}} \leq r^{\varepsilon(r)} = g(r) \to 0 \text{ as } r \to 0$$

by (2.4). As this holds for all $x \in X$, Lemma 2.1 implies that $H_h(X) = \infty$. $\square$

The next lemma follows easily from the previous one.

Lemma 2.4. *Let h be a gauge function satisfying* (1.1)*. Let*

$$D = \{z \in \mathbb{C} \colon 0 \leq \operatorname{Re} z \leq 1,\ 0 \leq \operatorname{Im} z \leq 1\} \tag{2.8}$$

and let $(P_k)_{k\in\mathbb{N}}$ and $(Q_k)_{k\in\mathbb{N}}$ be sequences of iterated function schemes on D, with

$$P_k = \{P_{k,1}, \ldots, P_{k,l_k}\} \quad \textit{and} \quad Q_k = \{Q_{k,1}, \ldots, Q_{k,m_k}\}.$$

Suppose that

(1) $P_{k,i}(D) \cap P_{k,j}(D) = \emptyset$ *for all $k \in \mathbb{N}$ and $i \neq j$.*

(2) *For $k \in \mathbb{N}$, $1 \leq i \leq l_k$ and $1 \leq j \leq m_k$, there exist $\tilde{b}_{k,i}, \overline{b}_{k,j} \in (0,1)$ with*

$$\sum_{i=1}^{l_k} \tilde{b}_{k,i} > 1 \tag{2.9}$$

such that for $z, w \in D$, we have $\tilde{b}_{k,i}\,|z-w| \leq |P_{k,i}(z) - P_{k,i}(w)|$ and $\overline{b}_{k,j}\,|z-w| \leq |Q_{k,j}(z) - Q_{k,j}(w)|$.

Then there exists an increasing sequence (n_i) of positive integers such that with $n_0 = 0$ the limit set X of the sequence (T_k) of iterated function schemes defined by

$$T_k = \begin{cases} P_i & \text{for } n_{i-1}+1 \leq k \leq n_i - 1, \\ Q_i & \text{for } k = n_i, \end{cases} \tag{2.10}$$

satisfies $H_h(X) = \infty$.

Proof. We have not assumed that Q_k satisfies condition (1) of Lemmas 2.3 and 2.4. But we may do so since otherwise we could work with $Q_k = \{Q_{k,1}\}$ so that $m_k = 1$, as this would make X smaller. In contrast, $l_k \geq 2$ by (2.9).

For $k \in \mathbb{N}$ we put $R_{2k-1} = P_k$ and $R_{2k} = (P_k)^{p_k} \circ Q_k$, where $p_k \in \mathbb{N}$. Then the R_{2k-1} satisfy the hypotheses of Lemma 2.3. We claim that the R_{2k} also satisfy these hypotheses if the p_k are chosen large enough. The conclusion then follows from Lemma 2.3.

Let $1 \leq j \leq m_k$ and $q_1, \ldots, q_{p_k} \in \{1, \ldots, l_k\}$. Then

$$\begin{aligned} &\left|P_{k,q_{p_k}} \circ \ldots \circ P_{k,q_1} \circ Q_{k,j}(z) - P_{k,q_{p_k}} \circ \ldots \circ P_{k,q_1} \circ Q_{k,j}(w)\right| \\ &\qquad \geq \overline{b}_{k,j} \prod_{n=1}^{p_k} \tilde{b}_{k,q_n}\,|z-w| =: c_{j,q_1,\ldots,q_{p_k}}\,|z-w|. \end{aligned}$$

Since

$$\sum_{j=1}^{m_k} \sum_{q_1,\dots,q_{p_k}\in\{1,\dots,l_k\}} c_{j,q_1,\dots,q_{p_k}} = \sum_{j=1}^{m_k} \sum_{q_1,\dots,q_{p_k}\in\{1,\dots,l_k\}} \left(\overline{b}_{k,j} \prod_{n=1}^{p_k} \tilde{b}_{k,q_n} \right)$$
$$\geq m_k \min_j \overline{b}_{k,j} \sum_{q_1,\dots,q_{p_k}\in\{1,\dots,l_k\}} \prod_{n=1}^{p_k} \tilde{b}_{k,q_n} = m_k \min_j \overline{b}_{k,j} \left(\sum_{i=1}^{l_k} \tilde{b}_{k,i} \right)^{p_k} > 1$$

by (2.9) if p_k is large enough, the above claim follows. □

Note that Lemma 2.2 implies that dim $X > s$ whenever $\sum_{i=1}^m b_i^s > 1$. The converse is not true in general, but it holds for some iterate of T if the maps in T are conformal on a domain containing D. Thus in this case condition (2) in Lemmas 2.3 and 2.4 is essentially equivalent to the statement that the corresponding iterated function schemes have limit sets of dimension greater than 1.

We prove this result for completeness.

Proposition 2.5. *Let $D \subset \mathbb{C}$ be compact. Let $T_1, \dots, T_m \colon D \to D$ be contractions that extend conformally to self-maps of some domain B that contains D. Suppose that the limit set X of the iterated function scheme $T = \{T_1, \dots, T_m\}$ satisfies dim $X > s$. Then, for large $p \in \mathbb{N}$, the iterated function scheme*

$$S = T^p = \{T_{i_p} \circ \dots \circ T_{i_1} : i_1, \dots, i_p \in \{1, \dots, m\}\}$$

has the property that, for $i = (i_1, \dots, i_p) \in \{1, \dots, m\}^p$ and $S_i = T_{i_p} \circ \dots \circ T_{i_1}$,

$$|S_i(z) - S_i(w)| \geq b_i\, |z - w| \quad \text{for } z, w \in D, \tag{2.11}$$

with constants $b_i \in (0, 1)$ satisfying $\sum_i b_i^s > 1$.

Proof. It is clear that $S = T^p$ and T have the same limit set X for all $p \in \mathbb{N}$. As the T_k extend to conformal self-maps of B, the same is true for the S_i. Thus for all i there exist some constant $b_i \in (0, 1)$ such that (2.11) holds. We define b_i as the largest number with this property. It can be deduced from the Koebe distortion theorem (Lemma 3.1) that there exists a constant K, depending only on the domain B, with $K b_i\, |z - w| \geq |S_i(z) - S_i(w)|$ for all $z, w \in D$ and all i. Because dim $X > s$ and the maps S_i are contractions, we have $\lim_{p\to\infty} \sum_i (\operatorname{diam} S_i(D))^s = \infty$. So for p large enough we obtain

$$\sum_i b_i^s \geq \sum_i \sup_{z,w\in D} \frac{|S_i(z) - S_i(w)|^s}{(K\, |z - w|)^s} \geq \sum_i \frac{(\operatorname{diam} S_i(D))^s}{(K \operatorname{diam}(D))^s} \to \infty,$$

which proves the proposition. □

The following simple lemma will be used in the proof of Theorem 1.

Lemma 2.6. *Let h be a gauge function satisfying $h(2t) \leq K\, h(t)$ for some $K > 0$ and all small t. Let $A \subset \mathbb{R}^n$ and $f \colon A \to \mathbb{R}^n$ be Lipschitz-continuous. If $H_h(A) < \infty$, then $H_h(f(A)) < \infty$.*

Proof. Let L be the Lipschitz constant of f; that is, $|f(z) - f(w)| \leq L\,|z - w|$ for $z, w \in A$. Choosing $m \in \mathbb{N}$ with $L \leq 2^m$ and putting $C = K^m$ we have $h(Lt) \leq C\,h(t)$ for small t. Let $\delta > 0$ and let (B_j) be a δ-cover of A. Then $(f(B_j))$ is an $L\delta$-cover of $f(A)$ and

$$\sum_j h(\operatorname{diam} f(B_j)) \leq \sum_j h(L \operatorname{diam} B_j) \leq C \sum_j h(\operatorname{diam} B_j)$$

so that $H_h(f(A)) \leq C\,H_h(A)$. □

It follows that if $f\colon A \to f(A)$ is a bilipschitz map, then $H_h(A) < \infty$ if and only if $H_h(f(A)) < \infty$.

3. Preliminaries for the proof of Theorem 1

For a function $f \in B$ we choose $R > |f(0)|$ such that $\operatorname{sing}(f^{-1}) \subset D(0, R)$ and put $G = \{z\colon\ |f(z)| > R\}$, $H = \{z\colon\ \operatorname{Re} z > \log R\}$ and $U = \exp^{-1}(G)$. Eremenko and Lyubich [15, section 2] showed that there exists a holomorphic, $2\pi i$-periodic function $F\colon U \to H$ with $\exp(F(z)) = f(\exp(z))$ for all $z \in U$. We call F the *logarithmic transform* of f. Moreover, they showed that for every connected component V of U, the map $F|_V\colon V \to H$ is bijective and the inverse map $\phi\colon H \to V$ satisfies

$$|\phi'(w)| \leq \frac{4\pi}{\operatorname{Re} w - \log R} \tag{3.1}$$

for every $w \in H$. In terms of F we obtain

$$|F'(z)| \geq \frac{1}{4\pi}(\operatorname{Re} F(z) - \log R) \tag{3.2}$$

for all $z \in U$. In the sequel, we will assume without loss of generality that $R = 1$. Let

$$\xi = \inf\{\operatorname{Re} z\colon\ z \in U\} = \log \inf\{|\zeta|:\ |f(\zeta)| = R\}.$$

As in [9] we consider the function $h\colon (\xi, \infty) \to (0, \infty)$,

$$h(x) = \max_{\operatorname{Re} z = x} \operatorname{Re} F(z), \tag{3.3}$$

and choose $z_x \in U$ with $\operatorname{Re} z_x = x$ such that $h(x) = \operatorname{Re} F(z_x)$. The function h is increasing and convex and thus differentiable except possibly at a countable set C of x-values, and for $x \in (\xi, \infty) \setminus C$ we have

$$h'(x) = F'(z_x), \tag{3.4}$$

cf. [38, p. 2562]. Recalling that $R = 1$ and $h(x) = \operatorname{Re} F(z_x)$ we deduce from (3.2) and (3.4) that

$$\frac{h'(x)}{h(x)} \geq \frac{1}{4\pi}. \tag{3.5}$$

Integration yields $\log h(x) \geq (x - \xi)/(4\pi)$ and thus

$$h(x) \geq 2x \tag{3.6}$$

for large x.

The following result is known as the Koebe distortion theorem and the Koebe one quarter theorem.

Lemma 3.1. *Let $g\colon D(a,r)\to\mathbb{C}$ be a univalent function, $0<\lambda<1$ and $z\in\overline{D(a,\lambda r)}\setminus\{a\}$. Then*

$$\frac{1}{(1+\lambda)^2}\left|g'(a)\right|\le\frac{|g(z)-g(a)|}{|z-a|}\le\frac{1}{(1-\lambda)^2}\left|g'(a)\right|$$

and

$$\frac{1-\lambda}{(1+\lambda)^3}\left|g'(a)\right|\le\left|g'(z)\right|\le\frac{1+\lambda}{(1-\lambda)^3}\left|g'(a)\right|.$$

Moreover,

$$g(D(a,r))\supset D(g(a),\tfrac{1}{4}\left|g'(a)\right|r).$$

Usually Koebe's theorems are stated only for the special case that $a=0$, $r=1$, $g(0)=0$ and $g'(0)=1$, but the above result follows easily from this special case.

For $a\in\mathbb{C}$ and $r>0$ we define the square

$$S(a,r)=\{z\in\mathbb{C}\colon\ |\mathrm{Re}(z-a)|<r,\ |\mathrm{Im}(z-a)|<r\}.$$

Lemma 3.2. *For sufficiently large x there exist $m\in\mathbb{N}$ and pairwise disjoint domains $W_1,W_2,\dots,W_m\subset S(F(z_x),\frac{1}{4}\operatorname{Re}F(z_x))$ satisfying*

$$\sum_{j=1}^{m}\operatorname{diam}W_j\ge 10^{-5}\frac{h(x)^3}{h'(x)} \tag{3.7}$$

such that F^2 is a univalent map from each W_j onto $S(F(z_x),\frac{1}{4}\operatorname{Re}F(z_x))$ whose inverse extends univalently to $S(F(z_x),\frac{1}{2}\operatorname{Re}F(z_x))$.

Proof. Let ϕ be the branch of F^{-1} which maps $F(z_x)$ to z_x and let

$$P_x=\phi(S(F(z_x),\tfrac{1}{4}\operatorname{Re}F(z_x))).$$

It follows from Koebe's theorem that

$$D(z_x,r_x)\subset P_x\subset D(z_x,R_x)$$

where

$$r_x=\frac{1}{4}\left|\phi'(F(z_x))\right|\cdot\frac{1}{4}\operatorname{Re}F(z_x)=\frac{1}{16}\frac{\operatorname{Re}F(z_x)}{|F'(z_x)|}=\frac{1}{16}\frac{h(x)}{h'(x)} \tag{3.8}$$

and, by (3.5),

$$R_x=4\left|\phi'(F(z_x))\right|\cdot\frac{1}{4}\operatorname{Re}F(z_x)=16r_x=\frac{h(x)}{h'(x)}\le 4\pi \tag{3.9}$$

for $x\notin C$. In particular, $P_x\subset H=\{z\in\mathbb{C}\colon\ \operatorname{Re}z>0\}$ for large x. For $k\in\mathbb{Z}$ we put

$$d_{x,k}=\left|\phi'(z_x+2\pi ik)\right|,\ \ v_{x,k}=\phi(z_x+2\pi ik)\ \ \text{and}\ \ V_{x,k}=\phi(P_x+2\pi ik).$$

Then, since $R = 1$,

$$d_{x,k} \leq \frac{4\pi}{\operatorname{Re} z_x} = \frac{4\pi}{x} \leq \frac{1}{4\pi} \tag{3.10}$$

for large x by (3.1). Again by Koebe's one quarter and distortion theorems,

$$D(v_{x,k}, t_{x,k}) \subset V_{x,k} \subset D(v_{x,k}, T_{x,k}) \tag{3.11}$$

where

$$t_{x,k} = \frac{1}{4} d_{x,k} r_x \tag{3.12}$$

and

$$T_{x,k} = 2 d_{x,k} R_x = 128 t_{x,k}.$$

By (3.9) and (3.10) we have

$$T_{x,k} \leq 2 \frac{4\pi}{x} 4\pi = \frac{32\pi^2}{x} \leq 1 \tag{3.13}$$

for large x. Koebe's distortion theorem and (3.10) also yield that

$$|v_{x,k+1} - v_{x,k}| \leq 4\pi d_{x,k} \leq \frac{16\pi^2}{x} \leq 1 \tag{3.14}$$

for large x. It is easy to see that $\operatorname{Re} v_{x,k} \to \infty$ as $k \to \infty$ or $k \to -\infty$. On the other hand, for large x there also exists $k_0 \in \mathbb{Z}$ such that

$$\operatorname{Re} v_{x,k_0} \leq x \leq \frac{1}{2} \operatorname{Re} F(z_x).$$

Let

$$k_1 = \max\left\{k \in \mathbb{Z} \colon \operatorname{Re} v_{x,k} \leq \tfrac{3}{4} h(x) + 1\right\}$$

and

$$k_2 = \min\left\{k \in \mathbb{Z} \colon k \geq k_1 \text{ and } \operatorname{Re} v_{x,k} > \tfrac{5}{4} h(x) - 1\right\}.$$

For $k_1 < k < k_2$ we then have $\frac{3}{4} h(x) + 1 \leq \operatorname{Re} v_{x,k} \leq \frac{5}{4} h(x) - 1$ and thus

$$V_{x,k} \subset \left\{z \in \mathbb{C} \colon \tfrac{3}{4} h(x) \leq \operatorname{Re} z \leq \tfrac{5}{4} h(x)\right\}$$

by (3.11) and (3.13). Now

$$\begin{aligned} \frac{1}{2} h(x) - 2 &\leq \operatorname{Re} v_{x,k_2} - \operatorname{Re} v_{x,k_1} = \sum_{k=k_1}^{k_2-1} (\operatorname{Re} v_{x,k+1} - \operatorname{Re} v_{x,k}) \\ &\leq \sum_{k=k_1}^{k_2-1} |v_{x,k+1} - v_{x,k}| \leq 4\pi \sum_{k=k_1}^{k_2-1} d_{x,k} \leq 4\pi \sum_{k=k_1+1}^{k_2-1} d_{x,k} + 1 \end{aligned} \tag{3.15}$$

by (3.10) and (3.14). Hence

$$\begin{aligned} \sum_{k=k_1+1}^{k_2-1} \operatorname{diam}(V_{x,k}) &\geq 2 \sum_{k=k_1+1}^{k_2-1} t_{x,k} = \frac{1}{2} r_x \sum_{k=k_1+1}^{k_2-1} d_{x,k} \\ &\geq \frac{1}{8\pi} r_x \left(\frac{1}{2} h(x) - 3\right) \geq \frac{h(x)^2}{10^3 h'(x)} \end{aligned} \tag{3.16}$$

by (3.8), (3.11), (3.12) and (3.15). We now put

$$N_x = \left\lfloor \frac{1}{2\pi}\left(\frac{1}{2}h(x) - 3\right)\right\rfloor .$$

Then

$$N_x \geq \frac{1}{20}h(x) \tag{3.17}$$

for large x. For $k_1 < k < k_2$ there exists $l_k \in \mathbb{Z}$ such that if $l_k \leq l < l_k + N_x$, then

$$v_{x,k} + 2\pi i l \in S(F(z_x), \tfrac{1}{4}\operatorname{Re} F(z_x) - 1)$$

and hence

$$V_{x,k} + 2\pi i l \subset S(F(z_x), \tfrac{1}{4}\operatorname{Re} F(z_x))$$

by (3.11) and (3.13). We now put $m = (k_2 - k_1 - 1)N_x$ and denote by $W_1, \dots, W_m$ the collection of the sets $V_{x,k} + 2\pi i l$ where $k_1 < k < k_2$ and $l_k \leq l < l_k + N_x$. We deduce from (3.16) and (3.17) that

$$\sum_{j=1}^{m} \operatorname{diam} W_j = N_x \sum_{k=k_1+1}^{k_2-1} \operatorname{diam} V_{x,k} \geq \frac{h(x)}{20} \cdot \frac{h(x)^2}{10^3 h'(x)} \geq 10^{-5}\frac{h(x)^3}{h'(x)}.$$

To prove that the inverse function of $F^2\colon W_j \to S(F(z_x), \frac{1}{4}\operatorname{Re} F(z_x))$ extends univalently to $S(F(z_x), \frac{1}{2}\operatorname{Re} F(z_x))$ we only note that the argument showing that $P_x \subset H$ for large x actually yields that

$$\phi(S(F(z_x), \tfrac{1}{2}\operatorname{Re} F(z_x))) \subset H$$

for large x. $\square$

The following growth lemma for real functions is well known, but for completeness we include the short proof. Usually it is stated for differentiable functions, but it also holds for absolutely continuous functions. Note that such functions are differentiable almost everywhere.

Lemma 3.3. *Let $g\colon [x_0, \infty) \to \mathbb{R}$ be an increasing, absolutely continuous function satisfying $\lim_{x\to\infty} g(x) = \infty$. Let $\delta > 0$. Then there exists a measurable subset E of $[x_0, \infty)$ having finite measure such that $g'(x) \leq g(x)^{1+\delta}$ for $x \notin E$.*

Proof. Suppose that $E = \{x \geq x_0 \colon g'(x) > g(x)^{1+\delta} \geq 1\} \neq \emptyset$. Then

$$\int_E dx \leq \int_E \frac{g'(x)}{g(x)^{1+\delta}}dx \leq \int_{x_1}^{\infty} \frac{g'(x)}{g(x)^{1+\delta}}dx = \frac{1}{\delta g(x_1)^{\delta}} \leq \frac{1}{\delta} < \infty,$$

where $x_1 = \inf E$. $\square$

We apply Lemma 3.3 to the function h defined by (3.3). Given $\varepsilon > 0$, we deduce from (3.7) that, in the situation of Lemma 3.2,

$$\sum_{j=1}^{m} \operatorname{diam} W_j \geq h(x)^{2-\varepsilon}$$

for x outside some set of finite measure. We summarize the above results as follows.

Lemma 3.4. *There exist $x_0 > 0$ and a subset E of $[x_0, \infty)$ of finite measure with the following property: if $x \in [x_0, \infty) \setminus E$, then there exist $m \in \mathbb{N}$ and pairwise disjoint sets $W_1, ..., W_m \subset S(F(z_x), \frac{1}{4} \operatorname{Re} F(z_x))$ satisfying*

$$\sum_{j=1}^{m} \operatorname{diam} W_j \geq h(x)^{3/2}. \tag{3.18}$$

such that $F^2 \colon W_j \to S(F(z_x), \frac{1}{4} \operatorname{Re} F(z_x))$ is a conformal map whose inverse extends univalently to the square $S(F(z_x), \frac{1}{2} \operatorname{Re} F(z_x))$.

4. Proof of Theorem 1

By (1.1) we have $h(t) = t^{1+\varepsilon(t)}$ where $\varepsilon(t) \to 0$ as $t \to 0$. Similarly as at the beginning of the proof of Lemma 2.3, we can assume without loss of generality that ε is positive and non-decreasing. Let $F \colon U \to H$ be the logarithmic transform of f as defined in section 3. The set corresponding to $\operatorname{Esc}(f, (p_n))$ is the set

$$Z = \{z \in U \colon \operatorname{Re} F^n(z) \to \infty, \operatorname{Re} F^n(z) \leq \log p_n \text{ for large } n\}.$$

Note that $\exp Z \subset \operatorname{Esc}(f, (p_n))$. We will show first that there exists a bounded subset Y of Z satisfying $H_h(Y) = \infty$. From this we will then deduce that $H_h(\operatorname{Esc}(f, (p_n))) = \infty$. Our main tool in the proof will be Lemma 2.4, so we have to construct a sequence of iterated function schemes.

Let x_0 and E be as in Lemma 3.4. Since E has finite measure, there exists $M \geq x_0$ with $\operatorname{meas}(E \cap [M, \infty)) < \frac{1}{2}$. So we can find a sequence (x_k) with $x_1 \geq M$, $x_1 \notin E$ and

$$x_k \in [h(x_{k-1}), h(x_{k-1}) + 1] \setminus E \quad \text{for } k \geq 2.$$

It follows from (3.6) that $x_k \to \infty$ if x_1 is chosen large enough. Let

$$S_k = S(F(z_{x_k}), \tfrac{1}{4} \operatorname{Re} F(z_{x_k})) = S(F(z_{x_k}), \tfrac{1}{4} h(x_k)).$$

Let $W_{k,1}, \ldots, W_{k,l_k}$ be the sets obtained from Lemma 3.4. The maps

$$\tilde{P}_{k,j} = (F^2|_{W_{k,j}})^{-1}$$

define an iterated function scheme $\tilde{P}_k$ on S_k. By conjugating $\tilde{P}_{k,j}$ with the affine map L_k that sends S_k to the square D defined by (2.8) we obtain an iterated function scheme $P_k = \{P_{k,1}, \ldots, P_{k,l_k}\}$ on D. In other words, we define $P_{k,j} = L_k \circ \tilde{P}_{k,j} \circ L_k^{-1}$ with $L_k \colon S_k \to D$,

$$L_k(z) = \frac{2}{h(x_k)}(z - F(z_{x_k})) + \frac{1}{2} + \frac{i}{2}.$$

Further, we have maps $\tilde{Q}_{k,j}\colon S_{k+1} \to V_{k,j} \subset S_k$ which are inverse branches of $F|_{V_{k,j}}$, for $1 \le j \le m_k$. Setting

$$Q_{k,j} = L_k \circ \tilde{Q}_{k,j} \circ L_{k+1}^{-1}$$

defines an iterated function scheme Q_k on D. It is obvious that condition (1) from Lemma 2.4 holds. It remains to verify condition (2). Since all the maps $\tilde{P}_{k,i}$ and $\tilde{Q}_{k,j}$ can be continued univalently to a square with twice the side length of S_k resp. S_{k+1}, the existence of positive numbers $\tilde{b}_{k,i}$ and $\overline{b}_{k,j}$ with $\tilde{b}_{k,i}\,|z-w| \le |P_{k,i}(z) - P_{k,i}(w)|$ and $\overline{b}_{k,j}\,|z-w| \le |Q_{k,j}(z) - Q_{k,j}(w)|$ follows immediately from the Koebe distortion theorem (Lemma 3.1). For the proof of (2.9), we will use (3.18). First note that, again by the Koebe distortion theorem, there exists an absolute constant $K > 0$ with

$$K\tilde{b}_{k,i} \ge \frac{|P_{k,i}(z) - P_{k,i}(w)|}{|z-w|} \ge \tilde{b}_{k,i}.$$

Choosing $z_0, w_0 \in D$ with

$$|P_{k,i}(z_0) - P_{k,i}(w_0)| = \max_{z,w\in D} |P_{k,i}(z) - P_{k,i}(w)| = \frac{2\operatorname{diam} W_{k,i}}{h(x_k)},$$

we obtain

$$\begin{aligned} K\tilde{b}_{k,i} &\ge \sup_{z,w\in D} \frac{|P_{k,i}(z) - P_{k,i}(w)|}{|z-w|} \ge \frac{|P_{k,i}(z_0) - P_{k,i}(w_0)|}{|z_0 - w_0|} \\ &\ge \frac{|P_{k,i}(z_0) - P_{k,i}(w_0)|}{\sup_{z,w\in D}|z-w|} = \sqrt{2}\frac{\operatorname{diam} W_{k,i}}{h(x_k)}, \end{aligned}$$

so

$$\tilde{b}_{k,i} \ge C\frac{\operatorname{diam} W_{k,i}}{h(x_k)}$$

for some absolute constant C. Using (3.18), we obtain

$$\sum_{i=1}^{l_k} \tilde{b}_{k,i} \ge C\frac{\sum_{i=1}^{l_k} \operatorname{diam} W_{k,i}}{h(x_k)} \ge C\, h(x_k)^{1/2} > 1$$

if x_1 was chosen large enough at the beginning. By Lemma 2.4, there exists an increasing sequence (n_i) such that the limit set X of (T_k), where T_k is defined as in (2.10), satisfies $H_h(X) = \infty$.

Put $Y = L_1^{-1}(X)$. By increasing (n_i) if necessary, we can achieve that $\operatorname{Re} F^k(z) \le \log p_k$ if $z \in Y$ and k is large enough. We also have $\operatorname{Re} F^n(z) \to \infty$ as $n \to \infty$ for $z \in Y$. To see this, let $z = L_1^{-1}(w)$, where $w \in X$. Let $k \in \mathbb{N}$ and put $i_k = \max\{i\colon n_i \le k\}$. Then

$$(T_{k,j_k}^{-1} \circ \ldots \circ T_{1,j_1}^{-1})(w) = L_{i_k+1}(F^{2k-i_k}(L_1^{-1}(w))) = L_{i_k+1}(F^{2k-i_k}(z)).$$

Since $i_k \to \infty$ and $\max_{z\in S_k} \operatorname{Re} z \to \infty$ as $k \to \infty$, we can deduce from this that $\operatorname{Re} F^{2k-i_k}(z) \to \infty$ as $k \to \infty$, from which we can easily deduce that $\operatorname{Re} F^n(z) \to \infty$ as $n \to \infty$. Altogether we thus have $Y \subset Z$.

Since $H_h(X)=\infty$ and infinite H_h-measure is invariant under affine mappings for any gauge function h, we have $H_h(Y)=\infty$. In order to deduce that $H_h(\operatorname{Esc}(f,(p_n)))=\infty$ we use Lemma 2.6. Recall that

$$\exp Y \subset \exp Z \subset \operatorname{Esc}(f,(p_n)). \tag{4.1}$$

Since Y is bounded and $H_h(Y)=\infty$, there exists $y_0\in\mathbb{R}$ with

$$H_h(Y\cap\{z\colon \operatorname{Im} z\in(y_0,y_0+\pi)\})=\infty. \tag{4.2}$$

Noting that ε is non-decreasing, we also see that

$$h(2t)=2t\cdot(2t)^{\varepsilon(2t)}\leq 3t\cdot t^{\varepsilon(2t)}\leq 3t\cdot t^{\varepsilon(t)}=3h(t)$$

for small t. Since exp restricted to $Y\cap\{z\colon \operatorname{Im} z\in(y_0,y_0+\pi)\}$ is a bilipschitz mapping, Lemma 2.6 and (4.2) yield

$$H_h(\exp(Y\cap\{z\colon \operatorname{Im} z\in(y_0,y_0+\pi)\}))=\infty.$$

An application of (4.1) finishes the proof. $\square$

Remark. The result of [4] yields a sequence (S_k) of squares tending to ∞ and associated iterated function schemes $\tilde{P}_k$ as in the above proof. Lemma 3.4 yields additional information about the "density" of such squares. However, this is not essential for the argument, since otherwise we could replace the $\tilde{Q}_{k,j}$ by inverse branches of some iterate of F.

5. Preliminaries for the proof of Theorem 2

The function f will have the property that it is bounded outside a narrow strip. There is a well-established technique to construct such functions using contour integrals, cf. [23, Chapter III, Problems 158-160], [32], [36] and, in particular, [26]. In order to apply this method we need some estimates concerning conformal mappings of strips. Let Ω be a domain of the form $\Omega=\{x+iy\colon |y|<\phi(x)\}$ with some non-negative function $\phi\colon\mathbb{R}\to\mathbb{R}$ and let $w\colon\Omega\to\{x+iy\colon |y|<\pi\}$ be a conformal map satisfying $w(x)\to\pm\infty$ as $x\to\pm\infty$. Put

$$\overline{H}(x)=\sup_{|y|<\phi(x)}\operatorname{Re} w(x+iy)\quad\text{and}\quad \underline{H}(x)=\inf_{|y|<\phi(x)}\operatorname{Re} w(x+iy).$$

The celebrated Ahlfors distortion theorem [1, §2], specialized to strips of the above form, says that

$$\underline{H}(x_2)-\overline{H}(x_1)\geq\pi\int_{x_1}^{x_2}\frac{dx}{\phi(x)}-8\pi\quad\text{if}\quad\int_{x_1}^{x_2}\frac{dx}{\phi(x)}>4.$$

We mention that Ahlfors denoted the "width" of a cross section (of more general strips) by $\theta(x)$. In our setting we have $\theta(x)=2\phi(x)$.

Ahlfors [1, §3] also proved an inequality in the opposite direction, provided that ϕ satisfies some regularity conditions. (It is easy to see that some additional hypotheses are necessary for such estimates.)

Suppose that ϕ is bounded, continuous and of bounded variation on every finite interval. Following Ahlfors we denote by $\phi_m(x_1,x_2)$ the minimum of ϕ and by $V(x_1, x_2)$ the total variation of ϕ^2 in the interval $[x_1, x_2]$. Noting that w extends continuously to ∂G, with ∂G being mapped bijectively onto $\{x + iy \colon |y| = \pi\}$, we put

$$\overline{x} = \operatorname{Re} w^{-1}(\overline{H}(x) + i\pi) \quad \text{and} \quad \underline{x} = \operatorname{Re} w^{-1}(\underline{H}(x) + i\pi).$$

With $L = \sup_{x\in\mathbb{R}} \phi(x)$ Ahlfors's result [1, p. 15] then takes the form

$$\overline{H}(x_2) - \underline{H}(x_1) \le \pi \int_{\underline{x}_1}^{\overline{x}_2} \frac{dx}{\phi(x)} + 8\pi L^2 \frac{\phi_m(\underline{x}_1, \overline{x}_2)^2 + V(\underline{x}_1, \overline{x}_2)}{\phi_m(\underline{x}_1, \overline{x}_2)^4}.$$

Suppose in addition that ϕ is decreasing. Then this simplifies to

$$\begin{aligned} \overline{H}(x_2) - \underline{H}(x_1) &\le \pi \int_{\underline{x}_1}^{\overline{x}_2} \frac{dx}{\phi(x)} + 8\pi L^2 \frac{\phi(\underline{x}_1)^2}{\phi(\overline{x}_2)^4} \\ &\le \pi \int_{\underline{x}_1}^{\overline{x}_2} \frac{dx}{\phi(x)} + \frac{8\pi L^4}{\phi(\overline{x}_2)^4}. \end{aligned} \tag{5.1}$$

Ahlfors [1, p. 15] also showed that

$$\int_{\underline{x}_1}^{x_1} \frac{dx}{\phi(x)} \le 8 \quad \text{and} \quad \int_{x_2}^{\overline{x}_2} \frac{dx}{\phi(x)} \le 8.$$

This implies that $8 \ge (\overline{x}_2 - x_2)/\phi(x_2)$ and hence

$$\overline{x}_2 \le x_2 + 8\phi(x_2). \tag{5.2}$$

We will also assume that $\phi(x) \le 1/x$ for large x. Assuming that $\underline{x}_1 \ge 0$ we can now deduce from (5.1) and (5.2) that

$$\overline{H}(x_2) - \underline{H}(x_1) \le \pi \frac{\overline{x}_2}{\phi(\overline{x}_2)} + \frac{8\pi L^4}{\phi(\overline{x}_2)^4} \le \frac{9\pi L^4}{\phi(\overline{x}_2)^4} \le \frac{9\pi L^4}{\phi(x_2 + 8\phi(x_2))^4}$$

for large x_2. For us only the case where x_1 is fixed and $x = x_2 \to \infty$ is of interest. We obtain the following result.

Lemma 5.1. *Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a positive, bounded, continuous and decreasing function satisfying $\phi(x) \le 1/x$ for large x. Let*

$$w \colon \{x + iy \colon |y| < \phi(x)\} \to \{x + iy \colon |y| < \pi\}$$

be a conformal map satisfying $w(x) \to \pm\infty$ as $x \to \pm\infty$. Then there exists a constant C such that

$$\sup_{|y|<\phi(x)} \operatorname{Re} w(x + iy) \le \frac{C}{\phi(x + 8\phi(x))^4}$$

for all large x.

Let ϕ be as above and put

$$S = \{x + iy \colon x > 0, |y| < \phi(x)\}. \tag{5.3}$$

The method of contour integrals described in the papers mentioned above consists of defining a function f by

$$f(z) = \frac{1}{2\pi i}\int_{\partial S}\frac{\exp\left(e^{w(\zeta)}\right)}{\zeta - z}d\zeta$$

for $z \in \mathbb{C}\backslash\overline{S}$ and analytic continuation of f to the whole plane by deforming the path of integration. The results of Rempe [26, Theorem 1.7] imply the following lemma.

Lemma 5.2. *Let ϕ and w be as in Lemma 5.1 and define S by* (5.3). *Then there exists $f \in B$ satisfying*

$$f(z) = \begin{cases} \exp\left(e^{w(z)}\right) + O(1/z) & \textit{for } z \in S, \\ O(1/z) & \textit{for } z \notin S. \end{cases}$$

It follows from Lemma 5.1 that the function f in Lemma 5.2 satisfies

$$|f(z)| \le \exp\left(\exp\left(\frac{C}{\phi(|z| + 8\phi(|z|))^4}\right)\right) \tag{5.4}$$

for some $C > 0$ if $|z|$ is large. Replacing f by εf with a small constant ε we may assume in addition that

$$\operatorname{sing}\left(f^{-1}\right) \subset \left\{z \in \mathbb{C}\colon\ |z| < \tfrac{1}{2}\right\}, \tag{5.5}$$

$$|f(z)| \le \tfrac{1}{2} \text{ for } |z| \le 1 \tag{5.6}$$

and

$$|f(z)| \le 1 \text{ for } z \notin S. \tag{5.7}$$

It is apparent from the above discussion that the behavior of $\phi(x)$ as $x \to -\infty$ is irrelevant for our purposes. In fact, it suffices to define the function ϕ on an interval $[x_0, \infty)$, as it can be continued to $\mathbb{R}$ by setting $\phi(x) = \phi(x_0)$ for $x < x_0$. We shall use the following result to define the function ϕ.

Lemma 5.3. *Let $\alpha\colon [x_0, \infty) \to (0, \infty)$ be decreasing and continuous and let $\beta\colon (0, 1] \to (0, \infty)$ be increasing and continuous. Then there exists a decreasing, continuous function $\phi\colon [x_0, \infty) \to (0, 1]$ satisfying $\phi(x + \alpha(x)) \le \beta(\phi(x))$ for large x.*

Proof. We may assume that the function σ given by $\sigma(x) = x + \alpha(x)$ is increasing as this can achieved by replacing α by a decreasing, continuous function $\alpha^*\colon [x_0, \infty) \to (0, \infty)$ satisfying $\alpha^*(x) \le \alpha(x)$ for all x. Similarly, we may assume that $\beta(x) < x$ for all x. We now define ϕ in the interval $[x_0, \sigma(x_0)]$ by $\phi(x_0) = 1$, $\phi(\sigma(x_0)) = \beta(\phi(x_0)) = \beta(1)$, and linear interpolation in $(x_0, \sigma(x_0))$. For $k \in \mathbb{N}$ we extend this to the interval $(\sigma^k(x_0), \sigma^{k+1}(x_0)]$ by putting $\phi(\sigma^k(x)) = \beta^k(\phi(x))$. Since $\sigma^k(x_0) \to \infty$ as $k \to \infty$, this defines a decreasing, continuous function $\phi\colon [x_0, \infty) \to (0, 1]$ satisfying $\phi(x + \alpha(x)) = \beta(\phi(x))$. $\square$

We will also use the following result known as the Besicovich covering lemma [13, Theorem 3.2.1]. Here $B(x, r)$ is the open ball of radius r around a point $x \in \mathbb{R}^n$.

Lemma 5.4. *Let $K \subset \mathbb{R}^n$ be bounded and $r\colon K \to (0, \infty)$. Then there exists an at most countable subset L of K satisfying*

$$K \subset \bigcup_{x \in L} B(x, r(x))$$

such that no point in $\mathbb{R}^n$ is contained in more than 4^{2n} of the balls $B(x, r(x))$, $x \in L$.

The following result is a simple consequence of the Besicovich covering lemma; see [6, Lemma 5.2] for a similar result concerning Hausdorff dimension.

Lemma 5.5. *Let $K \subset \mathbb{R}^n$ and let h be a gauge function. Suppose that for all $x \in K$ and $\varepsilon > 0$ there exists $\delta(x) \in (0, \varepsilon)$, $N(x) \in \mathbb{N}$ and balls $B_1, \dots, B_{N(x)}$ such that*

$$K \cap B(x, \delta(x)) \subset \bigcup_{j=1}^{N(x)} B_j \quad \textit{and} \quad \sum_{j=1}^{N(x)} h(\operatorname{diam} B_j) \le \varepsilon \delta(x)^n.$$

Then $H_h(K) = 0$.

Proof. We may assume that K is bounded, say $K \subset B(0, R)$. Let $\varepsilon > 0$. For $x \in K$, let $\delta(x) \in (0, \varepsilon)$, $N(x) \in \mathbb{N}$ and $B_1(x), \dots, B_{N(x)}(x)$ be as given in the hypothesis. Let L be as in Lemma 5.4. Then

$$\{B_j(x) \colon x \in L, 1 \le j \le N(x)\}$$

is an open cover of K. We may assume that $\operatorname{diam} B_j(x) \le 2\delta(x) < 2\varepsilon$ for $x \in L$ and $1 \le j \le N(x)$. Moreover,

$$\sum_{x \in L} \sum_{j=1}^{N(x)} h(\operatorname{diam} B_j(x)) \le \varepsilon \sum_{x \in L} \delta(x)^n \le \varepsilon\, 4^{2n} (R + \varepsilon)^n \omega_n,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Thus

$$H_h(K) \le \varepsilon\, 4^{2n} (R + \varepsilon)^n \omega_n,$$

and the conclusion follows. □

6. Proof of Theorem 2

Let (p_n) and h be as in the hypothesis. First we note that if $p_n \ge q_n$ for large n, then $\operatorname{Unb}(f, (p_n)) \subset \operatorname{Unb}(f, (q_n))$ and thus

$$H_h(\operatorname{Unb}(f, (p_n))) \le H_h(\operatorname{Unb}(f, (q_n))).$$

Hence it is no loss of generality to assume that

$$p_n \le n \quad \text{and} \quad p_n - p_{n-1} \ge \frac{6}{n^2} \tag{6.1}$$

for large n, since otherwise we could pass to the sequence (q_n) defined by

$$q_n = \min\left\{n, \inf_{k\geq n} p_k\right\} + 6\sum_{k=1}^{n}\frac{1}{k^2} - \pi^2,$$

which has the above properties and satisfies $q_n \leq p_n$.

We write our gauge function h in the form $h(t) = tg(t)$. Then (1.2) says that $g(t) \to 0$ as $t \to 0$. By (2.1) we may assume that g is increasing and satisfies $g(t) \geq t$ for all t, since otherwise we could replace $g(t)$ by $t + \sup_{s\leq t} g(s)$.

We consider the function $\tau\colon (0,1] \to (0,\infty)$,

$$\tau(t) = \left(\frac{t}{4}\exp\left(-\exp\left(\frac{1}{t^5}\right)\right)\right)^{1/t}.$$

We apply Lemma 5.3 to the function $\beta = g^{-1}\circ\tau$ and to a decreasing function $\alpha\colon [p_1,\infty) \to (0,1/4]$ satisfying $\alpha(x) \leq 1/n^2$ for $x \geq p_{n-1}$ and $n \geq 2$. For example, we may define α by putting $\alpha(p_{n-1}) = 1/n^2$ and interpolating linearly in the intervals $[p_{n-1}, p_n]$ for $n \geq 2$. We now choose $\phi\colon [p_1,\infty) \to (0,1]$ according to Lemma 5.3 and obtain

$$g\left(\phi\left(x + \frac{1}{n^2}\right)\right) \leq \tau(\phi(x)) \quad \text{for } x \geq p_{n-1}. \tag{6.2}$$

Since $g(t) \geq t$ this implies, together with (6.1), that

$$\phi(p_n) \leq g(\phi(p_n)) \leq \tau(\phi(p_{n-1})) \leq \frac{1}{4}\phi(p_{n-1})$$

and thus

$$\phi(p_n) \leq \frac{1}{4^n} \tag{6.3}$$

by induction. Combined with (6.1) this yields $\phi(n) \leq 4^{-n} \leq (n+1)^{-2}$ so that

$$\phi(x) \leq \frac{1}{x^2} \leq \frac{1}{x} \tag{6.4}$$

for $x \geq p_1$. Thus ϕ satisfies the hypotheses of Lemma 5.1 and we may choose $f \in B$ according to Lemma 5.2. As mentioned after Lemma 5.2, we may assume that (5.4)–(5.7) hold.

Fix $\xi \in \operatorname{Unb}(f,(p_n))$ and $\varepsilon > 0$. In order to apply Lemma 5.5, we have to show that there exist $\delta(\xi) \in (0,\varepsilon)$ and a positive integer $N(\xi)$ such that $D(\xi,\delta(\xi)) \cap \operatorname{Unb}(f,(p_n))$ can be covered by $N(\xi)$ disks $D_1,\ldots,D_{N(\xi)}$ such that

$$\sum_{j=1}^{N(\xi)} h(\operatorname{diam} D_j) \leq \varepsilon\,\delta(\xi)^2. \tag{6.5}$$

In order to show that such $\delta(\xi)$ and $N(\xi)$ exist we note first that there exist arbitrarily large n with

$$|f^n(\xi)| \geq p_n \quad \text{and} \quad |f^n(\xi)| \geq \frac{6}{n^2} + \max_{0\leq k\leq n-1} \left|f^k(\xi)\right|. \tag{6.6}$$

Indeed, since $(|f^m(\xi)|)_{m\in\mathbb{N}}$ is unbounded, there exist arbitrarily large m such that

$$|f^m(\xi)| \geq \frac{6}{m^2} + \max_{0\leq k\leq m-1} \left|f^k(\xi)\right|.$$

For m with this property we take

$$n = \min\{k \in \mathbb{N} \colon k \geq m \text{ and } \left|f^k(\xi)\right| \geq p_k\}.$$

Then n satisfies (6.6). For such n we put

$$t_n = |f^n(\xi)| - \frac{1}{n^2}, \quad s_n = |f^n(\xi)| - \frac{3}{n^2} \quad \text{and} \quad r_n = |f^n(\xi)| - \frac{5}{n^2}. \tag{6.7}$$

Then

$$t_n > s_n > r_n \geq p_n - \frac{5}{n^2} > p_{n-1}$$

and thus

$$g(\phi(t_n)) \leq \tau(\phi(s_n)) \tag{6.8}$$

by (6.2). Moreover, (6.3) yields

$$\phi(t_n) \leq \phi(s_n) \leq \phi(r_n) \leq \frac{1}{4^{n-1}} \leq \frac{1}{n^2} \leq \frac{1}{n}, \tag{6.9}$$

Put

$$l_n = t_n - s_n = \frac{2}{n^2}$$

and let $Q_n = S(f^n(\xi), 1/n^2)$ be the square of side length l_n centered at $f^n(\xi)$. We can cover $Q_n \cap S$ by N_n squares of side length $\phi(t_n)$, where S is given by (5.3) and

$$N_n \leq \frac{2l_n}{\phi(t_n)}. \tag{6.10}$$

Put

$$\rho_n = \frac{1}{|(f^n)'(\xi)|}.$$

We deduce from (5.5) and (5.6) that, for large n, the branch of the inverse function of f^n which maps $f^n(\xi)$ to ξ extends univalently to a square centered at $f^n(\xi)$ which has twice the side length of Q_n. Koebe's distortion theorem now shows that, for certain absolute constants c_1 and c_2, which in fact could be determined explicitly, we can cover $D(\xi, c_1\rho_n l_n) \cap f^{-n}(S)$ by N_n disks $D_1, \ldots, D_{N_n}$ of diameter at most

$c_2\rho_n\phi(t_n)$. Using (5.6) and (5.7) we see that $D(\xi, c_1\rho_n l_n)\cap \mathrm{Unb}(f,(p_k))$ is covered by these disks. We have

$$\begin{aligned}\sum_{j=1}^{N_n} h(\operatorname{diam} D_j) &\le N_n h(c_2\rho_n\phi(t_n)) = N_n c_2\rho_n\phi(t_n) g(c_2\rho_n\phi(t_n)) \\ &\le 2c_2\rho_n l_n g(c_2\rho_n\phi(t_n))\end{aligned} \tag{6.11}$$

by (6.10).

Put $\delta_n = c_1\rho_n l_n$. It is easy to see that $\rho_n\to 0$ and thus $\delta_n\to 0$. We will show that (6.5) holds for $\delta(\xi)=\delta_n$ and $N(\xi)=N_n$ if n is sufficiently large. In order to do so we note that, by (6.11), it suffices to show that

$$2c_2\rho_n l_n g(c_2\rho(n)\phi(t_n)) \le \varepsilon\,\delta_n^2,$$

which, by the definition of δ_n and l_n, is equivalent to

$$g(c_2\rho_n\phi(t_n)) \le \frac{\varepsilon\delta_n^2}{2c_2 l_n\rho_n} = \frac{c_1^2\varepsilon}{c_2 n^2}\rho_n.$$

Since $\rho_n\to 0$ we have $c_2\rho_n\le 1$ for large n and thus it suffices to show that

$$g(\phi(t_n)) \le \frac{c_1^2\varepsilon}{c_2 n^2}\rho_n \tag{6.12}$$

for large n. To estimate ρ_n we note that

$$\rho_n = \frac{1}{\prod_{k=0}^{n-1}|f'(f^k(\xi))|}$$

by the chain rule. Cauchy's integral formula implies that

$$\max_{|z|\le r}|f'(z)| \le \frac{1}{R-r}\max_{|z|\le R}|f(z)|$$

for $0<r<R$. Since

$$\left|f^k(\xi)\right| \le |f^n(\xi)| - \frac{6}{n^2} \le r_n - \frac{1}{n^2} \quad \text{for } 0\le k\le n-1$$

by (6.6) and (6.7), we deduce from (5.4) that

$$\left|f'(f^k(\xi))\right| \le n^2\max_{|z|=r_n}|f(z)| \le n^2\exp\left(\exp\left(\frac{C}{\phi(r_n+8\phi(r_n))^4}\right)\right).$$

Now (6.9) says that $n^2\le 1/\phi(s_n)$ and (6.9) also yields that

$$8\phi(r_n) \le \frac{8}{4^{n-1}} \le \frac{2}{n^2} = s_n - r_n$$

and hence $r_n + 8\phi(r_n)\le s_n$ for large n. We conclude that

$$\left|f'(f^k(\xi))\right| \le \frac{1}{\phi(s_n)}\exp\left(\exp\left(\frac{C}{\phi(s_n)^4}\right)\right) \le \frac{1}{\phi(s_n)}\exp\left(\exp\left(\frac{1}{\phi(s_n)^5}\right)\right),$$

provided n is sufficiently large. We obtain

$$\begin{aligned}\rho_n &\geq \left(\phi(s_n)\exp\left(-\exp\left(\frac{1}{\phi(s_n)^5}\right)\right)\right)^n\\ &= 4^n\left(\frac{\phi(s_n)}{4}\exp\left(-\exp\left(\frac{1}{\phi(s_n)^5}\right)\right)\right)^n.\end{aligned}$$

Using (6.9) we obtain $\rho_n \geq 4^n\tau(\phi(s_n)) \geq 4^n g(\phi(t_n))$ which implies (6.12) for large n.

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References

[1] L. Ahlfors, 'Untersuchungen zur Theorie der konformen Abbildung und der ganzen Funktionen', *Acta Soc. Sci. Fenn., Nova Ser. A,* 1, no. 9 (1930).

[2] I. N. Baker, 'The domains of normality of an entire function', *Ann. Acad. Sci. Fenn. Ser A I Math.* 1 (1975), 277–283.

[3] K. Barański, 'Hausdorff dimension of hairs and ends for entire maps of finite order', *Math. Proc. Cambridge Philos. Soc.* 145 (2008), 719–737.

[4] K. Barański, B. Karpińska and A. Zdunik, 'Hyperbolic dimension of Julia sets of meromorphic maps with logarithmic tracts', *Int. Math. Res. Not. IMRN* (2009), 615–624.

[5] W. Bergweiler, 'Iteration of meromorphic functions', *Bull. Amer. Math. Soc. (N. S.)* 29 (1993), 151–188.

[6] W. Bergweiler, 'Karpińska's paradox in dimension 3', *Duke Math. J.* 154 (2010), 599–630.

[7] W. Bergweiler and A. Hinkkanen, 'On semiconjugation of entire functions', *Math. Proc. Cambridge Philos. Soc.* 126 (1999), 565–574.

[8] W. Bergweiler and B. Karpińska, 'On the Hausdorff dimension of the Julia set of a regularly growing entire function', *Math. Proc. Cambridge Philos. Soc.* 148 (2010), 531–551.

[9] W. Bergweiler, B. Karpińska and G. M. Stallard, 'The growth rate of an entire function and the Hausdorff dimension of its Julia set', *J. London Math. Soc.* 80 (2009), 680–698.

[10] W. Bergweiler, P. J. Rippon and G. M. Stallard, 'Dynamics of meromorphic functions with direct or logarithmic singularities', *Proc. London Math. Soc.* 97 (2008), 368–400.

[11] C. J. Bishop, 'A transcendental Julia set of dimension 1', preprint.

[12] R. L. Devaney and M. Krych, 'Dynamics of exp(z)', *Ergodic Theory Dynam. Systems* 4 (1984), 35–52.

[13] M. de Guzmán, 'Real Variable Methods in Fourier Analysis', North-Holland Math. Studies 46, Amsterdam, New York, 1981.

[14] A. E. Eremenko, 'On the iteration of entire functions', in 'Dynamical Systems and Ergodic Theory', Banach Center Publ. 23, Warsaw 1989, pp. 339–345.

[15] A. E. Eremenko and M. Yu. Lyubich, 'Dynamical properties of some classes of entire functions', *Ann. Inst. Fourier* 42 (1992), 989–1020.

[16] K. Falconer, 'Fractal Geometry - Mathematical Foundations and Applications', John Wiley & Sons Ltd, Chichester, 1997.

[17] B. Karpińska and M. Urbański, 'How points escape to infinity under exponential maps', *J. London Math. Soc.* (2) 73 (2006), 141–156.

[18] J. C. Mayer, 'An explosion point for the set of endpoints of the Julia set of $\lambda\exp(z)$', *Ergodic Theory Dynam. Systems* 10 (1990), 177–183.

[19] C. McMullen, 'Area and Hausdorff dimension of Julia sets of entire functions', *Trans. Amer. Math. Soc.* 300 (1987), 329–342.

[20] R. Nevanlinna, 'Eindeutige analytische Funktionen', Springer, Berlin, Heidelberg, 1953.

[21] J. Peter, 'Hausdorff measure of Julia sets in the exponential family', *J. London Math. Soc.* (2) 82 (2010), 229–255.

[22] J. Peter, 'Hausdorff measure of escaping and Julia sets for bounded-type functions of finite order', *Ergodic Theory Dynam. Systems*. Available on CJO 2011, doi: 10.1017/S0143385711000745.

[23] G. Pólya and G. Szegö, 'Aufgaben und Lehrsätze aus der Analysis I', Springer, Berlin, 1925.

[24] F. Przytycki and M. Urbański, 'Fractals in the plane: ergodic theory methods', London Math. Soc. Lect. Note Ser. 371. Cambridge University Press, Cambridge, 2010.

[25] L. Rempe, 'Hyperbolic dimension and radial Julia sets of transcendental functions', *Proc. Amer. Math. Soc.* 137 (2009), 1411–1420.

[26] L. Rempe, 'Hyperbolic entire functions with full hyperbolic dimension and approximation by Eremenko-Lyubich functions', preprint, arXiv:1106.3439v2 [math.CV].

[27] L. Rempe, P.J. Rippon and G. M. Stallard, 'Are Devaney hairs fast escaping?', *J. Difference Equ. Appl.* 16 (2010), 739–762.

[28] L. Rempe and G. M. Stallard, 'Hausdorff dimensions of escaping sets of transcendental entire functions', *Proc. Amer. Math. Soc.* 138 (2010), 1657–1665.

[29] P. J. Rippon and G. M. Stallard, 'On questions of Fatou and Eremenko', *Proc. Amer. Math. Soc.* 133 (2005), 1119–1126.

[30] P. J. Rippon and G. M. Stallard, 'Slow escaping points of meromorphic functions', *Trans. Amer. Math. Soc.* 363 (2011), 4171–4201.

[31] P. J. Rippon and G. M. Stallard, 'Fast escaping points of entire functions', *Proc. London Math. Soc.*, doi: 10.1112/plms/pds001.

[32] G. Rottenfußer, J. Rückert, L. Rempe and D. Schleicher, 'Dynamic rays of bounded type entire functions', *Ann. of Math.* 173 (2011), 77–125.

[33] H. Schubert, 'Über die Hausdorff-Dimension der Juliamenge von Funktionen endlicher Ordnung', Dissertation, Christian-Albrechts-Universität zu Kiel, 2007.

[34] G. M. Stallard, 'Entire functions with Julia sets of zero measure', *Math. Proc. Cambridge Philos. Soc.* 108 (1990), 551–557.

[35] G. M. Stallard, 'The Hausdorff dimension of Julia sets of entire functions II', *Math. Proc. Cambridge Philos. Soc.* 119 (1996), 513–536.

[36] G. M. Stallard, 'The Hausdorff dimension of Julia sets of entire functions IV', *J. London Math. Soc.* (2) 61 (2000), 471–488.

[37] G. M. Stallard, 'Dimensions of Julia sets of transcendental meromorphic functions', in 'Transcendental Dynamics and Complex Analysis'. London Math. Soc. Lect. Note Ser. 348. Cambridge University Press, Cambridge, 2008, pp. 425–446.

[38] T. F. Tyler, 'Maximum curves and isolated points of entire functions', *Proc. Amer. Math. Soc.* 128 (2000), 2561–2568.

Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel, 24118 Kiel, Germany