Sample Midterm, MAT 550, March 29, 2012

Name

PROBLEM 1 (5 points each, 40 points total): Define each term or give a correct statement of the quoted result:

(1) Hahn Decomposition Theorem
(2) Lebesgue-Radon-Nykodym Theorem
(3) Hardy-Littlewood Maximal Function
(4) Lebesgue Set
(5) First Category set
(6) Closed Graph Theorem
(7) Hölder's Inequality
(8) Riesz-Thorin Interpolation Theorem

PROBLEM 2 (10 points each, 30 points total): Do each of the following.

(1) Give an example of a sequence that converges to the zero function in $L^1([0,1], dx)$, but does not converge at any point of $[0,1]$.
(2) If $1 < p < \infty$, give an example of a function in $L^p(\mathbb{R}, dx)$ that is not in any other L^q, $1 < q < \infty$, $q \neq p$.
(3) Give an example of an uncountable, compact set containing only irrational numbers.

PROBLEM 3 (15 points each, 30 points total): Do two of the following.

(1) We say that $f_n \to f$ in measure if for any $\epsilon > 0$, there is an n_0 so that $n > n_0$ implies
$$\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) < \epsilon.$$
Show that if $f_n \to f$ in $L^1([0,1], dx)$, then it also converges in measure. Give an example to show the converse is not true.
(2) Suppose f is a bounded, continuous function on $\mathbb{R}$ such that $\lim_{n\to\infty} f(nx) = 0$ for every $x > 0$ (the limit it over positive integers n). Use the Baire category theorem to show $\lim_{x\to\infty} f(x) = 0$ (the limit is over postive real numbers).
(3) Let ℓ^∞ denote bounded sequences $a = \{a_n\}_0^\infty$ with the supremum norm. Let ℓ^1 denote sequences $b = \{b_n\}_0^\infty$ with the norm $\sum_{n=0}^\infty |a_n|$. Show that ℓ^∞ is the dual space of ℓ^1 from the definition (do not quote a duality result from the text).
(4) Let $M \subset \ell^\infty$ be the set of sequence such that
$$L(a) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} a_n,$$
exists. Show that this is subspace and that the Hahn-Banach theorem applies to extend L to a linear functional on all of ℓ^∞. Prove this functional is not given by any ℓ^1 function.