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Trees and hairs for some hyperbolic entire maps of finite order

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Abstract Let f be an entire transcendental map of finite order, such that all the singularities of f^{-1} are contained in a compact subset of the immediate basin B of an attracting fixed point. It is proved that there exist geometric coding trees of preimages of points from B with all branches convergent to points from $\widehat{\mathbb{C}}$. This implies that the Riemann map onto B has radial limits everywhere. Moreover, the Julia set of f consists of disjoint curves (hairs) tending to infinity, homeomorphic to a half-line, composed of points with a given symbolic itinerary and attached to the unique point accessible from B (endpoint of the hair). These facts generalize the corresponding results for exponential maps.

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1 Introduction

For an entire transcendental (i.e. not polynomial) map $f : \mathbb{C} \to \mathbb{C}$ let $\mathrm{Sing}(f^{-1})$ denote the set of finite singularities of f^{-1}, which consists of critical and asymptotic values of f and their finite accumulation points. In this paper we consider the maps f satisfying the following conditions:

- $\mathrm{Sing}(f^{-1})$ is contained in a compact subset of an immediate attracting basin B.
- f has finite order.

Recall that an immediate attracting basin B is a domain containing an attracting fixed point $z_0 \in \mathbb{C}$, composed of points tending to z_0 under iteration of f. In particular, the assumptions imply $f \in \mathcal{B}$ for

$$\mathcal{B} = \{f : \mathrm{Sing}(f^{-1}) \text{ is bounded}\}.$$

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Hence, B is simply connected (see [13]). Let

$$P(f) = \bigcup_{n=0}^{\infty} f^n(\mathrm{Sing}(f^{-1})).$$

Note that f is hyperbolic in the sense that $J(f) \cap \overline{P(f)} = \emptyset$ and $\overline{P(f)}$ is compact. It follows that f has no wandering domains (see [5]) and Baker domains (see [13]), which implies that B is the only Fatou component of f and the Julia set $J(f)$ is equal to the boundary of B. Thus, the considered maps can be characterized as hyperbolic entire transcendental maps of finite order with one Fatou component.

Example 1 A classical example of maps fulfilling these assumptions is the family

$$E_\lambda(z) = \lambda e^z, \quad \lambda \in (0, 1/e)$$

and, more generally, $\lambda = z_0 e^{-z_0}$ for $z_0 \in \mathbb{C}$, $|z_0| < 1$. Then z_0 is an attracting fixed point, the set $\mathrm{Sing}(E_\lambda^{-1})$ consists only of the asymptotic value 0 and is contained in the immediate basin B of attraction to z_0. Another example is the sine family

$$S_\lambda(z) = \lambda \sin(z), \quad \lambda \in (0, 1).$$

In fact, it is easy to check that every map of the form

$$f(z) = \lambda g(z), \quad \lambda \in \mathbb{C} \setminus \{0\},$$

where $g \in \mathcal{B}$ and g has finite order, satisfies the above assumptions for sufficiently small $|\lambda|$.

In this paper we study the topological and combinatorial properties of $J(f)$ and the boundary behaviour of a Riemann mapping φ from the open unit disc $\mathbb{D}$ onto B. We prove:

Theorem B *A Riemann map onto B has radial limits at all points of the unit circle.*

(The proof is contained in Sect. 5.) This generalizes the result of Devaney and Goldberg [8], who proved the same for the exponential family from Example 1.

Note that despite having radial limits at all points, the Riemann map φ is highly discontinuous, e.g. every cluster of φ contains ∞ (see [2]). In fact, the clusters of φ are either singletons $\{\infty\}$ or the sets of points in the Julia set $J(f)$ sharing the same symbolic itinerary (together with ∞). Both cases happen on dense sets in the unit circle (see [4]).

To show Theorem B, we use the technique of geometric coding trees of preimages of points from B. A coding tree in B is the union of curves in B connecting a point $\zeta \in B$ to all its first preimages and the pull-backs of these curves by the branches of f^{-n}, $n > 0$. Choosing particular branches for each n, we define the branches of the coding tree, which are curves parameterized by $t \in [0, \infty)$, starting at ζ, with limit points for $t \to \infty$ contained in the boundary of B. See Sect. 4 for precise definitions.

Theorem B is a consequence of the following:

Theorem A *There exist coding trees of preimages of points from B with all branches convergent to points from* $\widehat{\mathbb{C}}$.

(A precise formulation and proof are contained in Sect. 5.) The same was proved in [8] for the exponential family from Example 1. In [16], Karpińska showed the result for hyperbolic entire transcendental maps with one Fatou component, whose tracts are "asymptotically straight" (the examples include e.g. maps considered by Stallard in [26]).

The dynamical plane for f can be decomposed into countably many parts — so-called tracts and fundamental domains (see Sect. 3 and 4). This enables to define symbolic itineraries for points from $J(f)$. In [10] (see also [6]), Devaney and Krych showed that for the exponential family from Example 1, the Julia set consists of disjoint curves (hairs) $h : [0, \infty) \to \mathbb{C}$ tending to ∞, composed of points sharing the same itinerary under f (see Sect. 6 for a precise definition). The point $h(0)$ is called the endpoint of the hair. Devaney and Goldberg showed in [8] that for the exponential family, the endpoint of the hair is the unique point in h, which is accessible from B. In [11], the existence of so-called Cantor bouquets in the Julia set was proved for a number of families of entire maps from $\mathcal{B}$, including the exponential and sine families from Example 1. In [1], Aarts and Oversteegen showed that for these two families the Julia set is homeomorphic to so-called straight brush in the plane. In particular, each hair is homeomorphic to the half-line $[0, \infty)$. Moreover, Viana proved in [27] that the hairs for the exponential family are C^∞-smooth.

Note that the above results were done for concrete families of periodic entire maps. In this paper we prove the existence of hairs for hyperbolic entire maps of finite order with one Fatou component without any assumptions on the form of the map. More precisely, we show:

Theorem C *The Julia set of f consists of disjoint hairs homeomorphic to the half-line $[0, \infty)$. The endpoints of the hairs are the unique points accessible from B.*

(A precise formulation and proof are contained in Sect. 6.)

Let us note that the hairs exist also for non-hyperbolic entire maps. For instance, the existence of hairs for so-called regular itineraries was proved in [6] for the general exponential family $E_\lambda(z) = \lambda e^z$, $\lambda \in \mathbb{C} \setminus \{0\}$. This includes also the case, when the Julia set is the whole plane. However, in this case the sets of points with non-regular itineraries can form indecomposable Knaster-type continua (see [7,9]).

Note that the hairs without endpoints are contained in the escaping set

$$I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}.$$

A symbolic classification of points from $I(f)$ and the existence of the curves in $I(f)$ (called dynamical rays) for the general exponential family was showed in [25]. The same was done for the family $ae^z + be^{-z}$, $a, b \in \mathbb{C}$, which contains the sine and cosine families under the change of coordinates (see [24]).

The problem of the existence of the hairs is related to the Eremenko conjecture, saying that $I(f)$ consists only of unbounded components (see [12]). The question was answered in some particular cases (see [17,22]). Recently, it was showed that for every map $f \in \mathcal{B}$ of finite order, the set $I(f) \cup \{\infty\}$ is path-connected (see [21,23]). Note that Theorem C from this paper implies the Eremenko conjecture in our case, which also follows from the mentioned result.

The plan of the paper is as follows. After preliminaries, in Sect. 3 we define logarithmic coordinates and prove several technical lemmas describing properties of the map F, which is the lift of f in the logarithmic coordinates. In Sect. 4 we describe the combinatorics of the Julia set and define the trees of preimages. Theorems A and B are formulated and proved in Sect. 5, while Theorem C is showed in Sect. 6.

This paper is the second part of the series of three papers concerning the properties of the Julia set of hyperbolic entire maps with one Fatou component. In the first part [4] we show (without the finite order assumption) that for given symbolic itinerary, if codes of the tracts of f are bounded and codes of the fundamental domains grow not faster than the iterates of an exponential function, then the itinerary is allowable, i.e. there exists a point in the Julia set with this itinerary. Moreover, we determine the cluster sets for φ and show that φ has unrestricted limit equal to ∞ at points of a dense uncountable set in the unit circle. In the third part [3] we prove (generalizing [15]) that under the finite order assumption the Hausdorff dimension of the set of endpoints of the hairs is equal to 2, while the union of the hairs without endpoints has Hausdorff dimension 1.

2 Preliminaries

Notation

For a set $A \subset \mathbb{C}$ the symbols $\overline{A}$, ∂A denote, respectively, the closure and boundary in $\mathbb{C}$. For $z \in \mathbb{C}$ and $A, B \subset \mathbb{C}$ let

$$\operatorname{dist}(z, A) = \inf\{|z - a| : a \in A\}, \quad \operatorname{dist}(A, B) = \inf\{|a - b| : a \in A, b \in B\}.$$

The open unit disc in $\mathbb{C}$ is denoted by $\mathbb{D}$ and the open disc of radius r centred at $z \in \mathbb{C}$ – by $\mathbb{D}_r(z)$. By an open simple arc we mean a set homeomorphic to $(0, 1)$, by a simple arc – a set homeomorphic to $[0, 1]$ and by a Jordan curve – a set homeomorphic to a circle. Let

$$H = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$$

and for $z_1, z_2 \in H$ denote by $\varrho_H(z_1, z_2)$ the hyperbolic distance between z_1 and z_2 in H for the hyperbolic metric $\varrho_H(z)\, dz = dz/\operatorname{Re}(z)$. Recall that

$$\varrho_H(z_1, z_2) = 2 \operatorname{asinh} \frac{|z_1 - z_2|}{2\sqrt{\operatorname{Re}(z_1)\operatorname{Re}(z_2)}}. \tag{1}$$

Boundary behaviour

We recall some basic facts concerning the boundary behaviour of a Riemann map. For a general exposition on the subject, refer e.g. to [20].

Let U be a simply connected domain in the Riemann sphere $\widehat{\mathbb{C}}$, whose boundary contains more than two points and let $z \in \widehat{\mathbb{C}}$ be a point from this boundary. We say that a curve $\gamma : [0, \infty) \to U$ lands at z, if $z = \lim_{t\to\infty} \gamma(t)$. In this case we say that z is accessible from U.

Let φ be a Riemann mapping from $\mathbb{D}$ onto U. The cluster set $C(\theta, \varphi)$, where $\theta \in [0, 2\pi]$, is a set of points $z \in \widehat{\mathbb{C}}$, for which there exists a sequence $w_n \in \mathbb{D}$ such that $w_n \to e^{i\theta}$ and $\varphi(w_n) \to z$. We say that φ has unrestricted limit equal to z at the point $e^{i\theta}$, if $C(\theta, \varphi) = \{z\}$, i.e. if $\varphi(w) \to z$ as $w \to e^{i\theta}$, $w \in \mathbb{D}$. The map φ has radial limit equal to $z \in \widehat{\mathbb{C}}$ at the point $e^{i\theta}$, if $z = \lim_{r\to 1^-} \varphi(re^{i\theta})$.

Recall a version of the classical Lindelöf theorem.

Lindelöf Theorem *Let $\gamma : [0, \infty) \to U$ be a curve landing at z. Then the curve $\varphi^{-1}(\gamma)$ in $\mathbb{D}$ lands at some point v of $\partial\mathbb{D}$. Moreover, φ has the radial limit at v equal to z. Hence, a point z is accessible from U if and only if z is the radial limit of φ at some point of the unit circle.*

Topology

The following theorems are classical results in the theory of continua (see e.g. [18]).

Mazurkiewicz-Moore Theorem *Let X be a complete, locally connected metric space. Then for every open connected set $V \subset X$ and every $x, y \in V$ there exists a homeomorphism h from $[0, 1]$ onto a subset of V, such that $h(0) = x$ and $h(1) = y$.*

Sierpiński Theorem *A metric compact space X is locally connected if and only if for every $\varepsilon > 0$ the space X is a union of a finite number of continua of diameters less than ε.*

3 Properties of logarithmic tracts

Consider a map f satisfying the assumptions stated in the beginning of the paper. Since B is simply connected, it is easy to find a simply connected domain $D \subset \mathbb{C}$, such that $\overline{D} \subset B$, ∂D is an analytic Jordan curve, $\overline{\mathrm{Sing}(f^{-1})} \subset D$ and $\overline{f(D)} \subset D$. Let T be a connected component of $f^{-1}(\mathbb{C} \setminus \overline{D})$. Then (see [11,13]) $\overline{T}$ is simply connected, ∂T is an analytic open simple arc, $\partial T \cup \{\infty\}$ is a Jordan curve and $\overline{T} \subset f(T) = \mathbb{C} \setminus \overline{D}$. The sets T are called *(exponential) tracts* of f. Note that by the Ahlfors Spiral Theorem (see [14]), entire maps of finite order have only a finite number of tracts. Enumerate the tracts of f by T^r, $r \in \mathcal{R}$, where $\mathcal{R} = \{1, \ldots, R\}$.

We use (slightly modified) logarithmic coordinates introduced by Eremenko and Lyubich (see [13,4]). Let

$$\Phi : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \widehat{\mathbb{C}} \setminus \overline{D}$$

be a Riemann mapping such that $\Phi(\infty) = \infty$. Since the closures of the tracts are simply connected, there exists an open simple arc

$$\alpha : (0, \infty) \to \mathbb{C} \setminus \overline{D} \tag{2}$$

disjoint from the closures of all tracts, such that $\alpha(t)$ tends to a point of ∂D (resp. to ∞) as t tends to 0^+ (resp. to ∞). Then $\Phi^{-1}(\mathbb{C} \setminus (\overline{D} \cup \alpha))$ is a simply connected domain, which does not contain 0, so the branches of logarithm can be defined on it. Denote these branches by $\log_s$, $s \in \mathbb{Z}$, such that $\log_s = \log_0 + 2\pi i s$ and let

$$H_s = \log_s(\Phi^{-1}(\mathbb{C} \setminus (\overline{D} \cup \alpha))).$$

By definition, H_s are pairwise disjoint domains in H, such that $H_s = H_0 + 2\pi i s$. Moreover,

$$H \cap \partial H_s = A_s \cup A_{s+1}, \tag{3}$$

for open simple arcs

$$A_s : (0, \infty) \to H,$$

which are the lifts of α by suitable branches of $\log \circ \Phi^{-1}$, such that $A_s = A_0 + 2\pi i s$, $A_s(t)$ tends to a point of ∂H as $t \to 0^+$ and $\mathrm{Re}(A_s) \to \infty$ as $t \to \infty$. Let

$$L_s^r = \log_s(\Phi^{-1}(T^r))$$

for $r \in \mathcal{R}$, $s \in \mathbb{Z}$. We call L_s^r a *logarithmic tract*. Then f can be lifted to the map

$$F : \bigcup_{r \in \mathcal{R}, s \in \mathbb{Z}} \overline{L_s^r} \to \overline{H}$$

such that

$$\Phi \circ \exp \circ F = f \circ \Phi \circ \exp, \tag{4}$$

F is periodic with period $2\pi i$, univalent on each logarithmic tract and maps its closure homeomorphically onto $\overline{H}$. Note that, by definition, logarithmic tracts contain no vertical segments of length 2π. Moreover, there exists $c_0 > 0$ such that

$$L_s^r \subset \{z : \mathrm{Re}(z) > c_0\} \tag{5}$$

for every r, s.

Now we present technical lemmas, which will be used in the proofs of the main results. Note that only Lemmas 2 and 4 use the finite order assumption.

Lemma 1 *Let $d > 0$. Then for every $c > 0$ there exists $M = M(c)$, such that $M(c) \to \infty$ as $c \to \infty$ and for every $z_1, z_2 \in H$, if $\varrho_H(z_1, z_2) > c$ and $|z_1 - z_2| < d \max(\mathrm{Re}(z_1), \mathrm{Re}(z_2))$, then*

$$\mathrm{Re}(z_1) > M \,\mathrm{Re}(z_2) \quad \text{or} \quad \mathrm{Re}(z_2) > M \,\mathrm{Re}(z_1).$$

Proof Assume $\mathrm{Re}(z_1) \geq \mathrm{Re}(z_2)$ (the other case is symmetric). Then, by assumptions and (1),

$$\begin{aligned} \sinh \frac{c}{2} &< \sinh \frac{\varrho_H(z_1, z_2)}{2} = \frac{|z_1 - z_2|}{2\sqrt{\mathrm{Re}(z_1)\,\mathrm{Re}(z_2)}} \\ &< \frac{d\,\mathrm{Re}(z_1)}{2\sqrt{\mathrm{Re}(z_1)\,\mathrm{Re}(z_2)}} = \frac{d}{2}\sqrt{\frac{\mathrm{Re}(z_1)}{\mathrm{Re}(z_2)}}, \end{aligned}$$

so $\mathrm{Re}(z_1) > M \,\mathrm{Re}(z_2)$ for $M = \frac{4}{d^2}\left(\sinh \frac{c}{2}\right)^2$. □

The following lemma describes the geometry of logarithmic tracts.

Lemma 2 *There exists $d_0 > 0$, such that for every logarithmic tract L_s^r and every $z_1, z_2 \in \overline{L_s^r}$,*

$$|z_1 - z_2| < d_0 \max(\mathrm{Re}(z_1), \mathrm{Re}(z_2)).$$

Proof By (3), for every $z \in \overline{L_s^r}$ there exist $t_1, t_2 > 0$, such that

$$\mathrm{Re}(A_0(t_1)) = \mathrm{Re}(A_0(t_2)) = \mathrm{Re}(z)$$

and

$$\mathrm{Im}(A_0(t_1)) + 2\pi s = \mathrm{Im}(A_s(t_1)) < \mathrm{Im}(z) < \mathrm{Im}(A_{s+1}(t_2)) = \mathrm{Im}(A_0(t_2)) + 2\pi(s+1).$$

Moreover, since f has finite order, the Ahlfors Spiral Theorem (see [14]) implies that

$$\sup_{t\in[c_0,\infty)} \frac{|\operatorname{Im}(A_0(t))|}{\operatorname{Re}(A_0(t))} = a < \infty.$$

This together with (5) gives

$$\begin{aligned}|\operatorname{Im}(z) - 2\pi s| &\le \max(|\operatorname{Im}(A_0(t_1))|, |\operatorname{Im}(A_0(t_2))| + 2\pi)\\ &\le a\operatorname{Re}(z) + 2\pi < \left(a + \frac{2\pi}{c_0}\right)\operatorname{Re}(z)\end{aligned}$$

for every $z \in \overline{L_s^r}$. Hence, for every $z_1, z_2 \in \overline{L_s^r}$ we get

$$\begin{aligned}|z_1 - z_2| &\le |\operatorname{Re}(z_1) - \operatorname{Re}(z_2)| + |\operatorname{Im}(z_1) - 2\pi s| + |\operatorname{Im}(z_2) - 2\pi s|\\ &< \left(1 + 2\left(a + \frac{2\pi}{c_0}\right)\right)\max(\operatorname{Re}(z_1), \operatorname{Re}(z_2)).\end{aligned}$$

□

Note that, geometrically, Lemma 2 means that each tract L_s^r is contained in a cone in H of a fixed angle, with vertex at $2\pi is$.

The next fact follows easily from (1), the univalency of f on logarithmic tracts and the fact that L_s^r does not contain vertical segments of length 2π (see [4] for details). It shows that F is uniformly expanding (note that, unlike the rational case, hyperbolicity does not imply expanding for a general entire map).

Lemma 3 *For every logarithmic tract L_s^r and every $z_1, z_2 \in L_s^r$,*

$$\varrho_H(F(z_1), F(z_2)) \ge \frac{1}{2\pi}|z_1 - z_2|, \tag{6}$$

$$\varrho_H(F(z_1), F(z_2)) \ge Q\varrho_H(z_1, z_2), \tag{7}$$

where $Q > 1$ is a constant independent on r, s, z_1, z_2.

The following lemma will be one of the main tools used in proving the results of the paper.

Lemma 4 *Let $d, \delta > 0$. Then there exists $C > 0$, such that for every logarithmic tract L_s^r and every $z_1, z_2 \in L_s^r$, if the following conditions are satisfied:*

- $\varrho_H(z_1, z_2) > C$,
- $|F(z_1) - F(z_2)| < d\max(\operatorname{Re}(F(z_1)), \operatorname{Re}(F(z_2)))$,
- $\operatorname{Re}(F(z_1)), \operatorname{Re}(F(z_2)) \ge \delta$,

then $\operatorname{Re}(z_1) \ne \operatorname{Re}(z_2)$, $\operatorname{Re}(F(z_1)) \ne \operatorname{Re}(F(z_2))$ *and*

$$\operatorname{Re}(z_1) > \operatorname{Re}(z_2) \iff \operatorname{Re}(F(z_1)) > \operatorname{Re}(F(z_2)).$$

Proof Consider z_1, z_2 satisfying the assumptions for a large constant C. By symmetry, we can assume $\operatorname{Re}(z_1) \ge \operatorname{Re}(z_2)$. By Lemma 2, we can use Lemma 1 for z_1, z_2, which implies $\operatorname{Re}(z_1) \ne \operatorname{Re}(z_2)$ and, moreover,

$$\operatorname{Re}(z_1) > M\operatorname{Re}(z_2)$$

for $M = M(C)$. This together with (6) and (1) gives

$$(M-1)\operatorname{Re}(z_2) < \operatorname{Re}(z_1) - \operatorname{Re}(z_2) \le |z_1 - z_2| \le 2\pi\varrho_H(F(z_1), F(z_2))$$
$$= 4\pi \operatorname{asinh} \frac{|F(z_1) - F(z_2)|}{2\sqrt{\operatorname{Re}(F(z_1))\operatorname{Re}(F(z_2))}} \le 4\pi \ln \frac{2|F(z_1) - F(z_2)|}{\sqrt{\operatorname{Re}(F(z_1))\operatorname{Re}(F(z_2))}} \tag{8}$$

(the latter estimation is due to $\varrho_H(F(z_1), F(z_2)) > C$, which holds by (7)).

Suppose $\operatorname{Re}(F(z_1)) \le \operatorname{Re}(F(z_2))$. Then, by (8) and the assumptions,

$$(M-1)\operatorname{Re}(z_2) < 4\pi \ln \frac{2d\operatorname{Re}(F(z_2))}{\sqrt{\operatorname{Re}(F(z_1))\operatorname{Re}(F(z_2))}} \le 4\pi \ln \frac{2d\operatorname{Re}(F(z_2))}{\delta}. \tag{9}$$

The finite order assumption implies easily that

$$\operatorname{Re}(F(z)) < e^{A\operatorname{Re}(z)}$$

for every $z \in L_s^r$ for some constant $A > 0$ independent of r, s, z. This together with (5) and (9) implies

$$(M-1)\operatorname{Re}(z_2) < 4\pi \ln \frac{2d}{\delta} + 4\pi A \operatorname{Re}(z_2) < \left(\frac{4\pi}{c_0} \ln \frac{2d}{\delta} + 4\pi A\right) \operatorname{Re}(z_2),$$

which gives

$$M < 1 + \frac{4\pi}{c_0} \ln \frac{2d}{\delta} + 4\pi A.$$

By Lemma 1, $M = M(C)$ is arbitrarily large for sufficiently large C, so we get a contradiction. This shows $\operatorname{Re}(F(z_1)) > \operatorname{Re}(F(z_2))$, which ends the proof. □

4 Symbolic dynamics and coding trees

Since for each r the map $f|_{T^r}$ is a cover of $\mathbb{C} \setminus \overline{D}$ of infinite degree, the set $T^r \setminus f^{-1}(\alpha)$ is the union of infinitely many disjoint simply connected domains T_s^r, $s \in \mathbb{Z}$, such that f maps T_s^r univalently onto $\mathbb{C} \setminus (\overline{D} \cup \alpha)$. The domains T_s^r are called *the fundamental domains* of f. Enumerate T_s^r such that

$$F(\log_s(\Phi^{-1}(T_{s_0}^r))) = H_{s_0}$$

for every $r \in \mathcal{R}$, $s, s_0 \in \mathbb{Z}$. (Then T_s^r are situated according to the order of $s \in \mathbb{Z}$.) Let

$$g_s^r = (f|_{T_s^r})^{-1} : \mathbb{C} \setminus (\overline{D} \cup \alpha) \to T_s^r$$

and for $\underline{r} = (r_0, r_1, \dots) \in \mathcal{R}^\infty$, $\underline{s} = (s_0, s_1, \dots) \in \mathbb{Z}^\infty$, $n \ge 1$ define

$$T_{s_0 \cdots s_n}^{r_0 \cdots r_n} = g_{s_0}^{r_0} \circ \cdots \circ g_{s_{n-1}}^{r_{n-1}}(T_{s_n}^{r_n}), \quad T_{\underline{s}}^{\underline{r}} = \bigcap_{n=0}^{\infty} \overline{T_{s_0 \cdots s_n}^{r_0 \cdots r_n}}.$$

By definition, $\overline{T_{s_0 \cdots s_n}^{r_0 \cdots r_n}} \subset T_{s_0 \cdots s_{n-1}}^{r_0 \cdots r_{n-1}}$, which implies that $T_{\underline{s}}^{\underline{r}} \cup \{\infty\}$ is a continuum. Moreover,

$$T_{s_0 \cdots s_n}^{r_0 \cdots r_n} = \{z \in \mathbb{C} : f^k(z) \in T_{s_k}^{r_k} \text{ for every } k = 0, \dots, n\}. \tag{10}$$

We say that a point $z \in \mathbb{C}$ has *itinerary* $(\underline{r}, \underline{s})$ under f, if $f^n(z) \in T_{s_n}^{r_n}$ for every $n \geq 0$. By (10),

$$T_{\underline{s}}^{\underline{r}} = \{z \in \mathbb{C} : z \text{ has itinerary } (\underline{r}, \underline{s})\}. \tag{11}$$

In particular, $f(T_{\underline{s}}^{\underline{r}}) = T_{\sigma(\underline{s})}^{\sigma(\underline{r})}$, where σ is the left-side shift on $\mathcal{R}^\infty$ or $\mathbb{Z}^\infty$, i.e. $\sigma(t_0, t_1, \ldots) = (t_1, t_2, \ldots)$. Moreover, since B is the only Fatou component of f, we have

$$J(f) = \bigcup_{(\underline{r}, \underline{s}) \in \mathcal{R}^\infty \times \mathbb{Z}^\infty} T_{\underline{s}}^{\underline{r}}.$$

Let

$$G_s^r = (F|_{L_s^r})^{-1} : H \to L_s^r$$

and for $n \geq 0$ define

$$\begin{aligned} L_{ss_0 \cdots s_n}^{r_0 \cdots r_n} &= G_s^{r_0} \circ \cdots \circ G_{s_{n-1}}^{r_n}(H_{s_n}) = \log_s(\Phi^{-1}(T_{s_0 \cdots s_n}^{r_0 \cdots r_n})), \\ L_{s\underline{s}}^{\underline{r}} &= \bigcap_{n=0}^{\infty} L_{ss_0 \cdots s_n}^{r_0 \cdots r_n} = \log_s(\Phi^{-1}(T_{\underline{s}}^{\underline{r}})), \end{aligned}$$

(here and in the sequel $s_{-1} = s$ and $s\underline{s} = (s, s_0, s_1, \ldots)$).

Trees of preimages

Take a point

$$\zeta \in B \setminus \overline{D}.$$

Since all its first preimages $w \in f^{-1}(\zeta)$ are contained in $B \setminus \overline{D}$, one can connect ζ to w by a curve $\gamma_w \subset B \setminus \overline{D}$, such that all branches of $f^{-n}, n \geq 0$ are defined on γ_w. A *(geometric) coding tree* $\mathcal{G}$ of preimages of ζ in $B \setminus \overline{D}$ is the family of the images of γ_w, $w \in f^{-1}(\zeta)$ under all branches of f^{-n}, $n \geq 0$. To define the codes of the branches, assume that

$$\zeta \notin \alpha$$

(if $\zeta \in \alpha$, we can perturb a little α near ζ to avoid it, which does not change the codes of the tracts). Then $f^{-1}(\zeta) = \{\zeta_s^r : r \in \mathcal{R}, s \in \mathbb{Z}\}$, where

$$\zeta_s^r = g_s^r(\zeta)$$

is the unique point of $f^{-1}(\zeta)$ in T_s^r. Denote by γ_s^r the unique curve from $\mathcal{G}$ connecting ζ to ζ_s^r. See Fig. 1.

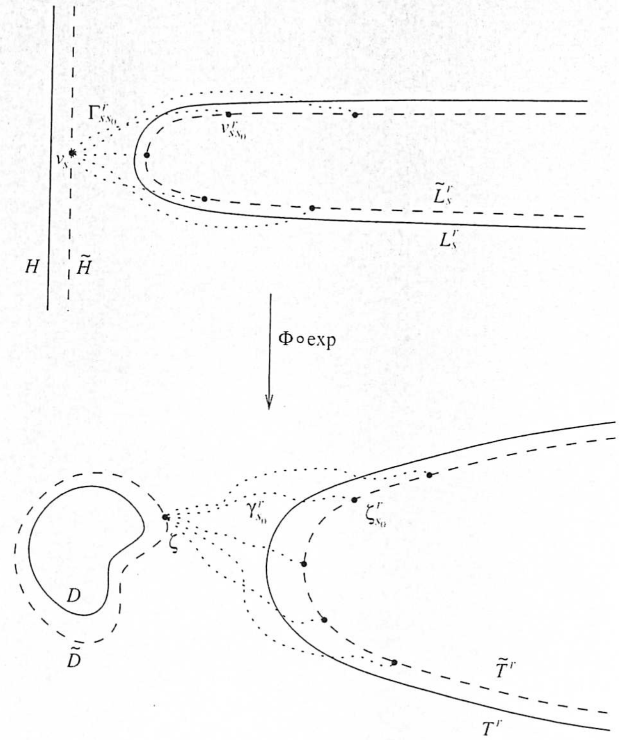
Similarly, for $n \geq 1$, $f^{-n}(\zeta) = \{\zeta_{s_0 \cdots s_n}^{r_0 \cdots r_n} : r_0, \ldots, r_n \in \mathcal{R}, s_0, \ldots, s_n \in \mathbb{Z}\}$, where

$$\zeta_{s_0 \cdots s_n}^{r_0 \cdots r_n} = g_{s_0}^{r_0} \circ \cdots \circ g_{s_{n-1}}^{r_{n-1}}(\zeta_{s_n}^{r_n})$$

is the unique point of $f^{-n}(\zeta)$ in $T_{s_0 \cdots s_n}^{r_0 \cdots r_n}$. Let $\gamma_{s_0 \cdots s_n}^{r_0 \cdots r_n}$ be the unique curve from $\mathcal{G}$ connecting $\zeta_{s_0 \cdots s_{n-1}}^{r_0 \cdots r_{n-1}}$ to $\zeta_{s_0 \cdots s_n}^{r_0 \cdots r_n}$. By definition,

$$f(\gamma_{s_0 \cdots s_n}^{r_0 \cdots r_n}(t)) = \gamma_{s_0 \cdots s_{n-1}}^{r_0 \cdots r_{n-1}}(t)$$

Fig. 1 The curves γ_s^r



for $n \geq 1$, $t \in [0,1]$. In particular, $f(\zeta_{s_0 \cdots s_n}^{r_0 \cdots r_n}) = \zeta_{s_0 \cdots s_{n-1}}^{r_0 \cdots r_{n-1}}$. Moreover, since T^r are components of $f^{-1}(\mathbb{C} \setminus \overline{D})$, we have

$$\gamma_{s_0 \cdots s_n}^{r_0 \cdots r_n} \subset T^{r_0} \subset \bigcup_{s_0' \in \mathbb{Z}} \overline{T_{s_0'}^{r_0}}$$

for $n \geq 1$, which implies

$$\gamma_{s_0 \cdots s_n}^{r_0 \cdots r_n} = g_{s_0}^{r_0} \circ \cdots \circ g_{s_{n-2}}^{r_{n-2}}(\gamma_{s_{n-1} s_n}^{r_{n-1} r_n}) \subset \bigcup_{s_{n-1}' \in \mathbb{Z}} \overline{T_{s_0 \cdots s_{n-2} s_{n-1}'}^{r_0 \cdots r_{n-1}}} \tag{12}$$

for $n \geq 2$.

A (geometric) branch of the tree $\mathcal{G}$ is a curve

$$\gamma_{\underline{s}}^{\underline{r}} : [0, \infty) \to B,$$

where

$$\gamma_{\underline{s}}^{\underline{r}}(t) = \gamma_{s_0 \cdots s_n}^{r_0 \cdots r_n}(t - n), \quad t \in [n, n+1), \ n = 0, 1, \ldots$$

for $\underline{r} = (r_0, r_1, \ldots) \in \mathcal{R}^\infty$, $\underline{s} = (s_0, s_1, \ldots) \in \mathbb{Z}^\infty$. Each branch $\gamma_{\underline{s}}^{\underline{r}}$ is a curve starting at ζ for $t = 0$, with limit points in $T_{\underline{s}}^{\underline{r}} \cup \{\infty\}$ as $t \to \infty$. We say that $\gamma_{\underline{s}}^{\underline{r}}$ converges to a point $z \in \partial B \cup \{\infty\}$, if $\gamma_{\underline{s}}^{\underline{r}}$ lands at z. Note that

$$\zeta_{s_0 \cdots s_n}^{r_0 \cdots r_n} = \gamma_{s_0 \cdots s_n}^{r_0 \cdots r_n}(1) = \gamma_{\underline{s}}^{\underline{r}}(n+1),$$

and

$$f(\gamma_{\underline{s}}^{\underline{r}}(t)) = \gamma_{\sigma(\underline{s})}^{\sigma(\underline{r})}(t-1)$$

for $t \in [1, \infty)$.

5 Convergence of the branches

In this section, we prove

Theorem A *For every $\zeta \in B \setminus \overline{D}$ there exists a coding tree $\mathcal{G}$ of preimages of ζ in $B \setminus \overline{D}$ with all branches convergent to points from $\widehat{\mathbb{C}}$. More precisely, for every $\underline{r} = (r_0, r_1, \ldots) \in \mathcal{R}^\infty$, $\underline{s} = (s_0, s_1, \ldots) \in \mathbb{Z}^\infty$, there exists a point $\zeta_{\underline{s}}^{\underline{r}} \in T_{\underline{s}}^{\underline{r}} \cup \{\infty\}$ such that the branch $\gamma_{\underline{s}}^{\underline{r}}$ of $\mathcal{G}$ converges to $\zeta_{\underline{s}}^{\underline{r}}$. In particular, $\zeta_{s_0 \cdots s_n}^{r_0 \cdots r_n} \to \zeta_{\underline{s}}^{\underline{r}}$ as $n \to \infty$. We have $\zeta_{\underline{s}}^{\underline{r}} = \infty$ if and only if $T_{\underline{s}}^{\underline{r}} = \emptyset$. If $\zeta_{\underline{s}}^{\underline{r}}, \zeta_{\underline{s}'}^{\underline{r}'} \neq \infty$, then $\zeta_{\underline{s}}^{\underline{r}} = \zeta_{\underline{s}'}^{\underline{r}'}$ if and only if $(\underline{r}, \underline{s}) = (\underline{r}', \underline{s}')$. Moreover, if $\zeta_{\underline{s}}^{\underline{r}} \neq \infty$, then it is the unique point from $T_{\underline{s}}^{\underline{r}}$, which is accessible from B.*

Remark In [4] it is proved that if $|s_n| < E_\lambda^n(x)$ for sufficiently large n and any $\lambda, x > 0$ (where $E_\lambda(x) = \lambda \exp(x)$), then $\zeta_{\underline{s}}^{\underline{r}} \neq \infty$. Moreover, there exists a sequence l_n, such that if $|s_n| > l_n$ for sufficiently large n, then $\zeta_{\underline{s}}^{\underline{r}} = \infty$.

Remark There exist coding trees of preimages of ζ in $B \setminus \overline{D}$, for which not all branches are convergent. Indeed, it is easy to see that if the curves γ_s^1 are chosen such that they contain points z_s, such that z_s tends to ∞ sufficiently fast for $s \to \infty$, then $g_0^1 \circ g_1^1 \circ \cdots \circ g_s^1(z_s) \to \infty$ for $s \to \infty$, so the branch $\gamma_{012\ldots}^{111\cdots}$ cannot converge to a finite point. On the other hand, $\gamma_{012\ldots}^{111\cdots}(s+1) = \zeta_{012\cdots s}^{1\cdots 1}$ converges to a point from $\mathbb{C}$ by the previous remark.

Theorem A implies the following result on the boundary behaviour of a Riemann map onto B.

Theorem B *A Riemann map from $\mathbb{D}$ onto B has radial limits at all points of $\partial\mathbb{D}$.*

Proof Let φ be the Riemann map. In [4] it was showed that there exists a countable set $\Theta_\infty \subset [0, 2\pi]$, such that φ has unrestricted limit equal to ∞ at all points $e^{i\theta}$, $\theta \in \Theta_\infty$, and for every $\theta \in [0, 2\pi] \setminus \Theta_\infty$, the point $e^{i\theta}$ is the landing point of the curve $\varphi^{-1}(\gamma_{\underline{s}}^{\underline{r}})$ for some $(\underline{r}, \underline{s}) \in \mathcal{R}^\infty \times \mathbb{Z}^\infty$. By Theorem A, the curve $\gamma_{\underline{s}}^{\underline{r}}$ converges to $\zeta_{\underline{s}}^{\underline{r}}$, so the Lindelöf Theorem implies that the radial limit of φ at $e^{i\theta}$ exists for $\theta \in [0, 2\pi] \setminus \Theta_\infty$ and is equal to the suitable $\zeta_{\underline{s}}^{\underline{r}}$. □

The proof of Theorem A consists of several steps. First we construct a suitable coding tree $\mathcal{G}$ and its lifts in the logarithmic coordinates. Then we prove two technical lemmas and show the convergence of the branches of the tree. Finally, we prove the additional facts stated in the theorem.

Definition of the curves γ_s^r.

Let

$$\tilde{D} = \mathbb{C} \setminus \Phi(\{z : |z| \geq 1 + \varepsilon\})$$

for a small $\varepsilon > 0$. Then $\tilde{D}$ is a bounded simply connected domain, $\overline{D} \subset \tilde{D} \subset \overline{\tilde{D}} \subset B$ and $\partial\tilde{D}$ is an analytic Jordan curve. Moreover, if ε is small enough, then $\overline{\tilde{D}}$ is disjoint from the closures of all tracts of f. Each tract T^r, $r \in \mathcal{R}$ contains the closure of a unique component $\tilde{T}^r$ of $f^{-1}(\mathbb{C} \setminus \overline{\tilde{D}})$, such that $\overline{\tilde{T}^r}$ is simply connected, $\partial\tilde{T}^r$ is an analytic open simple arc and $\partial\tilde{T}^r \cup \{\infty\}$ is a Jordan curve. Let

$$\tilde{H} = \{z : \mathrm{Re}(z) > \delta_0\} \tag{13}$$

and

$$\tilde{L}^r_s = G^r_s(\tilde{H}) = \log_s(\Phi^{-1}(\tilde{T}^r))$$

for $s \in \mathbb{Z}$. Choose a point $\zeta \in B \setminus \overline{D}$ such that

$$\zeta \in \partial\tilde{D} \setminus \alpha.$$

Note that in this case $\zeta^r_s \in \partial\tilde{T}^r$ for $s \in \mathbb{Z}$.

Let $\mathcal{G}^r_s$ for $r \in \mathcal{R}, s \in \mathbb{Z}$ be the class of all rectifiable simple arcs $\gamma : [0,1] \to B \setminus \overline{D}$, such that:

$$\gamma(0) = \zeta, \quad \gamma(1) = \zeta^r_s, \quad \gamma((0,1)) \subset B \Bigg\backslash \left(\overline{\tilde{D}} \cup \bigcup_{r' \in \mathcal{R}} \overline{\tilde{T}^{r'}}\right).$$

It is clear that $\mathcal{G}^r_s$ is non-empty. Choose curves

$$\gamma^r_s \in \mathcal{G}^r_s,$$

for $r \in \mathcal{R}, s \in \mathbb{Z}$, such that

$$\mathrm{length}(\gamma^r_s) < 1 + \inf_{\gamma \in \mathcal{G}^r_s} \mathrm{length}(\gamma). \tag{14}$$

See Fig. 1.

Define the tree $\mathcal{G}$ of preimages of ζ as in Sect. 4. Note that the definition of γ^r_s implies that for any $n < m$ and $t, t' \in [0,1]$, if $\gamma^{r_0\cdots r_n}_{s_0\cdots s_n}(t) = \gamma^{r'_0\cdots r'_m}_{s'_0\cdots s'_m}(t')$, then $m = n+1$, $r_0 = r'_0, s_0 = s'_0, \ldots, r_n = r'_n, s_n = s'_n, t = 1, t' = 0$. This shows that the branches $\gamma^{\underline{r}}_{\underline{s}}$ are homeomorphic to $[0, \infty)$.

To make use of the properties of the lifted map F, we will study the lifts of tree $\mathcal{G}$ in the logarithmic coordinates. Define

$$v_s = \log_s(\Phi^{-1}(\zeta))$$

for $s \in \mathbb{Z}$ and

$$v^{r_0\cdots r_n}_{ss_0\cdots s_n} = \log_s(\Phi^{-1}(\zeta^{r_0\cdots r_n}_{s_0\cdots s_n})) = G^{r_0}_s \circ \cdots \circ G^{r_n}_{s_{n-1}}(v_{s_n})$$

for $n \geq 0$ and $r_0, \ldots, r_n \in \mathcal{R}$, $s_0, \ldots, s_n \in \mathbb{Z}$. Since $\partial\tilde{D}$ is close to ∂D, we have

$$\mathrm{Re}(v_s) = \delta_0 < c_0 \tag{15}$$

for c_0 from (5). By (12), we can define

$$\Gamma^{r_0\cdots r_n}_{ss_0\cdots s_n}(t) = \log_s(\Phi^{-1}(\gamma^{r_0\cdots r_n}_{s_0\cdots s_n}(t)))$$

for $n \geq 1$, $t \in [0,1]$. Let $\Gamma^{r_0}_{ss_0}$ be the image of $\gamma^{r_0}_{s_0}$ under the inverse branch of $\Phi \circ \exp$, such that $\Gamma^{r_0}_{ss_0}(0) = v_s$. Then

$$F(\Gamma^{r_0 \cdots r_n}_{ss_0 \cdots s_n}(t)) = \Gamma^{r_0 \cdots r_{n-1}}_{ss_0 \cdots s_{n-1}}(t)$$

for $n \geq 1$. Moreover, by (12),

$$\Gamma^{r_0 \cdots r_n}_{ss_0 \cdots s_n} = G^{r_0}_s \circ \cdots \circ G^{r_{n-1}}_{s_{n-2}}(\Gamma^{r_n}_{s_{n-1}s_n}) \subset \bigcup_{s'_{n-1} \in \mathbb{Z}} \overline{L^{r_0 \cdots r_{n-1}}_{ss_0 \cdots s_{n-2}s'_{n-1}}}. \tag{16}$$

Let

$$\Gamma^{\underline{r}}_{s\underline{s}}(t) = \Gamma^{r_0 \cdots r_n}_{ss_0 \cdots s_n}(t-n), \quad t \in [n, n+1),\ n = 0, 1, \ldots$$

for $t \in [0,\infty)$. By (5), (15) and (16),

$$\mathrm{Re}(\Gamma^{\underline{r}}_{s\underline{s}}(t)) > \delta_0 \tag{17}$$

for every $t > 0$. The limit points of $\Gamma^{\underline{r}}_{s\underline{s}}(t)$ for $t \to \infty$ are contained in $L^{\underline{r}}_{s\underline{s}} \cup \{\infty\}$. Moreover,

$$v^{r_0 \cdots r_n}_{ss_0 \cdots s_n} = \Gamma^{r_0 \cdots r_n}_{ss_0 \cdots s_n}(1) = \Gamma^{\underline{r}}_{s\underline{s}}(n+1)$$

and

$$F(\Gamma^{\underline{r}}_{s\underline{s}}(t)) = \Gamma^{\sigma(\underline{r})}_{\underline{s}}(t)$$

for $t \in [1,\infty)$. Note that

$$\Gamma^{\underline{r}}_{s\underline{s}}([1,\infty)) = \log_s(\Phi^{-1}(\gamma^{\underline{r}}_{\underline{s}}([1,\infty)))) \subset L^{r_0}_s \tag{18}$$

and $\Phi \circ \exp$ is a homeomorphism on $\overline{L^{r_0}_s}$. Hence, $\gamma^{\underline{r}}_{\underline{s}}$ converges to a point from $T^{\underline{r}}_{\underline{s}}$ (resp. to ∞) if and only if $\Gamma^{\underline{r}}_{s\underline{s}}$ converges to a point from $L^{\underline{r}}_{s\underline{s}}$ (resp. to ∞).

The following lemma shows that the cone condition from Lemma 2 is fulfilled also for the branches $\Gamma^{\underline{r}}_{s\underline{s}}$.

Lemma 5 *There exists $d_1 > 0$, such that for every $s \in \mathbb{Z}$, $\underline{r} = (r_0, r_1, \ldots) \in \mathcal{R}^\infty$, $\underline{s} = (s_0, s_1, \ldots) \in \mathbb{Z}^\infty$ and every $z_1, z_2 \in \Gamma^{\underline{r}}_{s\underline{s}} \cup \overline{L^{r_0}_s}$,*

$$|z_1 - z_2| < d_1 \max(\mathrm{Re}(z_1), \mathrm{Re}(z_2)).$$

Proof Choose a point

$$w_0 \in \bigcup_{r' \in \mathcal{R}} \partial \tilde{L}^{r'}_0,$$

such that

$$\mathrm{Re}(w_0) = \inf\{\mathrm{Re}(z) : z \in \bigcup_{r' \in \mathcal{R}} \overline{\tilde{L}^{r'}_0}\}$$

Let

$$w_s = w_0 + 2\pi i s$$

for $s \in \mathbb{Z}$. By (5) and (15), we have

$$\mathrm{Re}(w_s) > \mathrm{Re}(v_s).$$

Since $|z_1 - z_2| \le |z_1 - w_s| + |z_2 - w_s|$, to prove the lemma it is sufficient to show that

$$|z - w_s| < \frac{d_1}{2} \operatorname{Re}(z) \tag{19}$$

for every $z \in \Gamma^{\underline{r}}_{\underline{s}\underline{s}} \cup \overline{L^{r_0}_s}$. By (18) and Lemma 2, we have (19) in the case $z \in \Gamma^{\underline{r}}_{\underline{s}\underline{s}}([1,\infty)) \cup \overline{L^{r_0}_s}$. Hence, we can assume

$$z \in \Gamma^{r_0}_{ss_0}([0,1)).$$

Since $\Gamma^{r_0}_{ss_0}([0,1))$ is disjoint from $\bigcup_{r' \in \mathcal{R}, s' \in \mathbb{Z}} \partial \tilde{L}^{r'}_{s'}$, there exists a curve

$$\tilde{A}_0 : [0,\infty) \to H_0,$$

homeomorphic to $[0,\infty)$, such that $\tilde{A}_0(0) = w_0$, $\operatorname{Re}(\tilde{A}_0(t)) > \operatorname{Re}(w_0)$ for every $t > 0$, $\tilde{A}_0(t) \to \infty$ as $t \to \infty$ and $\bigcup_{s' \in \mathbb{Z}} \tilde{A}_{s'}$ is disjoint from $\Gamma^{r_0}_{ss_0}([0,1))$ for

$$\tilde{A}_{s'} = \tilde{A}_0 + 2\pi i s'.$$

By definition, $\tilde{A}_{s'} \subset H_{s'}$ and $\tilde{A}_{s'}$ separates the half-plane $\{z : \operatorname{Re}(z) \ge \operatorname{Re}(w_0)\}$ into two components. Let

$$t_0 = \sup\{t \in [0,1] : \operatorname{Re}(\Gamma^{r_0}_{ss_0}(t)) \le \operatorname{Re}(w_s)\}.$$

Then

$$\Gamma^{r_0}_{ss_0}((t_0,1)) \subset \{z : \operatorname{Re}(z) > \operatorname{Re}(w_0)\} \Big\backslash \bigcup_{s' \in \mathbb{Z}} \tilde{A}_{s'}$$

and $\Gamma^{r_0}_{ss_0}(1) = v^{r_0}_{ss_0} \in H_s$. This easily implies that for every $z \in \Gamma^{r_0}_{ss_0}([t_0,1))$ there exist $t_1, t_2 > 0$, such that

$$\operatorname{Re}(\tilde{A}_s(t_1)) = \operatorname{Re}(\tilde{A}_s(t_2)) = \operatorname{Re}(z)$$

and

$$\operatorname{Im}(\tilde{A}_s(t_1)) - 2\pi = \operatorname{Im}(\tilde{A}_{s-1}(t_1)) < \operatorname{Im}(z) < \operatorname{Im}(\tilde{A}_{s+1}(t_2)) = \operatorname{Im}(\tilde{A}_s(t_2)) + 2\pi.$$

Hence, using Lemma 2 and (5), we get

$$\begin{aligned} |z - w_s| &\le \operatorname{Re}(z) - \operatorname{Re}(w_s) + |\operatorname{Im}(z) - \operatorname{Im}(w_s)| \\ &< \operatorname{Re}(z) + \max(|\operatorname{Im}(\tilde{A}_s(t_1)) - \operatorname{Im}(w_s)|, |\operatorname{Im}(\tilde{A}_s(t_2)) - \operatorname{Im}(w_s)|) + 2\pi \\ &< \left(1 + d_0 + \frac{2\pi}{c_0}\right) \operatorname{Re}(z). \end{aligned} \tag{20}$$

This proves (19) for $z \in \Gamma^{r_0}_{ss_0}([t_0,1))$.

It suffices to consider the remaining case

$$z \in \Gamma^{r_0}_{ss_0}([0,t_0)).$$

Modify the curve $\Gamma^{r_0}_{ss_0}$ to $\tilde{\Gamma}^{r_0}_{ss_0}$, replacing $\Gamma^{r_0}_{ss_0}([0,t_0])$ by the straight line segment connecting v_s to $\Gamma^{r_0}_{ss_0}(t_0)$ and let $\tilde{\gamma}^{r_0}_{s_0} = \Phi(\exp(\tilde{\Gamma}^{r_0}_{ss_0}))$. By the definition of w_s, we have $\tilde{\Gamma}^{r_0}_{ss_0}((0,1)) \subset \tilde{H} \setminus \bigcup_{r',s'} \overline{\tilde{L}^{r'}_{s'}}$ and $\tilde{\gamma}^{r_0}_{s_0}$ is a rectifiable simple arc, which implies $\tilde{\gamma}^{r_0}_{s_0} \in \mathcal{G}^{r_0}_{s_0}$. By (20) for $z = \Gamma^{r_0}_{ss_0}(t_0)$ we have

$$\operatorname{length}(\tilde{\Gamma}^{r_0}_{ss_0}([0,t_0])) = |\Gamma^{r_0}_{ss_0}(t_0) - v_s| < \left(1 + d_0 + \frac{2\pi}{c_0}\right) \operatorname{Re}(w_0) + |w_0 - v_0|,$$

so length$(\tilde{\gamma}_{s_0}^{r_0}([0,t_0]))$ is bounded by some constant independent of the choice of $\gamma_{s_0}^{r_0}$. On the other hand, if (19) does not hold for a sufficiently large d_1, then by (17), length$(\Gamma_{ss_0}^{r_0}([0,t_0]))$ and length$(\gamma_{s_0}^{r_0}([0,t_0]))$ are arbitrarily large, which contradicts (14). Hence, (19) holds in the case $z \in \Gamma_{ss_0}^{r_0}([0,t_0))$, which ends the proof. □

The following lemma forms the main step in the proof of Theorem A.

Lemma 6 *There exists $c_1 > 0$, such that for every $s \in \mathbb{Z}$, $\underline{r} = (r_0, r_1, \ldots) \in \mathcal{R}^\infty$, $\underline{s} = (s_0, s_1, \ldots) \in \mathbb{Z}^\infty$,*

- $\mathrm{Re}(v_{ss_0\cdots s_n}^{r_0\cdots r_n}) < c_1 \mathrm{Re}(z)$ *for every $n \ge 0$ and every $z \in L_{ss_0\cdots s_{n+1}}^{r_0\cdots r_{n+1}}$,*
- $\mathrm{Re}(\Gamma_{s\underline{s}}^{\underline{r}}(t_1)) < c_1 \mathrm{Re}(\Gamma_{s\underline{s}}^{\underline{r}}(t_2))$ *for every $0 \le t_1 < t_2$.*

In particular, $\mathrm{Re}(v_{ss_0\cdots s_n}^{r_0\cdots r_n}) < c_1 \mathrm{Re}(v_{ss_0\cdots s_m}^{r_0\cdots r_m})$ *for every $n < m$.*

Proof The proof is split into several cases. First, take

$$z \in L_{ss_0\cdots s_{n+1}}^{r_0\cdots r_{n+1}} \cup \Gamma_{s\underline{s}}^{\underline{r}}((n+1,\infty))$$

for some $n \ge 0$ and suppose that

$$\mathrm{Re}(v_{ss_0\cdots s_n}^{r_0\cdots r_n}) \ge c_1 \mathrm{Re}(z) \tag{21}$$

for a large constant $c_1 > 0$. Let

$$z_1^{(j)} = F^j(v_{ss_0\cdots s_n}^{r_0\cdots r_n}), \quad z_2^{(j)} = F^j(z)$$

for $j \in \{0, \ldots, n+1\}$. Note that by definition and (16), $z_1^{(j)}, z_2^{(j)} \in L_{s_{j-1}}^{r_j}$ for $j \le n$. Moreover, $z_1^{(n+1)} = v_{s_n} = \Gamma_{s\underline{s}}^{\underline{r}}(0)$ and $z_2^{(n+1)} \in \Gamma_{s_n s_{n+1}}^{r_{n+1}} \cup L_{s_n s_{n+1}}^{r_{n+1}}$. Therefore, by Lemma 5,

$$|z_1^{(j)} - z_2^{(j)}| < d_1 \max(\mathrm{Re}(z_1^{(j)}), \mathrm{Re}(z_2^{(j)})). \tag{22}$$

for $j \le n+1$. Moreover, by (5), (15) and (17),

$$\mathrm{Re}(z_1^{(j)}), \mathrm{Re}(z_2^{(j)}) \ge \delta_0. \tag{23}$$

Let C be the constant from Lemma 4 for $d = d_1$ and $\delta = \delta_0$. By (1) and (21), taking sufficiently large c_1 we can assume

$$\varrho_H(z_1^{(0)}, z_2^{(0)}) = \varrho_H(v_{ss_0\cdots s_n}^{r_0\cdots r_n}, z) > C,$$

so by (7),

$$\varrho_H(z_1^{(j)}, z_2^{(j)}) > C$$

for $j \le n$. This together with (22) and (23) implies that the assumptions of Lemma 4 are fulfilled for $\delta = \delta_0$ and $z_1 = z_1^{(j)}$, $z_2 = z_2^{(j)}$, $j = 0, \ldots, n$. Since by (21),

$$\mathrm{Re}(z_1^{(0)}) = \mathrm{Re}(v_{ss_0\cdots s_n}^{r_0\cdots r_n}) > \mathrm{Re}(z) = \mathrm{Re}(z_2^{(0)}),$$

Lemma 4 gives

$$\delta_0 = \mathrm{Re}(v_{s_n}) = \mathrm{Re}(z_1^{(n+1)}) > \mathrm{Re}(z_2^{(n+1)}) \ge \delta_0,$$

which is a contradiction. Hence,

$$\mathrm{Re}(v_{ss_0\cdots s_n}^{r_0\cdots r_n}) < c_1 \mathrm{Re}(z) \tag{24}$$

for every $n \geq 0$ and $z \in L^{r_0 \cdots r_{n+1}}_{ss_0 \cdots s_{n+1}} \cup \Gamma^{\underline{r}}_{s\underline{s}}((n+1, \infty))$. In particular, this gives the first assertion of the lemma.

Take now $0 \leq t_1 < t_2$ and suppose that

$$\operatorname{Re}(\Gamma^{\underline{r}}_{s\underline{s}}(t_1)) \geq \tilde{c}_1 \operatorname{Re}(\Gamma^{\underline{r}}_{s\underline{s}}(t_2)) \tag{25}$$

for a large $\tilde{c}_1$. Let m be the smallest integer not smaller than t_1. Note that $m = 0$ implies $t_1 = 0$, which is impossible due to (15) and (17). Hence, we have $m > 0$. Let

$$t_2' = \inf\{t \in (t_1, t_2) : \operatorname{Re}(\Gamma^{\underline{r}}_{s\underline{s}}(t)) = \operatorname{Re}(\Gamma^{\underline{r}}_{s\underline{s}}(t_1))/\sqrt{\tilde{c}_1}\}.$$

If $m \leq t_2'$, then by (25),

$$\operatorname{Re}(v^{r_0 \cdots r_{m-1}}_{ss_0 \cdots s_{m-1}}) = \operatorname{Re}(\Gamma^{\underline{r}}_{s\underline{s}}(m)) \geq \operatorname{Re}(\Gamma^{\underline{r}}_{s\underline{s}}(t_1))/\sqrt{\tilde{c}_1} \geq \sqrt{\tilde{c}_1}\operatorname{Re}(\Gamma^{\underline{r}}_{s\underline{s}}(t_2)),$$

so for sufficiently large $\tilde{c}_1$ we have a contradiction with (24). Therefore, we can assume $m > t_2'$, which gives

$$[t_1, t_2'] \subset [m-1, m).$$

Let

$$z_1^{(j)} = F^j(\Gamma^{\underline{r}}_{s\underline{s}}(t_1)), \quad z_2^{(j)} = F^j(\Gamma^{\underline{r}}_{s\underline{s}}(t_2'))$$

for $j \in \{0, \ldots, m-1\}$. Then

$$\operatorname{Re}(z_1^{(0)}) = \sqrt{\tilde{c}_1}\operatorname{Re}(z_2^{(0)}) > \operatorname{Re}(z_2^{(0)}).$$

Repeating the proof of (24), by Lemma 4 we get $\operatorname{Re}(z_1^{(m-1)}) > \operatorname{Re}(z_2^{(m-1)})$. Moreover, (1), (7) and (25) imply that $\varrho_H(z_1^{(m-1)}, z_2^{(m-1)})$ is arbitrarily large, if $\tilde{c}_1$ is large enough, so by Lemmas 1 and 5 we conclude

$$\operatorname{Re}(z_1^{(m-1)}) > M \operatorname{Re}(z_2^{(m-1)}) \tag{26}$$

for an arbitrarily large $M > 0$. Note that

$$z_1^{(m-1)} = \Gamma^{r_{m-1}}_{s_{m-2}s_{m-1}}(t_1 - m + 1), \quad z_2^{(m-1)} = \Gamma^{r_{m-1}}_{s_{m-2}s_{m-1}}(t_2' - m + 1),$$

where $t_1 - m + 1 < t_2' - m + 1$. For simplicity, write $\Gamma = \Gamma^{r_{m-1}}_{s_{m-2}s_{m-1}}$. Take $\tau_0 \in [0, t_2' - m + 1]$ such that

$$\operatorname{Re}(\Gamma(\tau_0)) = \sup\{\operatorname{Re}(\Gamma(t)) : t \in [0, t_2' - m + 1]\}$$

and let

$$\tau_1 = \sup\{t \in (0, \tau_0) : \operatorname{Re}(\Gamma(t)) = \operatorname{Re}(\Gamma(\tau_0))/M\},$$
$$\hat{\tau}_1 = \inf\{t \in (\tau_0, \infty) : \operatorname{Re}(\Gamma(t)) = \operatorname{Re}(\Gamma(\tau_0))/M\}.$$

By (26), $\Gamma((\tau_1, \tau_0))$, $\Gamma((\tau_0, \hat{\tau}_1))$ are disjoint open simple arcs intersecting the line

$$\ell = \{z : \operatorname{Re}(z) = \operatorname{Re}(\Gamma(\tau_0))/\sqrt{M}\}.$$

Assume $\sup\{\operatorname{Im}(z) : z \in \Gamma((\tau_1, \tau_0)) \cap \ell\} < \sup\{\operatorname{Im}(z) : z \in \Gamma((\tau_0, \hat{\tau}_1)) \cap \ell\}$ (the other case can be treated symmetrically) and take $\tau_2, \tau_3 \in (\tau_1, \tau_0)$, $\hat{\tau}_2, \hat{\tau}_3 \in (\tau_0, \hat{\tau}_1)$, such that

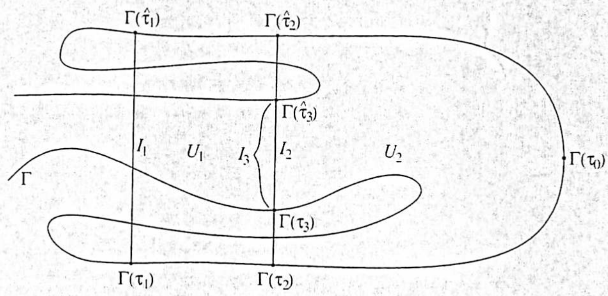


Fig. 2 The domains U_1, U_2

$\Gamma(\tau_2), \Gamma(\tau_3), \Gamma(\hat{\tau}_2), \Gamma(\hat{\tau}_3) \in \ell$ and

$$\begin{aligned}
\operatorname{Im}(\Gamma(\tau_2)) &= \sup\{\operatorname{Im}(z) : z \in \Gamma((\tau_1, \tau_0)) \cap \ell\},\\
\operatorname{Im}(\Gamma(\hat{\tau}_2)) &= \inf\{\operatorname{Im}(z) : z \in \Gamma((\tau_0, \hat{\tau}_1)) \cap \ell,\ \operatorname{Im}(z) > \operatorname{Im}(\Gamma(\tau_2))\},\\
\operatorname{Im}(\Gamma(\tau_3)) &= \sup\{\operatorname{Im}(z) : z \in \Gamma((0, \tau_0)) \cap \ell,\ \operatorname{Im}(z) \in [\operatorname{Im}(\Gamma(\tau_2)), \operatorname{Im}(\Gamma(\hat{\tau}_2))]\},\\
\operatorname{Im}(\Gamma(\hat{\tau}_3)) &= \inf\{\operatorname{Im}(z) : z \in \Gamma((\tau_0, \infty)) \cap \ell,\ \operatorname{Im}(z) \in [\operatorname{Im}(\Gamma(\tau_3)), \operatorname{Im}(\Gamma(\hat{\tau}_2))]\}.
\end{aligned}$$

See Fig. 2.

By (26), $\tau_2, \tau_3, \hat{\tau}_2, \hat{\tau}_3$ are well defined and

$$\begin{gathered}
\operatorname{Re}(\Gamma(\tau_1)) = \operatorname{Re}(\Gamma(\hat{\tau}_1)), \quad \operatorname{Re}(\Gamma(\tau_2)) = \operatorname{Re}(\Gamma(\tau_3)) = \operatorname{Re}(\Gamma(\hat{\tau}_2)) = \operatorname{Re}(\Gamma(\hat{\tau}_3)),\\
\operatorname{Re}(\Gamma(\tau_0)) = M \operatorname{Re}(\Gamma(\tau_1)) = \sqrt{M} \operatorname{Re}(\Gamma(\tau_2)).
\end{gathered} \tag{27}$$

Let I_j for $j = 1, 2, 3$ be the vertical segment without endpoints connecting $\Gamma(\tau_j)$ to $\Gamma(\hat{\tau}_j)$. Note that

$$I_3 \subset I_2 \quad \text{and} \quad I_3 \cap \Gamma = \emptyset. \tag{28}$$

Since Γ is a simple arc, by the definition of $\tau_j, \hat{\tau}_j$, the sets $I_1 \cup \Gamma([\tau_1, \tau_2]) \cup I_2 \cup \Gamma([\hat{\tau}_2, \hat{\tau}_1])$, $I_2 \cup \Gamma([\tau_2, \hat{\tau}_2])$, $I_1 \cup \Gamma([\tau_1, \hat{\tau}_1])$ are three Jordan curves. Let U_1, U_2, U be, respectively, the bounded domains cut out by these curves. We have $\partial U_1 \cap \partial U_2 = I_2$, so one the three possibilities holds: $\overline{U_2} \subset U_1 \cup \overline{I_2}$, $\overline{U_1} \subset U_2 \cup \overline{I_2}$ or $U_1 \cap U_2 = \emptyset$. By the definition of τ_0 and the fact that U_1 is bounded, we have $\overline{U_1} \subset \{z : \operatorname{Re}(z) \leq \operatorname{Re}(\Gamma(\tau_0))\}$, so $\Gamma(\tau_0) \notin U_1$ and the first possibility cannot hold. Similarly, $\overline{U_2} \subset \{z : \operatorname{Re}(z) \geq \operatorname{Re}(\Gamma(\tau_1))\}$, so $I_1 \cap U_2 = \emptyset$, which excludes the second possibility. Hence, we have $U_1 \cap U_2 = \emptyset$, so

$$U = U_1 \cup I_2 \cup U_2. \tag{29}$$

See Fig. 2.

Consider first the case, when there exists a point

$$w \in I_3 \cap \partial \tilde{L}_{\tilde{s}}^{\tilde{r}_0}$$

for some $\tilde{r}_0 \in \mathcal{R}$, $\tilde{s} \in \mathbb{Z}$. Since $\operatorname{Re}(F(w)) = \delta_0$, there exists $\tilde{s}_0 \in \mathbb{Z}$, such that $|F(w) - v_{\tilde{s}_0}| \leq \pi$. By (1) and (7), $\varrho_H(w, v_{\tilde{s}\tilde{s}_0}^{\tilde{r}_0}) < 2 \operatorname{asinh}(\pi/2\delta_0)$, so $\operatorname{Re}(v_{\tilde{s}\tilde{s}_0}^{\tilde{r}_0}) > \tilde{c} \operatorname{Re}(w)$ for some $\tilde{c} > 0$ (independent of M). Since by (27), $\operatorname{Re}(w) = \operatorname{Re}(\Gamma(\tau_3)) = \operatorname{Re}(\Gamma(\tau_0))/\sqrt{M} > \sqrt{M} \operatorname{Re}(\Gamma(\tau_1))$, this implies

$$\operatorname{Re}(v_{\tilde{s}\tilde{s}_0}^{\tilde{r}_0}) > \tilde{c}\sqrt{M} \operatorname{Re}(\Gamma(\tau_1)). \tag{30}$$

Moreover, $v_{\tilde{s}\tilde{s}_0}^{\tilde{r}_0} \in U$, because $w \in U$ by (28) and (29), and $\partial \tilde{L}_{\tilde{s}}^{\tilde{r}_0}$ is disjoint from $\Gamma([\tau_1, \hat{\tau}_1])$.

It is easy to check that there exist $\tilde{r}_1, \tilde{r}_2, \ldots \in \mathcal{R}$, $\tilde{s}_1, \tilde{s}_2, \ldots \in \mathbb{Z}$, such that $\Gamma_{\underline{s}\tilde{\underline{s}}}^{\tilde{\underline{r}}}$ converges to ∞ for

$$\tilde{\underline{r}} = (\tilde{r}_0, \tilde{r}_1, \ldots), \quad \tilde{\underline{s}} = (\tilde{s}_0, \tilde{s}_1, \ldots)$$

(it is enough to take $\tilde{s}_n$ growing sufficiently fast, see the remark after Theorem A in this paper and Theorem B from [4]). Since $\Gamma_{\underline{s}\tilde{\underline{s}}}^{\tilde{\underline{r}}}(1) = v_{\underline{s}\tilde{s}_0}^{\tilde{r}_0} \in U$, there exists $t_0 > 1$, such that $\Gamma_{\underline{s}\tilde{\underline{s}}}^{\tilde{\underline{r}}}(t_0) \in \partial U$. On the other hand, $\Gamma_{\underline{s}\tilde{\underline{s}}}^{\tilde{\underline{r}}}((1,\infty))$ is disjoint from Γ, so $\Gamma_{\underline{s}\tilde{\underline{s}}}^{\tilde{\underline{r}}}(t_0) \in I_1$, which together with (30) implies

$$\mathrm{Re}(v_{\underline{s}\tilde{s}_0}^{\tilde{r}_0}) > \tilde{c}\sqrt{M}\,\mathrm{Re}(\Gamma_{\underline{s}\tilde{\underline{s}}}^{\tilde{\underline{r}}}(t_0)).$$

This contradicts (24) for sufficiently large M.

It remains to consider the remaining case

$$I_3 \cap \bigcup_{r' \in \mathcal{R}, s' \in \mathbb{Z}} \partial \tilde{L}_{s'}^{r'} = \emptyset. \tag{31}$$

Modify the curve Γ to $\tilde{\Gamma}$, replacing $\Gamma((\tau_3, \hat{\tau}_3))$ by I_3 and let $\tilde{\gamma} = \Phi(\exp(\tilde{\Gamma}))$. Note that by (28), the growth of argument on the curve $\exp(\Gamma((\tau_3, \hat{\tau}_3)))$ is less than 2π, so $\mathrm{length}(I_3) < 2\pi$, and $\tilde{\gamma}$ is a rectifiable simple arc. Hence, by (31), $\tilde{\gamma} \in \mathcal{G}_{s_{m-1}}^{r_{m-1}}$. Moreover, $\mathrm{length}(\tilde{\gamma}((\tau_3, \hat{\tau}_3))) < c e^{\mathrm{Re}(\Gamma(\tau_3))}$ for some $c > 0$ (independent of M). On the other hand, since $\tau_0 \in (\tau_3, \hat{\tau}_3)$, by (27) we have $\mathrm{length}(\gamma_{s_{m-1}}^{r_{m-1}}((\tau_3, \hat{\tau}_3))) > c'(e^{\sqrt{M}\,\mathrm{Re}(\Gamma(\tau_3))} - e^{\mathrm{Re}(\Gamma(\tau_3))})$ for some $c' > 0$ (independent of M), which contradicts (14) for sufficiently large M. This shows that (25) cannot hold, which ends the proof. □

Convergence of the curves $\gamma_{\underline{s}}^{\underline{r}}$

Suppose $\gamma_{\underline{s}}^{\underline{r}}$ does not converge to any point from $\widehat{\mathbb{C}}$. Then $\Gamma_{s\underline{s}}^{\underline{r}}$ does not converge to any point from $\widehat{\mathbb{C}}$, so there exist two distinct limit points $x, y \in H$ of $\Gamma_{s\underline{s}}^{\underline{r}}(t)$ for $t \to \infty$. By (16), $x, y \in L_{s\underline{s}}^{\underline{r}}$. Note that for every $k > 0$ the map F^k is defined and univalent on an open set containing x, y. Moreover, $F^k(x)$ and $F^k(y)$ are contained in the same logarithmic tract. Hence, by Lemmas 1 and 2 and (7), for sufficiently large k we have

$$\mathrm{Re}(F^k(x)) > 2c_1\,\mathrm{Re}(F^k(y)) \quad \text{or} \quad \mathrm{Re}(F^k(y)) > 2c_1\,\mathrm{Re}(F^k(x))$$

for the constant c_1 from Lemma 6. By symmetry, we can assume $\mathrm{Re}(F^k(x)) > 2c_1\,\mathrm{Re}(F^k(y))$. Since $F^k(x), F^k(y)$ are two distinct limit points of the curve $\Gamma_{\sigma^k(s\underline{s})}^{\sigma^k(\underline{r})}$, there exist $t_1, t_2 \in [0, \infty)$ such that $t_1 < t_2$ and

$$\mathrm{Re}(\Gamma_{\sigma^k(s\underline{s})}^{\sigma^k(\underline{r})}(t_1)) > c_1\,\mathrm{Re}(\Gamma_{\sigma^k(s\underline{s})}^{\sigma^k(\underline{r})}(t_2)),$$

which contradicts Lemma 6. Hence, $\Gamma_{s\underline{s}}^{\underline{r}}$ converges to a point $v_{s\underline{s}}^{\underline{r}} \in L_{s\underline{s}}^{\underline{r}} \cup \{\infty\}$, so $\gamma_{\underline{s}}^{\underline{r}}$ converges to a point $\zeta_{\underline{s}}^{\underline{r}} \in T_{\underline{s}}^{\underline{r}} \cup \{\infty\}$. In particular, $v_{ss_0\cdots s_n}^{r_0\cdots r_n} \to v_{s\underline{s}}^{\underline{r}}$ and $\zeta_{s_0\cdots s_n}^{r_0\cdots r_n} \to \zeta_{\underline{s}}^{\underline{r}}$ as $n \to \infty$.

Note that we have proved the convergence assuming $\zeta \in \partial\tilde{D}$. To prove the same for other points, consider $\hat{\zeta} \in B \setminus \overline{D}$. Perturbing a little (if necessary) the curve α from (2), we can assume $\zeta, \hat{\zeta} \notin \alpha$. Connect ζ to $\hat{\zeta}$ by a curve $\gamma \subset B \setminus (\overline{D} \cup \alpha)$ and let

$$\hat{\gamma}_s^r = \gamma \cup \gamma_s^r \cup g_s^r(\gamma)$$

for $r \in \mathcal{R}, s \in \mathbb{Z}$. Then $\hat{\gamma}_s^r$ is a curve in $B\setminus(\overline{D}\cup\alpha)$ connecting $\hat{\zeta}$ to $\gamma_s^r(\hat{\zeta})$. Let $\hat{\mathcal{G}} = \{\hat{\gamma}_{s_0\cdots s_n}^{r_0\cdots r_n}\}$ be the tree of preimages of $\hat{\zeta}$ defined by the curves $\hat{\gamma}_s^r$ and let $\hat{\Gamma}_{ss_0\cdots s_n}^{r_0\cdots r_n}$ be the lifts of $\hat{\gamma}_{s_0\cdots s_n}^{r_0\cdots r_n}$ under $\log_s \circ \Phi^{-1}$. Then by (7), for every $z \in \hat{\Gamma}_{ss_0\cdots s_n}^{r_0\cdots r_n}$ we have

$$\sup\{\varrho_H(z,w) : w \in \Gamma_{ss_0\cdots s_n}^{r_0\cdots r_n}\} < cQ^{-n}$$

for some constants $c > 0, Q > 1$. This easily implies the convergence of the branches of $\hat{\mathcal{G}}$.

Additional facts

Now we show $\zeta_{\underline{s}}^{\underline{r}} = \infty \iff T_{\underline{s}}^{\underline{r}} = \emptyset$. (This was proved in [4] without the finite order assumption, but under this assumption the argument is much simpler.) If $T_{\underline{s}}^{\underline{r}} = \emptyset$, then obviously $\zeta_{\underline{s}}^{\underline{r}} = \infty$. Suppose $T_{\underline{s}}^{\underline{r}} \neq \emptyset$ and $\zeta_{\underline{s}}^{\underline{r}} = \infty$. Then $L_{s\underline{s}}^{\underline{r}} \neq \emptyset$ and $\mathrm{Re}(v_{ss_0\cdots s_n}^{r_0\cdots r_n}) \to \infty$, so we can find $z \in L_{s\underline{s}}^{\underline{r}}$ such that $\mathrm{Re}(v_{ss_0\cdots s_n}^{r_0\cdots r_n}) > c_1 \mathrm{Re}(z)$ for the constant c_1 from Lemma 6, which is a contradiction.

If $\zeta_{\underline{s}}^{\underline{r}}, \zeta_{\underline{s}'}^{\underline{r}'} \neq \infty$ and $(\underline{r},\underline{s}) \neq (\underline{r}',\underline{s}')$, then $\zeta_{\underline{s}}^{\underline{r}} \neq \zeta_{\underline{s}'}^{\underline{r}'}$, because they are in two disjoint sets $T_{s_0\cdots s_k}^{r_0\cdots r_k}, T_{s_0'\cdots s_k'}^{r_0'\cdots r_k'}$, where $k = \min\{n : (r_n,s_n) \neq (r_n',s_n')\}$.

To end the proof of Theorem A, note that $\zeta_{\underline{s}}^{\underline{r}}$ is accessible from B since it is the landing point of $\gamma_{\underline{s}}^{\underline{r}}$. On the other hand, in [4] it is proved that in $T_{\underline{s}}^{\underline{r}}$ there is at most one accessible point.

Theorem A and Lemma 6 give immediately

Corollary 1 *For every $(\underline{r},\underline{s}) \in \mathcal{R}^\infty \times \mathbb{Z}^\infty$, $s \in \mathbb{Z}$ and every $z \in L_{s\underline{s}}^{\underline{r}}$,*

$$\mathrm{Re}(z) \geq \frac{1}{c_1}\mathrm{Re}(v_{s\underline{s}}^{\underline{r}}).$$

6 Hairs and endpoints

Definition Adapting the definition from [6], we say that a curve $h : [0,\infty) \to \mathbb{C}$ is *a hair* of the code $(\underline{r},\underline{s}) \in \mathcal{R}^\infty \times \mathbb{Z}^\infty$ attached to a point $z \in \mathbb{C}$ if:

- $h(0) = z$,
- $\lim_{t\to\infty} h(t) = \infty$,
- Every point from h has itinerary $(\underline{r},\underline{s})$ under f,
- For every $t > 0$ we have $\lim_{n\to\infty} f^n(h(t)) = \infty$.

The point z is called *the endpoint* of the hair.

In this section we prove:

Theorem C *For every $(\underline{r},\underline{s}) \in \mathcal{R}^\infty \times \mathbb{Z}^\infty$, if the set $T_{\underline{s}}^{\underline{r}}$ is non-empty, then it is a hair of the code $(\underline{r},\underline{s})$ attached to $\zeta_{\underline{s}}^{\underline{r}}$. Moreover, $T_{\underline{s}}^{\underline{r}}$ is homeomorphic to the half line $[0,\infty)$, with $\zeta_{\underline{s}}^{\underline{r}}$ corresponding to 0. By Theorem A, $\zeta_{\underline{s}}^{\underline{r}}$ is the unique point from $T_{\underline{s}}^{\underline{r}}$, which is accessible from B.*

Remark The set of endpoints

$$\mathcal{E} = \{\zeta_{\underline{s}}^{\underline{r}} : (\underline{r},\underline{s}) \in \mathcal{R}^\infty \times \mathbb{Z}^\infty\}$$

has interesting properties. Note that $\mathcal{E} \cup \{\infty\}$ is the set of radial limits of the Riemann map φ. The set $\mathcal{E}$ is totally disconnected (which follows e.g. from the fact that the radial limit of φ is equal to ∞ on a dense set in the unit circle — see [4]), but $\mathcal{E} \cup \{\infty\}$ is connected by the result from [19]. Therefore, the point ∞ is a topological explosion point for this set. Moreover, the topological dimension of $\mathcal{E}$ is equal to 1 (see [19]), while its Hausdorff dimension is equal to 2 (see [3]).

The proof of Theorem C is split into several lemmas. Let

$$\tilde{L}^{r_0\cdots r_n}_{ss_0\cdots s_{n-1}} = G^{r_0}_s \circ \cdots \circ G^{r_n}_{s_{n-1}}(\tilde{H})$$

for $\tilde{H}$ from (13), $n \geq 0$ and $r_0, \ldots, r_n \in \mathcal{R}$, $s, s_0, \ldots, s_{n-1} \in \mathbb{Z}$ (recall that we set $s_{-1} = s$). Note that by definition, we have

$$v^{r_0\cdots r_n}_{ss_0\cdots s_n} \in \partial \tilde{L}^{r_0\cdots r_n}_{ss_0\cdots s_{n-1}}$$

for every $s_n \in \mathbb{Z}$ and, by (5) and (15),

$$\bigcup_{\substack{r_{n+1}\in\mathcal{R},\\ s_n, s_{n+1}\in\mathbb{Z}}} \overline{L^{r_0\cdots r_{n+1}}_{ss_0\cdots s_{n+1}}} \subset \tilde{L}^{r_0\cdots r_n}_{ss_0\cdots s_{n-1}} \subset \bigcup_{s_n\in\mathbb{Z}} \overline{L^{r_0\cdots r_n}_{ss_0\cdots s_n}},$$

so

$$L^{\underline{r}}_{s\underline{s}} = \bigcap_{n=0}^{\infty} \tilde{L}^{r_0\cdots r_n}_{ss_0\cdots s_{n-1}}.$$

Lemma 7 *There exist $c_2 > 0, Q > 1$, such that for every $n \geq 0$, every $r_0, \ldots, r_n \in \mathcal{R}$, $s, s_0, \ldots, s_{n-1} \in \mathbb{Z}$ and every $z \in \overline{\tilde{L}^{r_0\cdots r_n}_{ss_0\cdots s_{n-1}}}$ there exists $s_n \in \mathbb{Z}$ such that*

$$|z - v^{r_0\cdots r_n}_{ss_0\cdots s_n}| < c_2 Q^{-n}.$$

Moreover, if $z \in \partial \tilde{L}^{r_0\cdots r_n}_{ss_0\cdots s_{n-1}}$, then z can be connected to $v^{r_0\cdots r_n}_{ss_0\cdots s_n}$ by a curve in $\partial \tilde{L}^{r_0\cdots r_n}_{ss_0\cdots s_{n-1}}$ of diameter less than $c_2 Q^{-n}$.

Proof Denote the diameter in the Euclidean metric (resp. in ϱ_H) by diam (resp. $\mathrm{diam}_{\varrho_H}$). Since $F^n(z) \in \overline{\tilde{L}^{r_n}_{s_{n-1}}} \subset L^{r_n}_{s_{n-1}}$ and logarithmic tracts contain no vertical segments of length 2π, there exists a point $w \in \partial \tilde{L}^{r_n}_{s_{n-1}}$, which can be connected to $F^n(z)$ by a vertical segment β_1 in $\overline{\tilde{L}^{r_n}_{s_{n-1}}}$ of length less than π. Moreover, since $\mathrm{Re}(F(w)) = \delta_0$, we can take $s_n \in \mathbb{Z}$, such that $|F(w) - v_{s_n}| \leq \pi$. Let β_2 be the vertical segment connecting $F(w)$ and v_{s_n}. Then $\beta = \beta_1 \cup G^{r_n}_{s_{n-1}}(\beta_2)$ connects $F^n(z)$ to $v^{r_n}_{s_{n-1}s_n}$ in $\overline{\tilde{L}^{r_n}_{s_{n-1}}}$ and by (1) and (5),

$$\mathrm{diam}_{\varrho_H}(\beta_j) \leq 2\,\mathrm{asinh}\,\frac{\pi}{2\delta_0}$$

for $j = 1, 2$, so by (7),

$$\mathrm{diam}_{\varrho_H}(\beta) < 4\,\mathrm{asinh}\,\frac{\pi}{2\delta_0}.$$

Then $G^{r_0}_s \circ \cdots \circ G^{r_{n-1}}_{s_{n-2}}(\beta)$ connects z to $v^{r_0\cdots r_n}_{ss_0\cdots s_n}$ in $\overline{\tilde{L}^{r_0\cdots r_n}_{ss_0\cdots s_{n-1}}}$ and by (6) and (7),

$$\mathrm{diam}(G^{r_0}_s \circ \cdots \circ G^{r_{n-1}}_{s_{n-2}}(\beta)) < 8\pi\,\mathrm{asinh}\,\frac{\pi}{2\delta_0}\,Q^{1-n}$$

for $Q > 1$. In particular, $|z - v^{r_0\cdots r_n}_{ss_0\cdots s_n}| < 8\pi \operatorname{asinh}(\pi/2\delta_0)Q^{1-n}$. Moreover, if $z \in \partial\tilde{L}^{r_0\cdots r_n}_{ss_0\cdots s_{n-1}}$, then $F^n(z) \in \partial\tilde{L}^{r_n}_{s_{n-1}}$, so we can take $w = F^n(z)$. Then $\beta = G^{r_n}_{s_{n-1}}(\beta_2)$ connects $F^n(z)$ to $v^{r_n}_{s_{n-1}s_n}$ in $\partial\tilde{L}^{r_n}_{s_{n-1}}$, so $G^{r_0}_s \circ \cdots \circ G^{r_{n-1}}_{s_{n-2}}(\beta)$ connects z to $v^{r_0\cdots r_n}_{ss_0\cdots s_n}$ in $\partial\tilde{L}^{r_0\cdots r_n}_{ss_0\cdots s_{n-1}}$, which ends the proof. □

Lemma 8 *Let Z_n, $n \ge 1$ be a sequence of continua in $\widehat{\mathbb{C}}$. Assume that there exist points $z_n \in Z_n$, such that $z_n \to z$ as $n \to \infty$ for some $z \in \widehat{\mathbb{C}}$. Then the set*

$$Z = \left\{ w \in \widehat{\mathbb{C}} : w = \lim_{k\to\infty} w_k \quad \text{for some } w_k \in Z_{n_k},\ n_k \to \infty \right\}$$

is a continuum.

Proof The proof is standard, we present it for completeness. Let $w^{(m)} \in Z$ for $m \ge 1$, such that $w^{(m)} \to w \in \widehat{\mathbb{C}}$ as $m \to \infty$. Then $w^{(m)} = \lim_{k\to\infty} w_k^{(m)}$ for some $w_k^{(m)} \in Z_{n_k^{(m)}}$, so choosing k_m sufficiently large, we have $|w_{k_m}^{(m)} - w^{(m)}| < 1/m$ and $n_{k_m}^{(m)} \to \infty$ as $m \to \infty$, so $w_{k_m}^{(m)} \to w$ and $w \in Z$. This shows that Z is closed in $\widehat{\mathbb{C}}$, so in fact it is compact.

Suppose Z is not connected. Then $Z = Z' \cup Z''$, where Z', Z'' are non-empty and disjoint compact sets. We can assume $z \in Z'$. There exist open sets $U', U'' \subset \widehat{\mathbb{C}}$, such that $Z' \subset U'$, $Z'' \subset U''$ and the closures in $\widehat{\mathbb{C}}$ of U' and U'' are disjoint. Choose a point $w \in Z''$. Since $w = \lim_{k\to\infty} w_k$ for some $w_k \in Z_{n_k}$ and $z = \lim_{n\to\infty} z_n$ for $z_n \in Z_n$, the sets Z_{n_k} intersect U' and U'' for large k. By the connectivity of Z_{n_k}, they must intersect the boundary of U at some points u_k, and by compactness $u_{k_j} \to u$ for a subsequence k_j. This means $u \in Z$, so Z intersects the boundary of U, which is a contradiction. Hence, Z is connected, so it is a continuum. □

The following lemma will be the main tool in the proof of Theorem C.

Lemma 9 *There exists $c_3 > 0$, such that for every $s \in \mathbb{Z}$, $(\underline{r}, \underline{s}) \in \mathcal{R}^\infty \times \mathbb{Z}^\infty$ and every $R_1, R_2 > 0$ with $R_2/R_1 > c_3$, if $X, Y \subset L^{\underline{r}}_{s\underline{s}}$ are two continua intersecting the lines $\ell_1 = \{z : \operatorname{Im}(z) = R_1\}$ and $\ell_2 = \{z : \operatorname{Im}(z) = R_2\}$, then $X \cap Y \ne \emptyset$.*

Proof The proof is (roughly) similar to the one of Lemma 6. Suppose X, Y are disjoint for a large constant c_3. Let

$$\ell_1 = \{z : \operatorname{Im}(z) = R_1\}, \quad \ell_2 = \{z : \operatorname{Im}(z) = R_2\}, \quad S = \{z : \operatorname{Re}(z) \in [R_1, R_2]\}.$$

First, we will show that $X \cap S$ contains a continuum X' intersecting ℓ_1, ℓ_2. Note that every component of $X \cap S$ intersects $\partial S = \ell_1 \cup \ell_2$ (this follows easily from the fact that components of a compact space are minimal intersections of open-closed subsets of the space, see e.g. [18]). Hence, defining

$$X_1 = \overline{\bigcup\{Z : Z \text{ is a component of } X \cap S \text{ and } Z \cap \ell_1 \ne \emptyset\}} \cup (X \cap \{z : \operatorname{Re}(z) \le R_1\}),$$

$$X_2 = \overline{\bigcup\{Z : Z \text{ is a component of } X \cap S \text{ and } Z \cap \ell_2 \ne \emptyset\}} \cup (X \cap \{z : \operatorname{Re}(z) \ge R_2\}),$$

we have $X = X_1 \cup X_2$ and X_1, X_2 are compact and non-empty, so $X_1 \cap X_2 \ne \emptyset$ by the connectivity of X. Take $z \in X_1 \cap X_2$. Then $z = \lim_{n\to\infty} z_n^{(1)} = \lim_{n\to\infty} z_n^{(2)}$ for some $z_n^{(1)} \in Z_n^{(1)}$, $z_n^{(2)} \in Z_n^{(2)}$, where $Z_n^{(1)}, Z_n^{(2)}$ are components of $X \cap S$, such that $Z_n^{(1)} \cap \ell_1, Z_n^{(2)} \cap \ell_2 \neq \emptyset$. By Lemma 8, there exist continua $Z^{(1)}, Z^{(2)} \subset X \cap S$, such that $Z^{(1)} \cap \ell_1 \neq \emptyset$, $Z^{(2)} \cap \ell_2 \neq \emptyset$ and $z \in Z^{(1)} \cap Z^{(2)}$. Then $X' = Z^{(1)} \cup Z^{(2)}$ is a continuum in $X \cap S$ intersecting ℓ_1, ℓ_2. Similarly, $Y \cap S$ contains a continuum Y' intersecting ℓ_1, ℓ_2. Replacing X, Y respectively by X', Y', we can assume

$$X, Y \subset S. \tag{32}$$

Note that since X, Y are connected, they intersect the line

$$\ell = \{z : \mathrm{Im}(z) = \sqrt{R_1 R_2}\}.$$

By symmetry, we can assume $\sup\{\mathrm{Im}(z) : z \in X \cap \ell\} < \sup\{\mathrm{Im}(z) : z \in Y \cap \ell\}$ and take

$$z_X \in X \cap \ell, \quad z_Y \in Y \cap \ell,$$

such that

$$\mathrm{Im}(z_X) = \sup\{\mathrm{Im}(z) : z \in X \cap \ell\},$$
$$\mathrm{Im}(z_Y) = \inf\{\mathrm{Im}(z) : z \in Y \cap \ell,\ \mathrm{Im}(z) > \mathrm{Im}(z_X)\}.$$

Denote by I the vertical segment without endpoints connecting z_X and z_Y. By definition,

$$I \cap (X \cup Y) = \emptyset. \tag{33}$$

Let $\underline{r} = (r_0, r_1, \ldots)$, $\underline{s} = (s_0, s_1, \ldots)$. We define a point

$$v = v^{r_0 \cdots r_n}_{s s_0 \cdots s_{n-1} \tilde{s}_n}$$

for some $n > 1$ and $\tilde{s}_n \in \mathbb{Z}$ in the following way. If $I \subset L^{\underline{r}}_{s\underline{s}}$, then take z_0 to be the centre of I and $a > 0$ so small that the disc $\mathbb{D}_a(z_0)$ is disjoint from $X \cup Y \cup \ell_1 \cup \ell_2$. By Lemma 7, there exists $n > 1$ and $\tilde{s}_n \in \mathbb{Z}$, such that $v = v^{r_0 \cdots r_n}_{s s_0 \cdots s_{n-1} \tilde{s}_n}$ is contained in this disc. If $I \not\subset L^{\underline{r}}_{s\underline{s}}$, then, since $z_X \in L^{\underline{r}}_{s\underline{s}}$, there exists $z_0 \in I \cap \partial \tilde{L}^{r_0 \cdots r_n}_{s s_0 \cdots s_{n-1}}$ for some n. Take $a > 0$ such that $\mathbb{D}_a(z_0)$ is disjoint from $X \cup Y \cup \ell_1 \cup \ell_2$. Again by Lemma 7, there exists $n > 1$ and $\tilde{s}_n \in \mathbb{Z}$, such that $v = v^{r_0 \cdots r_n}_{s s_0 \cdots s_{n-1} \tilde{s}_n}$ is connected to z_0 by a curve in $\partial \tilde{L}^{r_0 \cdots r_n}_{s s_0 \cdots s_{n-1}}$ of diameter less than a. By construction, in both cases

$$I \text{ and } v \text{ are in the same component of } H \setminus (X \cup Y \cup \ell_1 \cup \ell_2). \tag{34}$$

Moreover, if a is small enough, then

$$\frac{1}{2}\sqrt{R_1 R_2} < \mathrm{Re}(v) < 2\sqrt{R_1 R_2}. \tag{35}$$

It is easy to check that there exist $\tilde{r}_{n+1}, \tilde{r}_{n+2}, \ldots \in \mathcal{R}$, $\tilde{s}_{n+1}, \tilde{s}_{n+2}, \ldots \in \mathbb{Z}$, such that the branch $\Gamma^{\tilde{\underline{r}}}_{s\tilde{\underline{s}}}$ defined in the proof of Theorem A converges to infinity for

$$\tilde{\underline{r}} = (r_0, \ldots, r_n, \tilde{r}_{n+1}, \tilde{r}_{n+2}, \ldots), \quad \tilde{\underline{s}} = (s_0, \ldots, s_{n-1}, \tilde{s}_n, \tilde{s}_{n+1}, \ldots)$$

(see the proof of Theorem A). For simplicity, write Γ for $\Gamma^{\tilde{\underline{r}}}_{s\tilde{\underline{s}}}$. Note that $v = \Gamma(n+1)$. Moreover, Γ is disjoint from $L^{\underline{r}}_{s\underline{s}}$, so $\mathrm{dist}(\Gamma, X \cup Y) > 0$. This together with (32), (33) and the connectivity of X and Y implies that there exist simple arcs

$$\omega_X, \omega_Y : [0,1] \to S,$$

such that:

- $\sup_{z \in \omega_X} \operatorname{dist}(z, X), \sup_{z \in \omega_Y} \operatorname{dist}(z, Y) < \varepsilon$ for a small $\varepsilon > 0$,
- $\omega_X(0), \omega_Y(0) \in \ell_1, \quad \omega_X(1), \omega_Y(1) \in \ell_2$,
- $\operatorname{Re}(\omega_X(t)), \operatorname{Re}(\omega_Y(t)) \in (R_1, R_2)$ for $t \in (0, 1)$,
- $z_X \in \omega_X, z_Y \in \omega_Y$,
- $\omega_X \cap \omega_Y = \emptyset$,
- $(\omega_X \cup \omega_Y) \cap (I \cup \Gamma) = \emptyset$.

Let $I_1 \subset \ell_1, I_2 \subset \ell_2$ be the straight line segments connecting, respectively, $\omega_X(0)$ to $\omega_Y(0)$ and $\omega_X(1)$ to $\omega_Y(1)$. Then $I_1 \cup \omega_X \cup I_2 \cup \omega_Y$ is a Jordan curve. Let U be the domain cut out by this curve, such that $I \subset U$. Then $U \setminus I$ has two components U_1, U_2, such that $I_1 \subset \partial U_1, I_2 \subset \partial U_2$. See Fig. 3. By (34), if we take ε sufficiently small, then

$$v \in U.$$

Now we show that

$$U \text{ is bounded.} \tag{36}$$

To prove this, suppose that U is unbounded. Then one of the domains U_1, U_2 (say it is U_1) is bounded and the other is unbounded. Hence, $\mathbb{C} \setminus U_2$ is a bounded compact set containing ∂U. On the other hand, $\partial(\mathbb{C} \setminus U_2) = \partial U_2$, which is a compact set in $S \setminus \ell_1$. Hence, $\operatorname{dist}(\mathbb{C} \setminus U_2, \ell_1) > 0$, so $\mathbb{C} \setminus U_2$ cannot contain points from I_1, which is a subset of ∂U — a contradiction. This shows (36).

Hence, U is a bounded domain in S. Note that $R_1 > \delta_0$ (because ℓ_1 contains points from $L^{\underline{r}}_{s\underline{s}}$), so $\overline{U} \subset \tilde{H}$ and $\Gamma(0) = v_s \notin \overline{U}$. Moreover, $\Gamma(t) \notin \overline{U}$ for large t, since Γ converges to ∞. On the other hand, $v = \Gamma(n+1) \in U$ and Γ is disjoint from $\omega_X \cup \omega_Y$. Hence, $\Gamma([0, n+1))$ and $\Gamma((n+1, \infty))$ must intersect $I_1 \cup I_2$. By (35),

$$\frac{\operatorname{Re}(v)}{R_1}, \frac{R_2}{\operatorname{Re}(v)} > \frac{\sqrt{c_3}}{2},$$

so if c_3 is sufficiently large, then by Lemma 6,

$$\Gamma([0, n+1)) \cap I_2, \Gamma((n+1, \infty)) \cap I_1 = \emptyset$$

and $\Gamma([0, n+1))$ (resp. $\Gamma((n+1, \infty))$) must intersect I_1 (resp. I_2). Hence, for

$$t_1 = \inf\{t > 0 : \Gamma(t) \in \partial U\}, \qquad t_1' = \sup\{t < n+1 : \Gamma(t) \in I_1\},$$
$$t_2 = \sup\{t > 0 : \Gamma(t) \in \partial U\}, \qquad t_2' = \inf\{t > n+1 : \Gamma(t) \in I_2\}$$

we have

$$\Gamma(t_1), \Gamma(t_1') \in I_1, \quad \Gamma(t_2), \Gamma(t_2') \in I_2$$

and

$$\Gamma((t_1', t_2')) \subset U, \quad \Gamma((0, t_1)) \cup \Gamma((t_2, \infty)) \subset \tilde{H} \setminus \overline{U}.$$

See Fig. 3.

Note that by Theorem A, we have $L^{\tilde{\underline{r}}}_{s\tilde{\underline{s}}} = \emptyset$, because Γ converges to $v^{\tilde{\underline{r}}}_{s\tilde{\underline{s}}} = \infty$. Hence, $(\tilde{\underline{r}}, \tilde{\underline{s}}) \neq (\underline{r}, \underline{s})$. This together with (16) implies that Γ is disjoint from $L^{r_0 \cdots r_m}_{ss_0 \cdots s_m}$ for some m. Since $L^{r_0 \cdots r_m}_{ss_0 \cdots s_m}$ is an open connected subset of $\tilde{H}$ and $z_X, z_Y \in L^{\underline{r}}_{s\underline{s}} \subset L^{r_0 \cdots r_m}_{ss_0 \cdots s_m}$, there exists a simple arc

$$\xi : [0, 1] \to \tilde{H}$$

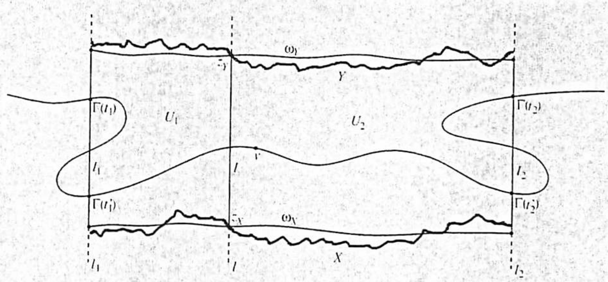


Fig. 3 The curves ω_X, ω_Y and the domains U_1, U_2

such that $\xi(0) = z_X$, $\xi(1) = z_Y$ and ξ is disjoint from Γ. Let η_X (resp. η_Y) be the open simple arc in ∂U of endpoints $\Gamma(t_1), \Gamma(t_2)$ containing z_X (resp. z_Y). Since $\partial U \setminus (\eta_X \cup \eta_Y) \subset \Gamma$, there exist $0 \le \tau_1 < \tau_2 \le 1$, such that $\xi(\tau_1) \in \eta_X, \xi(\tau_2) \in \eta_Y$ and $\xi((\tau_1, \tau_2))$ is disjoint from ∂U. Hence, $\xi((\tau_1, \tau_2)) \subset U$ or $\xi((\tau_1, \tau_2)) \subset \tilde{H} \setminus \overline{U}$.

If $\xi((\tau_1, \tau_2)) \subset U$, then $\xi((\tau_1, \tau_2))$ dissects U into two components V_1, V_2, such that $\Gamma(t_1') \in \partial V_1 \setminus \partial V_2$, $\Gamma(t_2') \in \partial V_2 \setminus \partial V_1$. Since the curve $\Gamma((t_1', t_2'))$ is contained in U, it must intersect $\xi((\tau_1, \tau_2))$, which is a contradiction.

If $\xi((\tau_1, \tau_2)) \subset \tilde{H} \setminus \overline{U}$, then $\xi((\tau_1, \tau_2))$ dissects $\tilde{H} \setminus \overline{U}$ into two components W_1, W_2, such that $\Gamma(t_1) \in \partial W_1 \setminus \partial W_2$, $\Gamma(t_2) \in \partial W_2 \setminus \partial W_1$. One of the components W_1, W_2 is bounded and the other is unbounded. Since $\mathrm{Re}(\Gamma(0)) = \delta_0$ and $\mathrm{Re}(\Gamma(t)) \to \infty$ as $t \to \infty$, both curves $\Gamma((0, t_1))$ and $\Gamma((t_2, \infty))$ are contained in the unbounded component. On the other hand, one of the points $\Gamma(t_1), \Gamma(t_2)$ is in the boundary of the bounded component and outside the boundary of the unbounded one. Hence, $\Gamma((0, t_1))$ or $\Gamma((t_2, \infty))$ must intersect $\xi((\tau_1, \tau_2))$, which is a contradiction. □

Corollary 2 *For every* $(\underline{r}, \underline{s}) \in \mathcal{R}^\infty \times \mathbb{Z}^\infty$, $s \in \mathbb{Z}$, *the set* $L_{s\underline{s}}^{\underline{r}} \cup \{\infty\}$ *does not contain a 3-star, i.e. a set homeomorphic to* $\{te^{2\pi i\theta/3} : t \in [0, 1], \theta \in \{0, 1, 2\}\}$.

Proof Suppose $L_{s\underline{s}}^{\underline{r}} \cup \{\infty\}$ contains a 3-star $\mathcal{S}$. Let z be "the centre" and l_1, l_2, l_3 "the arms" of $\mathcal{S}$, i.e. l_1, l_2, l_3 are simple arcs in $\mathcal{S}$ such that $l_1 \cup l_2 \cup l_3 = \mathcal{S}$ and $l_1 \cap l_2 \cap l_3 = \{z\}$. If $z = \infty$, then l_1, l_2 contain, respectively, continua X, Y satisfying the conditions of Lemma 9, which gives a contradiction. Suppose $z \neq \infty$. Then, diminishing $\mathcal{S}$ we can assume $\infty \notin \mathcal{S}$. Let $z_1, z_2, z_3 \neq z$ be the endpoints of l_1, l_2, l_3 respectively. By Lemmas 1 and 2 and (7), there exists $k > 0$, such that for $j = 1, 2, 3$ we have $\mathrm{Re}(F^k(z_j))/\mathrm{Re}(F^k(z)) > c_3$ or $\mathrm{Re}(F^k(z))/\mathrm{Re}(F^k(z_j)) > c_3$, where c_3 is the constant from Lemma 9. Note that F^k is defined and univalent on some neighbourhood of $L_{s\underline{s}}^{\underline{r}}$. Hence, $F^k(l_j)$ are simple arcs in $L_{s\underline{s}}^{\underline{r}}$ with exactly one common point $F^k(z)$. Moreover, there exists $j_1, j_2 \in \{1, 2, 3\}$, $j_1 \neq j_2$, such that

$$\frac{\mathrm{Re}(F^k(z_{j_1}))}{\mathrm{Re}(F^k(z))}, \frac{\mathrm{Re}(F^k(z_{j_2}))}{\mathrm{Re}(F^k(z))} > c_3 \quad \text{or} \quad \frac{\mathrm{Re}(F^k(z))}{\mathrm{Re}(F^k(z_{j_1}))}, \frac{\mathrm{Re}(F^k(z))}{\mathrm{Re}(F^k(z_{j_2}))} > c_3.$$

This clearly contradicts Lemma 9. □

Lemma 10 *For every* $(\underline{r}, \underline{s}) \in \mathcal{R}^\infty \times \mathbb{Z}^\infty$, $s \in \mathbb{Z}$, *the set* $L_{s\underline{s}}^{\underline{r}} \cup \{\infty\}$ *is locally connected.*

Proof Suppose $L_{s\underline{s}}^{\underline{r}} \cup \{\infty\}$ is not locally connected. Since $L_{s\underline{s}}^{\underline{r}} \cup \{\infty\}$ is compact, for every $\varepsilon > 0$ it admits a finite cover $\mathcal{U}$ composed of discs of diameters ε in the spherical metric. Denote by $\mathcal{D}_\varrho(z)$ (resp. $\overline{\mathcal{D}}_\varrho(z)$) the open (resp. closed) disc in the spherical metric centred at $z \in \widehat{\mathbb{C}}$ of radius ϱ. Then the family

$$\mathcal{Z} = \{Z : Z \text{ is a component of } (L^{\underline{r}}_{\underline{s}\underline{s}} \cup \{\infty\}) \cap \overline{\mathcal{D}}_{2\varepsilon}(z)$$
$$\text{such that } \mathcal{D}_\varepsilon(z) \in \mathcal{U} \text{ and } Z \cap \mathcal{D}_\varepsilon(z) \neq \emptyset\}$$

is a cover of $L^{\underline{r}}_{\underline{s}\underline{s}} \cup \{\infty\}$ by continua of diameters not greater than 2ε. By the Sierpiński Theorem (see Sect. 2), there exists an arbitrarily small $\varepsilon > 0$ such that $\mathcal{Z}$ is infinite. Since $\mathcal{U}$ is finite, there exists $z \in \widehat{\mathbb{C}}$ and an infinite sequence of disjoint components Z_n, $n = 1, 2, \ldots$, of $(L^{\underline{r}}_{\underline{s}\underline{s}} \cup \{\infty\}) \cap \overline{\mathcal{D}}_{2\varepsilon}(z)$, such that $Z_n \cap \mathcal{D}_\varepsilon(z) \neq \emptyset$. We can assume $\infty \notin Z_n$. Note that $L^{\underline{r}}_{\underline{s}\underline{s}} \cup \{\infty\}$ is not contained in $\overline{\mathcal{D}}_{2\varepsilon}(z)$, if ε is small enough, so by connectivity, Z_n intersects the boundaries of $\mathcal{D}_\varepsilon(z)$ and $\mathcal{D}_{2\varepsilon}(z)$. Hence, taking a subsequence, we can assume that there exist points $u_n \in Z_n \cap \partial\mathcal{D}_\varepsilon(z)$ and $w_n \in Z_n \cap \partial\mathcal{D}_{2\varepsilon}(z)$, such that $u_n \to u$, $w_n \to w$ for some $u, w \in \widehat{\mathbb{C}}$. Note that Z_n are disjoint continua in $L^{\underline{r}}_{\underline{s}\underline{s}}$. If one of the points u, w is equal to ∞, then we clearly have contradiction with Lemma 9. Suppose $u, w \in \mathbb{C}$. Since $u \neq w$, by Lemmas 1 and 2 and (7), there exists $k > 0$, such that $\mathrm{Re}(F^k(u)) > c_3 \mathrm{Re}(F^k(w))$ or $\mathrm{Re}(F^k(w)) > c_3 \mathrm{Re}(F^k(u))$ for the constant c_3 from Lemma 9. Since F^k is defined and univalent on some neighbourhood of $L^{\underline{r}}_{\underline{s}\underline{s}}$, the sets $F^k(Z_n)$ for sufficiently large n are disjoint continua in $L^{\underline{r}}_{\underline{s}\underline{s}}$ fulfilling the conditions from Lemma 9, which gives a contradiction. □

Now we prove Theorem C. Take $\underline{r} = (r_0, r_1, \ldots) \in \mathcal{R}^\infty$, $\underline{s} = (s_0, s_1, \ldots) \in \mathbb{Z}^\infty$ such that $T^{\underline{r}}_{\underline{s}}$ is non-empty and consider $L^{\underline{r}}_{\underline{s}\underline{s}}$ for $s \in \mathbb{Z}$. By Lemma 10, $L^{\underline{r}}_{\underline{s}\underline{s}} \cup \{\infty\}$ is locally connected, so by the Mazurkiewicz-Moore Theorem (see Sect. 2), we can connect $v^{\underline{r}}_{\underline{s}\underline{s}}$ to ∞ by a simple arc $\eta \subset L^{\underline{r}}_{\underline{s}\underline{s}} \cup \{\infty\}$. We will show that $L^{\underline{r}}_{\underline{s}\underline{s}} \cup \{\infty\} = \eta$. Suppose otherwise and take $z \in (L^{\underline{r}}_{\underline{s}\underline{s}} \cup \{\infty\}) \setminus \eta$. Again by the Mazurkiewicz-Moore Theorem, we can connect z to $v^{\underline{r}}_{\underline{s}\underline{s}}$ by a simple arc

$$\xi : [0, 1] \to L^{\underline{r}}_{\underline{s}\underline{s}} \cup \{\infty\},$$

such that $\xi(0) = z$, $\xi(1) = v^{\underline{r}}_{\underline{s}\underline{s}}$. Let

$$t_0 = \inf\{t \in [0, 1] : \xi(t) \in \eta\}.$$

If $\xi(t_0) \neq v^{\underline{r}}_{\underline{s}\underline{s}}, \infty$, then $\xi([0, t_0]) \cup \eta$ is a 3-star, which contradicts Corollary 2. If $\xi(t_0) = \infty$, then $\xi([0, t_0])$ and η contain, respectively, disjoint continua X, Y fulfilling the conditions from Lemma 9, which again gives a contradiction. Therefore, we can assume $\xi(t_0) = v^{\underline{r}}_{\underline{s}\underline{s}}$, which means $t_0 = 1$ and $\xi \cap \eta = \{v^{\underline{r}}_{\underline{s}\underline{s}}\}$. By Lemmas 1 and 2 and (7), there exists $k > 0$, such that

$$\mathrm{Re}(F^k(z)) > \max(c_1, c_3)\, \mathrm{Re}\left(v^{\sigma^k(\underline{r})}_{\sigma^k(\underline{s}\underline{s})}\right) \quad \text{or} \quad \mathrm{Re}\left(v^{\sigma^k(\underline{r})}_{\sigma^k(\underline{s}\underline{s})}\right) > \max(c_1, c_3)\, \mathrm{Re}(F^k(z))$$

for the constants c_1, c_3 from Corollary 1 and Lemma 9, respectively. Moreover, $F^k(\xi)$ and $F^k(\eta \setminus \{\infty\}) \cup \{\infty\}$ are simple arcs in $L^{\sigma^k(\underline{r})}_{\sigma^k(\underline{s}\underline{s})} \cup \{\infty\}$, such that

$$F^k(\xi) \cap (F^k(\eta \setminus \{\infty\}) \cup \{\infty\}) = \left\{v^{\sigma^k(\underline{r})}_{\sigma^k(\underline{s}\underline{s})}\right\}.$$

If $\mathrm{Re}(F^k(z)) > c_3 \mathrm{Re}(v^{\sigma^k(\underline{r})}_{\sigma^k(\underline{s}\underline{s})})$, then $F^k(\xi)$ and $F^k(\eta \setminus \{\infty\}) \cup \{\infty\}$ contain, respectively, continua X, Y fulfilling the conditions of Lemma 9, which gives a contradiction. On the other hand, $\mathrm{Re}(F^k(v^{\underline{r}}_{\underline{s}\underline{s}})) > c_1 \mathrm{Re}(F^k(z))$ contradicts Corollary 1.

In this way we have showed that $L^{\underline{r}}_{s\underline{s}} \cup \{\infty\} = \eta$, which implies that $L^{\underline{r}}_{s\underline{s}}$ is a curve homeomorphic to $[0, \infty)$, starting from $v^{\underline{r}}_{s\underline{s}}$ and tending to ∞. Analogously, $T^{\underline{r}}_{\underline{s}}$ is a curve homeomorphic to $[0, \infty)$, starting from $\zeta^{\underline{r}}_{\underline{s}}$ and tending to ∞.

To complete the proof of Theorem C, it remains to use (11), and to show that

$$f^n(z) \to \infty \text{ as } n \to \infty$$

for every $z \in T^{\underline{r}}_{\underline{s}} \setminus \{\zeta^{\underline{r}}_{\underline{s}}\}$. To do it, it is sufficient to prove that for every $z \in L^{\underline{r}}_{s\underline{s}} \setminus \{v^{\underline{r}}_{s\underline{s}}\}$ we have $\mathrm{Re}(F^n(z)) \to \infty$. By Lemmas 1 and 2 and (7), there exists a sequence $M_n > 0$ such that $M_n \to \infty$ and

$$\mathrm{Re}(F^n(z)) > M_n \,\mathrm{Re}\left(v^{\sigma^n(\underline{r})}_{\sigma^n(s\underline{s})}\right) \quad \text{or} \quad \mathrm{Re}\left(v^{\sigma^n(\underline{r})}_{\sigma^n(s\underline{s})}\right) > M_n \,\mathrm{Re}(F^n(z))$$

for every $n > 0$. Since $F^n(z) \in L^{\sigma^n(\underline{r})}_{\sigma^n(s\underline{s})}$, the case $\mathrm{Re}(v^{\sigma^n(\underline{r})}_{\sigma^n(s\underline{s})}) > M_n \,\mathrm{Re}(F^n(z))$ is impossible for sufficiently large n due to Corollary 1. Hence,

$$\mathrm{Re}(F^n(z)) > M_n \,\mathrm{Re}\left(v^{\sigma^n(\underline{r})}_{\sigma^n(s\underline{s})}\right) > M_n \delta_0$$

for sufficiently large n, so $\mathrm{Re}(F^n(z)) \to \infty$.

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