A property of the derivative of an entire function

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July 21, 2011

Abstract

We prove that the derivative of a non-linear entire function is unbounded on the preimage of an unbounded set.

MSC 2010: 30D30. Keywords: entire function, normal family.

1 Introduction and results

The main result of this paper is the following theorem conjectured by Allen Weitsman (private communication):

Theorem 1. *Let f be a non-linear entire function and M an unbounded set in $\mathbf{C}$. Then $f'(f^{-1}(M))$ is unbounded.*

We note that there exist entire functions f such that $f'(f^{-1}(M))$ is bounded for every bounded set M, for example, $f(z) = e^z$ or $f(z) = \cos z$.

Theorem 1 is a consequence of the following stronger result:

Theorem 2. *Let f be a transcendental entire function and $\varepsilon > 0$. Then there exists $R > 0$ such that for every $w \in \mathbf{C}$ satisfying $|w| > R$ there exists $z \in \mathbf{C}$ with $f(z) = w$ and $|f'(z)| \geq |w|^{1-\varepsilon}$.*

*Supported by the Deutsche Forschungsgemeinschaft, Be 1508/7-1, and the ESF Networking Programme HCAA.

†Supported by NSF grant DMS-1067886.

The example $f(z)=\sqrt{z}\sin\sqrt{z}$ shows that that the exponent $1-\varepsilon$ in the last inequality cannot be replaced by 1. The function $f(z)=\cos\sqrt{z}$ has the property that for every $w\in\mathbf{C}$ we have $f'(z)\to 0$ as $z\to\infty$, $z\in f^{-1}(w)$.

We note that the Wiman–Valiron theory [20, 12, 4] says that there exists a set $F\subset[1,\infty)$ of finite logarithmic measure such that if

$$|z_r|=r\notin F \quad \text{and} \quad |f(z_r)|=\max_{|z|=r}|f(z)|,$$

then

$$f(z)\sim\left(\frac{z}{z_r}\right)^{\nu(r,f)}f(z_r) \quad \text{and} \quad f'(z)\sim\frac{\nu(r,f)}{r}f(z)$$

for $|z-z_r|\leq r\nu(r,f)^{-1/2-\delta}$ as $r\to\infty$. Here $\nu(r,f)$ denotes the central index and $\delta>0$. This implies that the conclusion of Theorem 2 holds for all w satisfying $|w|=M(r,f)$ for some sufficiently large $r\notin F$. However, in general the exceptional set in the Wiman–Valiron theory is non-empty (see, e.g., [3]) and thus it seems that our results cannot be proved using Wiman–Valiron theory.

Acknowledgment. We thank Allen Weitsman for helpful discussions.

2 Preliminary results

One important tool in the proof is the following result known as the Zalcman Lemma [21]. Let

$$g^{\#}=\frac{|g'|}{1+|g|^2}$$

denote the spherical derivative of a meromorphic function g.

Lemma 1. *Let F be a non-normal family of meromorphic functions in a region D. Then there exist a sequence (f_n) in F, a sequence (z_n) in D, a sequence (ρ_n) of positive real numbers and a non-constant function g meromorphic in $\mathbf{C}$ such that $\rho_n\to 0$ and $f_n(z_n+\rho_n z)\to g(z)$ locally uniformly in $\mathbf{C}$. Moreover, $g^{\#}(z)\leq g^{\#}(0)=1$ for $z\in\mathbf{C}$.*

We say that $a\in\overline{\mathbf{C}}$ is a *totally ramified* value of a meromorphic function f if all a-points of f are multiple. A classical result of Nevanlinna says that a non-constant function meromorphic in the plane can have at most 4 totally ramified values, and that a non-constant entire function can have at most 2 finite totally ramified values. Together with Zalcman's Lemma this yields the following result [5, 13, 14]; cf. [22, p. 219].

Lemma 2. *Let F be a family of functions meromorphic in a domain D and M a subset of $\overline{\mathbf{C}}$ with at least 5 elements. Suppose that there exists $K \geq 0$ such that for all $f \in F$ and $z \in D$ the condition $f(z) \in M$ implies $|f'(z)| \leq K$. Then F is a normal family.*

If all functions in F are holomorphic, then the conclusion holds if M has at least 3 elements.

Applying Lemma 2 to the family $\{f(z+c) : c \in \mathbf{C}\}$ where f is an entire function, we obtain the following result.

Lemma 3. *Let f be an entire function and M a subset of $\mathbf{C}$ with at least 3 elements. If f' is bounded on $f^{-1}(M)$, then $f^{\#}$ is bounded in $\mathbf{C}$.*

It follows from Lemma 3 that the conclusion of Theorems 1 and 2 holds for all entire functions for which $f^{\#}$ is unbounded.

We thus consider entire functions with bounded spherical derivative. The following result is due to Clunie and Hayman [6]. Let

$$M(r,f) = \max_{|z|\leq r} |f(z)| \quad \text{and} \quad \rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}$$

denote the maximum modulus and the order of f.

Lemma 4. *Let f be an entire function for which $f^{\#}$ is bounded. Then $\log M(r,f) = O(r)$ as $r \to \infty$. In particular, $\rho(f) \leq 1$.*

We will include a proof of Lemma 4 after Lemma 6.

The following result is due to Valiron [20, III.10] and H. Selberg [17, Satz II].

Lemma 5. *Let f be a non-constant entire function of order at most 1 for which 1 and -1 are totally ramified. Then $f(z) = \cos(az+b)$, where $a, b \in \mathbf{C}$, $a \neq 0$.*

We sketch the proof of Lemma 5. Put $h(z) = f'(z)^2/(f(z)^2-1)$. Then h is entire and the lemma on the logarithmic derivative [9, p.94, (1.17)], together with the hypothesis that $\rho(f) \leq 1$, yields that $m(r,h) = o(\log r)$ and hence that h is constant. This implies that f has the form given. Another proof is given in [10]

The next lemma can be extracted from the work of Pommerenke [16, Sect. 5], see [8, Theorem 5.2].

Lemma 6. *Let f be an entire function and $C > 0$. If $|f'(z)| \leq C$ whenever $|f(z)| = 1$, then $|f'(z)| \leq C|f(z)|$ whenever $|f(z)| \geq 1$.*

Lemma 6 implies the theorem of Clunie and Hayman mentioned above (Lemma 4). For the convenience of the reader we include a proof of a slightly more general statement, which is also more elementary than the proofs of Clunie, Hayman and Pommerenke; see also [1, Lemma 1].

Let $G = \{z : |f(z)| > 1\}$ and $u = \log |f|$. Then $|f'/f| = |\nabla u|$ and our statement which implies Lemmas 4 and 6 is the following.

Proposition. *Let G be a region in the plane, u a harmonic function in $\overline{G}$, positive in G, and such that for $z \in \partial G$ we have $u(z) = 0$ and $|\nabla u(z)| \leq 1$. Then $|\nabla u(z)| \leq 1$ for $z \in G$, and $u(z) \leq |z| + O(1)$ as $z \to \infty$.*

Proof. It is enough to consider the case of unbounded G with non-empty boundary. For $a \in G$, consider the largest disc B centered at a and contained in G. The radius $d = d(a)$ of this disc is the distance from a to ∂G. There is a point $z_1 \in \partial B$ such that $u(z_1) = 0$. Put $z(r) = a + r(z_1 - a)$, where $r \in (0, 1)$. Harnack's inequality gives

$$\frac{u(a)}{d(1+r)} \leq \frac{u(z(r))}{d(1-r)} = \frac{u(z(r)) - u(z_1)}{d(1-r)}.$$

Passing to the limit as $r \to 1$ we obtain

$$u(a) \leq 2d(a)|\nabla u(z_1)| \leq 2d(a).$$

This holds for all $a \in G$. Now we take the gradient of both sides of the Poisson formula and, noting that $u(a + d(a)e^{it}) \leq 2d(a + d(a)e^{it}) \leq 4d(a)$, obtain the estimate

$$|\nabla u(a)| \leq \frac{1}{\pi d(a)} \int_{-\pi}^{\pi} |u(a + d(a)e^{it})| dt \leq 8.$$

So ∇u is bounded in G. As the complex conjugate of ∇u is holomorphic in G and $|\nabla u(z)| \leq 1$ at all boundary points z of G, except infinity, the Phragmén–Lindelöf theorem [15, III, 335] gives that $|\nabla u(z)| \leq 1$ for $z \in G$. This completes the proof of the Proposition.

We recall that for a non-constant entire function f the maximum modulus $M(r) = M(r, f)$ is a continuous strictly increasing function of r. Denote by

φ the inverse function of M. Clearly, for $|w| > |f(0)|$ the equation $f(z) = w$ has no solutions in the open disc of radius $\varphi(|w|)$ around 0. The following result of Valiron ([18, 19], see also [7]) says that for functions of finite order this equation has solutions in a somewhat larger disc.

Lemma 7. *Let f be a transcendental entire function of finite order and $\eta > 0$. Then there exists $R > |f(0)|$ such that for all $w \in \mathbf{C}, |w| \geq R$, the equation $f(z) = w$ has a solution z satisfying $|z| < \varphi(|w|)^{1+\eta}$.*

We note that Hayman ([11], see also [2, Theorem 3]) has constructed examples which show that the assumption about finite order is essential in this lemma.

3 Proof of Theorem 2

Suppose that the conclusion is false. Then there exists $\varepsilon > 0$, a transcendental entire function f and a sequence (w_n) tending to ∞ such that $|f'(z)| \leq |w_n|^{1-\varepsilon}$ whenever $f(z) = w_n$. By Lemma 3, the spherical derivative of f is bounded, and we may assume without loss of generality that

$$f^{\#}(z) \leq 1 \quad \text{for } z \in \mathbf{C}. \tag{1}$$

We may also assume that $f(0) = 0$. It follows from (1) that $|f'(z)| \leq 2$ if $|f(z)| = 1$, and thus Lemma 6 yields

$$\left|\frac{f'(z)}{f(z)}\right| \leq 2 \quad \text{if } |f(z)| \geq 1. \tag{2}$$

It also follows from (1), together with Lemma 4, that $\rho(f) \leq 1$. We may thus apply Lemma 7 and find that if $\eta > 0$ and if n is sufficiently large, then there exists ξ_n satisfying

$$|\xi_n| \leq \varphi(|w_n|)^{1+\eta} \quad \text{and} \quad f(\xi_n) = w_n.$$

We put

$$\tau_n = \varphi(|w_n|)^{1+2\eta}$$

and define

$$\Phi_n(z) = \frac{w_n - 2f(\tau_n z)}{w_n} = 1 - 2\frac{f(\tau_n z)}{w_n}.$$

Then $\Phi_n(0) = 1$, $\Phi_n(\xi_n/\tau_n) = -1$, and $\xi_n/\tau_n \to 0$ as $n \to \infty$. Thus the sequence (Φ_n) is not normal at 0, and we may apply Zalcman's Lemma (Lemma 1) to it. Replacing (Φ_n) by a subsequence if necessary, we thus find that

$$g_n(z) = \Phi_n(z_n + \rho_n z) = 1 - \frac{2}{w_n} f(\tau_n z_n + \tau_n \rho_n z) \to g(z)$$

locally uniformly in $\mathbf{C}$, where $|z_n| \leq 1$, $\rho_n > 0$, $\rho_n \to 0$, and g is a non-constant entire function with bounded spherical derivative. With $\zeta_n = \tau_n z_n$ and $\mu_n = \tau_n \rho_n$ we have

$$g_n(z) = 1 - \frac{2}{w_n} f(\zeta_n + \mu_n z), \tag{3}$$

and

$$g_n'(z) = -\frac{2\mu_n}{w_n} f'(\zeta_n + \mu_n z). \tag{4}$$

We may assume that $\rho_n \leq 1$ and hence $|\zeta_n| \leq \tau_n$ and $\mu_n \leq \tau_n$ for all n.

If $g_n(z) = 1$, then $f(\zeta_n + \mu_n z) = 0$, hence $|f'(\zeta_n + \mu_n z)| \leq 1$ by (1). Since $\mu_n \leq \tau_n$, we deduce that

$$|g_n'(z)| \leq \frac{2\tau_n}{w_n} \quad \text{if } g_n(z) = 1. \tag{5}$$

If $g_n(z) = -1$, then $f(\zeta_n + \mu_n z) = w_n$, and hence $|f'(\zeta_n + \mu_n z)| \leq |w_n|^{1-\varepsilon}$ by our assumption. Thus

$$|g_n'(z)| \leq \frac{2\mu_n}{|w_n|} |w_n|^{1-\varepsilon} \leq \frac{2\tau_n}{|w_n|^{\varepsilon}} \quad \text{if } g_n(z) = -1. \tag{6}$$

It follows from the definition of τ_n that

$$\tau_n = o(|w_n|)^{\delta}) \quad \text{as } n \to \infty, \tag{7}$$

for any given $\delta > 0$.

We deduce from (5), (6) and (7) that $g'(z) = 0$ whenever $g(z) = 1$ or $g(z) = -1$. Since g has bounded spherical derivative, we conclude from Lemmas 3 and 4 that $g(z) = \cos(az + b)$. Without loss of generality, we may assume that $g(z) = \cos z$ so that $g'(z) = -\sin z$. In particular, there exist sequences (a_n) and (b_n) both tending to 0, such that $g_n(a_n) = 1$ and $g_n'(b_n) = 0$. From (5) we deduce that

$$|g_n'(a_n)| \leq \frac{2\tau_n}{|w_n|}. \tag{8}$$

Noting that $g''(z) = -\cos z$ we find that

$$g_n'(a_n) = g_n'(a_n) - g_n'(b_n) = \int_{b_n}^{a_n} g_n''(z)dz \sim b_n - a_n \tag{9}$$

as $n \to \infty$, and thus

$$|b_n - a_n| \leq \frac{3\tau_n}{|w_n|} \tag{10}$$

for large n, by (8). This implies that

$$|g_n(b_n) - 1| = |g_n(b_n) - g_n(a_n)| = \left|\int_{a_n}^{b_n} g_n'(z)dz\right| \leq 2|b_n - a_n| \leq \frac{6\tau_n}{|w_n|} \tag{11}$$

for large n.

We put

$$h_n(z) = g_n(z + b_n) - g_n(b_n)$$

and note that $h_n(0) = 0$, $h_n'(0) = g_n'(b_n) = 0$ and

$$h_n(z) \to \cos z - 1 \quad \text{as } n \to \infty.$$

It follows that

$$\frac{h_n(z)}{z^2} \to \frac{\cos z - 1}{z^2} \quad \text{as } n \to \infty,$$

which implies that there exists $r > 0$ such that

$$\frac{1}{4} \leq \frac{|h_n(z)|}{|z^2|} \leq \frac{3}{4} \quad \text{for } |z| \leq r. \tag{12}$$

and large n.

Now we fix any $\gamma \in (0, 1/2)$ and put

$$c_n = b_n + \frac{1}{|w_n|^\gamma}.$$

Then

$$g_n(c_n) - 1 = h_n(|w_n|^{-\gamma}) + g(b_n) - 1$$

and thus, using (11) and (12) we obtain for large n:

$$|g_n(c_n) - 1| \leq \left|h_n(|w_n|^{-\gamma})\right| + |g(b_n) - 1| \leq \frac{3}{4|w_n|^{2\gamma}} + \frac{6\tau_n}{|w_n|} \leq \frac{1}{|w_n|^{2\gamma}}. \tag{13}$$

Similarly

$$|g_n(c_n) - 1| \geq \left|h_n(|w_n|^{-\gamma})\right| - |g(b_n) - 1| \geq \frac{1}{5|w_n|^{2\gamma}}. \tag{14}$$

On the other hand, arguing as in (9), we have

$$g_n'(c_n) = g_n'(c_n) - g_n'(b_n) = \int_{b_n}^{c_n} g_n''(z)dz \sim b_n - c_n = -\frac{1}{|w_n|^{\gamma}},$$

and thus

$$|g_n'(c_n)| \geq \frac{1}{2|w_n|^{\gamma}} \tag{15}$$

for large n. Put $v_n = \zeta_n + \mu_n c_n$. Then

$$f(v_n) = \frac{w_n}{2}(1 - g_n(c_n)) \quad \text{and} \quad f'(v_n) = \frac{w_n}{2\mu_n} g_n'(c_n),$$

by (3) and (4). Hence

$$\frac{1}{10}|w_n|^{1-2\gamma} \leq |f(v_n)| \leq \frac{1}{2}|w_n|^{1-2\gamma}, \tag{16}$$

by (13) and (14) while

$$|f'(v_n)| \geq \frac{|w_n|^{\gamma}}{2\mu_n}.$$

Since $|f(v_n)| \geq 1$ for large n, by (16), this contradicts (2) and (7).

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