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THE RIEMANN MAPPING PROBLEM*

PETER ULLRICH

In his doctoral dissertation of 1851 Bernhard Riemann (1826–1866) enunciated the following theorem [96, Art. 21, 72]:

> "Zwei gegebenene einfach zusammenhängende ebene Flächen können stets so auf einander bezogen werden, dass jedem Punkte der einen Ein mit ihm stetig fortrückender Punkt der andern entspricht und ihre entsprechenden kleinsten Theile ähnlich sind" (= "Two given simply connected plane surfaces can always be related in such a way that each point of the one [surface] corresponds to One [= one and only one] point of the other which moves continuously together with it and [that] the corresponding smallest parts [of the surfaces] are similar").

In this "Riemann mapping theorem", as it is called today, he wanted the mapping function to be bijective, continuous, and inducing "similarity of the smallest parts", for which the modern technical term is "conformal". As to similarity of smallest parts, Riemann had proved before in his thesis [96, Art. 3–4, 37–38] that each (injective) function which fulfils the Cauchy-Riemann differential equations, i.e., is holomorphic, satisfies this property. (This fact was already known

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to Carl Friedrich Gauß (1777–1855), whose Copenhagen "Preisschrift" [43] is one of the two articles of other mathematicians that Riemann quoted in his dissertation [96, 38, 75].)

Riemann's proof of the above theorem, in particular its defects, and the efforts of subsequent mathematicians like Hermann Amandus Schwarz (1843–1921), Carl Neumann (1832–1925), and Henri Poincaré (1854–1912) to prove it in a sound way have been the subject of numerous historical studies, starting already with comments in the second editon of Riemann's collected works, cf. [98, 79, (5)]. Classical expositions are the "Encyklopädie" articles by Heinrich Burkhardt (1861–1914) and W. Franz Meyer (1856–1934) [18, Nr. 25–31] and by Leon Lichtenstein (1878–1933) [76, Nr. 34–49]. For modern accounts see, e.g., [17, 337–343], [42, 367–372], and [118]. In this very journal, in particular, two detailed studies have recently been published, one by Gray [44] and one by Tazzioli [114].

Obviously, the Riemann mapping theorem has generalizations to several directions: First, one may drop the condition of simple connectivity, still in the situation of *one* complex variable, cf. Section 1. Furthermore, denoting – as throughout this article – by a "domain" a connected open subset of the n-dimensional number space $\mathbb{C}^n$ over the field of complex numbers $\mathbb{C}$, one may ask the following question for *several* complex variables:

> Let two domains D and D' in $\mathbb{C}^n$ be given. Under which circumstances is it possible to map D onto D' biholomorphically, i.e., when is there a bijective, holomorphic – or, if one prefers the term, analytic – mapping f from D to D' whose inverse mapping f^{-1} is holomorphic, too? (In fact, the last condition follows from the other ones.)
>
> In particular: When can one biholomorphically map a domain D in $\mathbb{C}^n$ onto the (open) unit ball
>
> $$B_n := \left\{(z_1, \ldots, z_n) \in \mathbb{C}^n;\ \sum_{\nu=1}^{n} |z_\nu|^2 < 1\right\} \ ?$$

Sections 2-5 of the present study will be concerned with the research that has been done on this "Riemann mapping problem", starting from an article in 1907 by Poincaré on "Les fonctions analytiques de deux variables et la représentation conforme" (= "The analytic functions of two variables and conformal representation") [90], published in these very "Rendiconti del Circolo Matematico di Palermo".

1. Remarks on the situation of one variable.

But before really lifting the lid off Pandora's box, one should realize that even the classical one-variable version – "Each simply connected domain in $\mathbb{C}$ which is not equal to $\mathbb{C}$ can biholomorphically be mapped onto the unit disc B_1" – is not as obvious as one might get the impression from the stream-lined proofs in modern textbooks, going back to ideas of Lipót Fejér (1880–1959) and Frigyes Riesz (1880–1956) [92], Constantin Carathéodory (1873–1950) [22], and Alexander Ostrowski (1893–1986) [79].

In order to estimate the boldness of Riemann's mathematical concepts one may read Schwarz who wrote on the first page of his 1869 paper [102, 105, resp. 65]

> "Herr Mertens, mit dem ich im Wintersemester 1863–64 die Vorlesungen des Herrn Weierstrass über die Theorie der analytischen Functionen hörte, machte gelegentlich mir gegenüber die Bemerkung, es sei doch eigenthümlich, dass Riemann von einer Function, welche z.B. die Fläche eines ebenen geradlinigen Dreiecks auf die Fläche eines Kreises conform abbildet, bereits die Existenz nachgewiesen habe, während die wirkliche Bestimmung einer solchen Function wegen der in den Ecken liegenden Unstetigkeiten der Begrenzungslinie die Kräfte der Analysis zur Zeit noch zu übersteigen scheine." (= "Mr. Mertens, together with whom I heard the lectures of Mr. Weierstrass on the theory of analytic functions in the winter term 1863–64, occasionally made the remark to me that it was odd that Riemann had already proved the existence of a function which conformally maps, e.g., the area of a plane straight-sided triangle onto the area of a circle whereas for the present time the effective determination of such a function seems to surmount the power of analysis because of the discontinuities of the boundary line in the vertices.")

(For a discussion of [102] cf. [114, 105–111].)

Just in order to make clear that these were not problems of mediocre student consider that during the years from 1810 to 1933 only thirteen students received the best possible mark of the doctorate in mathematics at Berlin University, and Franz Mertens (1840–1927) and Schwarz were among them [12, 173].

There is even a story about Karl Weierstraß (1815–1897), which has been transmitted by Arnold Sommerfeld (1868–1951) [109, Kap. IV, § 19, 124]:

> "Charakteristisch für die Aufnahme, die die Riemannsche Dissertation ursprünglich gefunden hat, ist folgendes Erlebnis Adolf Wüllners, des langjährigen verdienten Vertreters der Experimentalphysik an der Technischen Hochschule in Aachen: Wüllner traf in den siebziger Jahren auf dem Rigi mit Weierstrass und Helmholtz zusammen. Weierstrass hatte die Riemannsche Dissertation zum Ferienstudium mitgenommen und klagte, daß ihm, dem Funktionentheoretiker, die Riemannschen Methoden schwer verständlich seien. Helmholtz bat sich die Schrift aus und sagte beim nächsten Zusammentreffen, ihm schienen die Riemannschen Gedankengänge völlig naturgemäß und selbstverständlich zu sein." (= "The following experience of Adolf Wüllner, the long-time merited exponent of experimental physics at the Aachen Institute of Technology, is characteristic of the reception that Riemann's dissertation has originally found: In the 1870's Wüllner met Weierstrass and Helmholtz on the Rigi. Weierstrass had taken Riemann's dissertation with him for summer reading and complained that to him, the function theorist, the Riemannian methods were difficult to understand. Helmholtz asked for the paper and said at the next meeting that to him the Riemannian lines of thought were totally natural and self-evident.")

In fact, the physicist Hermann von Helmholtz (1821–1894) deeply believed in the methods that Riemann used: Among mathematicians there had been objections to Riemann's use of the so-called Dirichlet's principle already since the end of the 1850's [18, 494, footnote 157]. Schwarz's article of 1869, which has been quoted above, also contains, right at its end, the first printed criticism of this principle [102,

120 resp. 82–83]. And on July 14, 1870 Weierstraß read a counterexample to Dirichlet's principle in its general formulation to the Berlin Academy [120]. (For a more detailed account of the history of Dirichlet's principle see [14, 295–303], also [18, Nr. 24–31].) Helmholtz, however, remained unimpressed: "Für uns Physiker bleibt das Dirichletsche Prinzip ein Beweis." (= "For us physicists, Dirichlet's principle remains a proof.") [60, 264].

Mathematicians saw things a litte bit different. For example, David Hilbert (1862–1943) stressed several times that Weierstraß had rightly criticized Riemann's use of Dirichlet's principle, even in those articles where he – successfully – undertook the "Versuch der Wiederbelebung des Dirichletschen Prinzips" (= "attempt of reviving Dirichlet's principle") [53, 185 resp. 64 resp. 11], also [54, 161 resp. 15]. (See [115] for the contents of Hilbert's work on Dirichlet's principle, also [44, 76–77].)

In fact, Riemann himself was well aware of the criticism. Felix Klein (1849–1925) reported on Riemann's reaction ([60, 264], cf. also [58, 492, footnote 8]):

> "Er erkannte die Berechtigung und Richtigkeit der Weierstraßschen Kritik zwar voll an; sagte aber, wie mir Weierstraß bei Gelegenheit erzählte: 'er habe das Dirichletsche Prinzip nur als ein bequemes Hilfsmittel herangeholt, das gerade zur Hand war – seine Existenztheoreme seien trotzdem richtig'." (= "He, indeed, fully acknowledged the justification and correctness of the Weierstrassian criticism, but said, as Weierstraß occasionally told me: 'he had made use of Dirichlet's principle only as an easy resort which was just at hands – nevertheless his existence theorems were true'.")

All the more one has to appreciate the mathematical imagination of Riemann who saw the general mapping theorem for simply connected domains in the complex plane, even though he showed it by using means whose validity was not clear at his time. At the end of his doctoral dissertation [96, Art. 22] he even gave some vague comments on the situation if one drops the condition of simple connectivity. In his published papers, however, he never got back to this topic, his collected works containing only a manuscript [97] which was (re)constructed by Heinrich Weber (1842–1913) from some sketchy sheets of paper. And in the book on Riemann's lectures on partial differential equations edited by Weber it is written that the solution of the task to map a doubly connected domain onto an annulus "nur in besonders einfachen Fällen möglich ist" (= "is only possible in especially simple cases") [99, 1:355]. In fact, in the multiply connected setting even the statement of the theorem is not as nice and simple as before.

Since each biholomorphic map is, in particular, a homeomorphism, two domains in the complex plane $\mathbb{C}$ that can biholomorphically be mapped onto one another must have the same *degree of connectivity*, i.e., the number of bounded connected components of the complement plus 1. So one could only hope to generalize the Riemann mapping theorem in the way that for each p there is a "canonical domain" onto which each domain in $\mathbb{C}$ with degree of connectivity p can biholomorphically be mapped, the unit disc being such a "canonical domain" for $p=1$ according to Riemann's result.

However, as already Friedrich Schottky (1851–1935) noted in 1875 in his doctoral thesis, which was refereed by Weierstraß and published two years later in revised form [101], there are obstructions to such a result: Schottky studied domains (of finite degree of connectivity) in terms of the associated "charakteristische Gleichung" (= "characteristic equation") of the corresponding Riemann surface and showed that for the existence of a biholomorphic map between two such domains it is necessary that one can transform the equations of the domains into one another by substitutions defined over the real numbers [101, 321]. In particular, even two annuli, the simplest type of doubly connected domains, can only biholomorphically be mapped onto one another if they have the same ratio of radii (with some obvious conventions for the case that the outer radius is $+\infty$ or the inner is 0) [101, 326].

On the other, positive side, by using explicit information on elliptic functions, Schottky could prove that *each* doubly connected domain can biholomorphically be mapped onto a, possibly degenerate, annulus [101, 324–326]. Furthermore, he also considered the general problem of mapping domains of finite connectivity biholomorphically onto *circle domains*, i.e., domains whose boundary consists of circular

lines (maybe degenerated to a point), and formulated this "Schottky problem" in terms of a linear differential equation of second degree [101, 331–351]. (In fact, in the original version of his thesis he had argued for the positive solution of this problem by "counting constants" i.e., functions and the differential equations they have to fulfil, but he had to delete this for the published version on Weierstraß' insistence [59, 3:579].)

Work on this problem was taken up again in 1902 by R. Le-Vavasseur [74], but the mathematician who really plunged into these mapping problems was Paul Koebe (1882–1945). Concerning his general activities on the Riemann mapping problem and the uniformization problem the reader may consult [44, 68–89]. The following remarks will mainly concentrate on his contributions to the Schottky problem. First, in an article of 1906 [61], which was the written account of his talk at the annual meeting of the Deutsche Mathematiker-Vereinigung in 1905 at Meran, he showed that biholomorphic maps between two circle domains of finite degree of connectivity can only be fractional linear transformations, by this generalizing Schottky's result on annuli. In this paper he also wrote on the Schottky problem [61, 150]:

> "Für den Nachweis, daß es in jedem Falle einen den Bedingungen der Aufgabe genügenden kreisförmig begrenzten Bereich gibt, kommen außer der Arbeit des Herrn Schottky namentlich die von Herrn Poincaré in der ersten Bänden der *Acta mathematica* veröffentlichten Arbeiten in Betracht." (= " Besides Mr. Schottky's work the articles published by Mr. Poincaré in the first volumes of *Acta mathematica* come into consideration for the proof that there always exists a domain with circular boundary which fulfills the conditions of the task.")

which sounds as if its proof would just be a straightforward application of the continuity method Poincaré had developed in [87] (or [88]). In September of the same year, however, he presented to the next annual assembly of the Deutsche Mathematiker-Vereinigung (at Stuttgart) a proof that each domain of connectivity $p=3$ can be mapped biholomorphically onto a circular domain [62, 125, Satz V], which only was the next open special case, the cases $p=1$ and $p=2$ having been settled by Riemann and Schottky, respectively. In fact, now his comments on the continuity method of Poincaré (and Klein) are much more sceptical [62, 116]. In particular, he refers to the criticism Robert Fricke (1861-1930) had given on this method in his talk at the Third International Congress of Mathematicians at Heidelberg in 1904 [40].

As one learns from [63, 115, footnote 3], however, already in summer 1905 Koebe had given a talk to the Mathematisches Seminar at Berlin where he indicated a solution even of the general Schottky problem by his own method, without having found complete convergence proofs, alas. (In [69, 442] Koebe also mentions a letter of Easter 1906 written to Klein.) Officially, the proof of the Schottky problem was announced by Koebe in 1908 in a "Voranzeige" (= "previous notice") in the "Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen" [63, 115], but only presented – as could be expected, taking into account Koebe's sense for publicity – at another annual meeting of the Deutsche Mathematiker-Vereinigung, this time in 1910 at Königsberg [66]. Of course, there only a sketch could be indicated, more details can be found in his note [67] and his articles [68], [69]. A full worked-out exposition of his solution to the Schottky problem was published in 1920 [70, 282–296].

This problem, as fomulated by Schottky in 1875 [101], had restricted to domains of finite connectivity. But already in an article of 1908 Koebe conjectured what is now called "Koebe's *Kreisnormierungsproblem*": Each domain in the complex plane ℂ, without regard to its degree of connectivity, can be mapped biholomorphically onto a circle domain and this mapping is unique up to fractional linear transformations [64, 358].

Koebe himself took up work on this problem immediately in the long series of papers which he published on the uniformization problem. In particular, he considered the case when the domain under consideration has certain symmetry properties, where he could show that the *Kreisnormierungsproblem* has a positive solution if the domain is symmetrical with respect to the real axis and the real axis meets each component of its boundary [71]. Also students of Koebe like Horst Denneberg and Herbert Grötzsch contributed on the problem. For example, Denneberg solved it in 1932 for the situation that an upper bound exists for the diameter of each component of the

boundary and a non-zero lower bound for the distance of any two such components [36]. However, in 1951 Kurt Strebel noticed [111, 5] that a result of Lars V. Ahlfors and Arne Beurling [1] implies that the unicity statement does not hold in the general case. This remark also seemed a little bit contradictory to the usual philosophy of the existence proof, using exhaustion and a limit argument. But it did neither keep Strebel nor others from thinking on the question of existence (cf. [41, 40–41] and the list of references [49, 404–406]). And, in fact, very recently there have been new impressive results on the *Kreisnormierungsproblem* by Zheng-Xu He and Oded Schramm.

In 1991 they showed that each domain in ℂ whose complement consists of only countably many components can – uniquely up to fractional linear transformations – biholomorphically be mapped onto a circle domain [49]. Their next paper [50] generalized the unicity statement to domains whose boundary is a countable union of subsets with finite 1-dimensional Hausdorff measure, and in the third article [51] they showed the existence of the conformal mapping function for arbitrary domains in ℂ provided that the familiy of components of the complement with non-circular boundary is countable and closed.

We leave the area of one complex variable by just mentioning that there are, of course, also other types of possible "canonical domains" like parallel slit domains, which are the (extended) complex plane with parallel line segments deleted: That each finitely connected domain in ℂ can biholomorphically be mapped onto such a slit domain has been shown by Hilbert in 1909 [55], the details were worked out by Richard Courant (1888–1972) in his doctoral thesis [35] with Koebe giving his version of the story in [65]. The reader interested in these types of uniformisation problems may consult [17, 357–358], [41], and [42, 374–379] for a survey.

2. Poincaré's article of 1907.

In the beginning of the 20th century the study on the mapping problem for *several* complex variables started. One might ask why it took over half a century since Riemann's enunciation of the mapping theorem for one variable until the question of mapping domains biholomorphically onto the unit ball was generalized from one to several complex variables.

Of course, there is the general time lag. The foundations of complex analysis in one variable had firmly been laid during the second third of the 19th century, whereas the corresponding results for several variables were not taken care of until the turn of the century. But perhaps even more important: physical intuition and geometric reasoning, which led Riemann to the statement of his theorem, are obviously not so easy to use in the context of at least two complex, i.e., four real variables. For example, for strictly more than one complex variable biholomorphic maps do no longer guarantee "similarity of smallest parts", i.e., conformality. (Hence, following Francesco Severi (1879–1961), these maps are often called "pseudo-conformal".) The situation is different from the one treated by Riemann in so far as one can *not* in general biholomorphically map even a simply connected domain D in $\mathbb{C}^n$ for $n > 1$ onto the corresponding unit ball B_n.

As mentioned already in the introduction, the first one to take up research on this problem was Poincaré in 1907 [90], who considered the following situation: Given two bounded domains D and D' in the space $\mathbb{C}^2$ of two complex variables, let S and S' denote their respective boundaries, which are hypersurfaces of real dimension three and are, in fact, supposed to be real analytic and non-singular [90, 194, 201 resp. 255–256, 264]. He studied three kinds of problems in this situation:

a) the local problem ("problème local", [90, 186 resp. 245]): Fix a point p of the boundary S and a point p' of S'. Is it possible to find a small part s of S containing p and a small part s' of S' containing p' and a bijective mapping from s onto s' which holomorphically extends to a neighborhood of s, analogously its inverse map?

Poincaré argued by counting constants that in general even this local problem is not solvable [90, 186–187 resp. 245–246]. In order to study the solvability of this problem he considered the automorphism group $\mathrm{Aut}\, s$ of s, i.e., the group of all bijective transformations of the surface s onto itself which are holomorphic in a neighborhood of s [90, 187–188 resp. 246–247]. Namely, suppose that there exists a map U from s to s' of the type under consideration. Then for each $T \in \mathrm{Aut}\, s'$ the map $U^{-1} \circ T \circ U$ is an automorphism of s and one easily sees that the mapping $T \mapsto U^{-1} \circ T \circ U$ is a group isomorphism between $\mathrm{Aut}\, s'$ and $\mathrm{Aut}\, s$. In particular, a necessary condition for the existence of such a map U is that these automorphism groups are isomorphic. Furthermore, one has that (in case of existence) the map U is unique if and only if one of the – and hence both – automorphism groups just consist of the identity map [90, 187 resp. 247].

b) the mixed problem ("problème mixte", [90, 201 resp. 264]): In the above situation, is it possible to find a bijection from *the whole of* S onto *the whole of* S' which extends to a map holomorphic on a neighborhood of S, but *a priori* not necessarily on the whole of D?

Naturally, for the solution of this mixed problem it is necessary that one can solve the local problem throughout the whole of S. Hence it is, in general, not solvable, too.

c) the global problem ("problème étendu", [90, 186, 213 resp. 245, 279]), which is next to the Riemann mapping problem as formulated in the introduction: Again in the above situation, is it possible to find a bijection from D plus its boundary S, i.e., the closure $\overline{D}$ of D, onto D' plus S' which holomorphically extends to a neighborhood of $\overline{D}$?

At first sight, this problem seems even harder than the mixed one. But, in fact, if the mixed problem is solvable then (since one is in the case of strictly more than one variable) one can extend the mapping from the neighborhood of S to the whole of D by means of the "Kugelsatz" of Fritz Hartogs (1874–1943), which Poincaré was well aware of in 1907 and proved anew [90, 213–214 resp. 279–280]. (Note that Hartogs' result had been published only three years before [47, 67–68], the article [48] that Poincaré really quoted [90, 213 resp. 279, footnote 1] one year before.)

Using this result Poincaré was able to give a necessary and sufficient condition in terms of differential equations and exactness of differentials for the existence of a map of the type under consideration if D equals the two-dimensional unit ball B_2, hence S the hypersphere in $\mathbb{C}^2$, and the boundary S' of D' is "infiniment voisine" (= "infinitely close") to the hypersphere [90, 217–220 resp. 284–289]. He, however, mentioned no explicit examples of domains which can or cannot be mapped onto B_2 resp. the hypersphere. In particular:

Nowadays, the standard example of two simply connected domains in $\mathbb{C}^n$ for $n>1$ which cannot biholomorphically be mapped onto one another are the unit ball B_n and the unit polycylinder

$$P_n := \left\{(z_1, \ldots, z_n) \in \mathbb{C}^n;\ \max_{1\le\nu\le n} |z_\nu| < 1\right\}.$$

A proof of this fact which uses automorphism groups runs as follows (for other proofs cf., e.g., [9, 235–236], [78, 71–73], [93, 24–25], [95, 184]:

Suppose they could be mapped. Then, by the argument discussed above, $\operatorname{Aut} B_n$ and $\operatorname{Aut} P_n$ had to be isomorphic, not only as groups, but also as Lie groups. But, for example, for $n=2$ the automorphism group of B_2 is a connected real Lie group of dimension 8 whereas for P_2 it is a real Lie group of dimension 6 consisting of two components.

Obviously, this argument is very close to the hands of Poincaré in [90], in particular, since one finds the automorphism group of B_2 there [90, 207 resp. 272]. Contrary to claims in modern mathematical texts (e.g., [9, 234], [78, 70], [93, 24, 40–41], a little bit more cautious [110, 33]), however, neither statement nor proof can be found in Poincaré's paper of 1907:

As mentioned above, one indeed finds the argument of the isomorphy of automorphism groups in the article (even if just for hypersurfaces and not for domains), but, at least for Riemann surfaces, this is not a really new idea with Poincaré (cf. e.g., Klein in 1881 [57, 76 resp. 568]). Furthermore, the object that Poincaré determined in [90, 207–212 resp. 272–278] is not the automorphism group of B_2, but of its boundary, the hypersphere, even more restrictive: the group of those automorphisms of the hypersphere that holomorphically extend to a neighborhood of it. Of course, *a posteriori* all automorphisms of B_2 give rise to such an automorphism of the hypersphere, but this fact is by no means clear before knowing the exact form of the elements of $\operatorname{Aut} B_2$. Most important, one does not find a polycylinder in [90], obviously because of the edges of the boundary, which cause

problems for Poincaré's consideration of smooth hypersurfaces.

But, in any case, Poincaré's article set the agenda for the research on the Riemann mapping problem in several complex variables, on the one hand, his idea to consider the problem from a local point of view, mapping small portions of the boundary of one domain onto that of the other (cf. Section 3), and on the other hand his taking into account the automorphism groups (cf. Sections 4 and 5).

3. The local Riemann mapping problem.

Let two real hypersurfaces s resp. s' in a complex number space be given, pieces of the boundary of D and D', respectively. The idea is to look for invariants which are preserved under biholomorphic maps between neighborhoods of s and s', respectively, just like the automorphism groups Poincaré had studied. Even more, one is interested in a *complete* set of these invariants, i.e., enough of them so that one also has the other direction: If all these invariants coincide for s and s' then there exists such a mapping between them.

It would last some time until the first complete set of such invariants was published, even in the case of only two complex variables: In 1931 Beniamino Segre (1903-1977) assigned an ordinary differential equation of second order to each real analytic hypersurface and could show that two such hypersurfaces can be mapped onto one another if and only if their corresponding differential equations can be transformed into one another [106], [107]. Apparently motivated by this result Élie Cartan (1869–1951) took up the problem for himself and, in 1932, published an article consisting of two parts [25], [26] in which, by use of Pfaffian forms, he defined a complete set of numerical invariants for real analytic hypersurfaces [25, Chapitre III, esp. 87 resp. 1301].

The general case had still longer to wait. In order to explain the results, consider a real analytic hypersurface, say, around the origin O of $\mathbb{C}^n$, which is given as the zero set of a real analytic function $p(z, \overline{z}, u, v)$ with not all first partial derivatives vanishing at O, where $z=(z_1, \ldots, z_{n-1})$ denotes the first $n-1$ complex variables,

$\bar{z}=(\bar{z}_1,\dots,\bar{z}_{n-1})$ their complex conjugates and u the real and v the imaginary part of the n-th variable. Then, after a linear change of coordinates, one can express v as $v=F(z,\bar{z},u)$ with F real analytic in a neighborhood of O and F and all its first partial derivatives vanishing at the origin O. Assume that the Levi form

$$\sum_{\mu,\nu=1}^{n} \frac{\partial^2 p}{\partial z_\mu \partial \bar{z}_\nu} dz_\mu d\bar{z}_\nu$$

of p is non-degenerate (or, equivalently, the corresponding form for F).

In 1962 Noburo Tanaka studied the special case that v is a function of z (and $\bar{z}$) alone, independent of u. For this situation he could both give a complete list of invariants and verify this property [112]. Three years later he stated a list for the general case [113], but did not furnish the highly non-trivial proofs. They were only given in 1974 in a collaboration of Shiing-Shen Chern (1911) and Jürgen Moser (1928): Already in 1973 Moser had announced [77, 111–112, Theorem] that for each hypersurface of the above type there is a unique biholomorphic mapping which transforms the above equation for v into a normal form $v^*=\langle z^*,z^*\rangle+N$ with N real analytic in a neighborhood of O and vanishing at O of prescribed type (for details of the definition cf. [77, 111] or [34, 233]). Chern had constructed a connection for these hypersurfaces and considered the covariant derivatives of the curvature of this connection, which are biholomorphic invariants of the hypersurface [34, §5] (and, in the case $n=2$, specialize to the invariants Cartan had found). In their joint paper Chern and Moser then could prove that these differential geometric invariants form a complete set [34, 258–259, Theorem 4.6]. (In fact, the invariants can be defined even if the hypersurface is only C^∞-smooth, i.e., the defining function p is arbitrarily often differentiable. But then the set is not complete any longer.)

A remark on the connection between the local and the original, global Riemann mapping problem should be added: In order to apply the above conditions on the existence of maps of pieces of the boundary for getting information on the possibility to map the domains

themselves, one has to know that a biholomorphic map extends in a sufficiently smooth way up to the boundary.

A look back to the one variable situation seems to be adequate: With Riemann this was no question since he did only consider maps between domains *plus* the respective boundary [96, Art. 21, 72]. (In fact, his line of proof inherently involved the boundary of the domain under consideration.) Also Schwarz formulated the theorem in the same way still in 1870 [103, 109] although his approximative method of proof made it necessary to discuss the behavior of the mapping function separately for interior and boundary (cf. [103, 125–126]).

The first theorem of its own concerning the extension of mapping functions to the boundary was given by Paul Painlevé (1863–1933), who simply stated in his *Thése* [80] of 1887 that if the boundary of the domain D is a C^1-curve, i.e., given by a continuously differentiable function, with the exception of finitely many angular points, then each biholomorphic mapping from D to B_1 extends to a homeomorphism between the respective closures $\overline{D}$ and $\overline{B_1}$. Axel Harnack (1851–1888), however, expressed his doubts on this statement in [46]. Hence Painlevé gave a proof of this fact in a Comptes Rendus note of 1891 [81], where he also noted a generalization [81, 466] which in particular implies that if in the above situation the boundary is C^{N+1} – that means $N+1$-times countinuously differentiable – then the mapping function to B_1 is C^N up to the boundary of D. In particular, if the boundary of D is C^∞ then the mapping function is C^∞ at the boundary. (Of course, in this connection one should not forget to mention Carathéodory who generalized Painlevé's original result to the situation that the boundary of D is just a Jordan curve [19]. For further results of this kind in the one variable situation, cf. [119].)

The several variables analogue of the last mentioned result of Painlevé, that a biholomorphic mapping between two (bounded) domains D and D' with C^∞-boundary extends C^∞ to the respective closures, has long been conjectured. And, accidentally, it was proved by Charles Fefferman [38] in the very year 1974 of the Chern-Moser result under the additional – but for several complex variables not too surprising – condition that D and D' are strictly pseudo-convex, i.e., that the Levi form of the function which locally defines the boundary is positive definite. As Sergey I. Pinchuk [86, 375, Corollary 1] and, independently, Hans Lewy [75] immediately showed by use of the reflection principle, Fefferman's result implies that for bounded strictly pseudo-convex domains with real analytic boundary – the case in which the Chern-Moser set of invariants is complete! – each biholomorphic mapping even extends holomorphically past the boundary, hence giving the situation one needs. Some years later, in 1985, M. Salah Baouendi, Howard Jacobowitz, and François Trèves succeeded in relieving the condition of strict to weak pseudo-convexity [3, 397, Corollary 7.2], i.e., the Levi form just has to be positive *semi*-definite. (In fact, holomorphic extension past the boundary even holds for proper instead of biholomorphic maps between bounded weakly pseudo-convex domains with real analytic boundary, as both Baouendi, Steven R. Bell, and Linda Preiss Rothschild [2, 506, Theorem 7], [4, 483, Corollary 2] and Klas Diederich and John E. Fornæss [37, 681, Theorem 1] proved in 1987. For the general situation of proper mappings the reader is referred to the surveys [10] and [39]; in particular, as to the generalization of Painlevé's original result to several variables, one finds in Section 1 of [39] the present state of the art how much smoothness of the boundary one has to invest in order to get a certain amount of smoothness of the mapping function at the boundary.)

4. Circular domains.

It was mentioned before that Poincaré did not prove in his 1907 paper [90] that there is no biholomorphic map between the unit ball B_n and the unit polycylinder P_n for n strictly greater than 1. In fact, the first one to prove this result was Karl Reinhardt (1895–1941) in his "Habilitationsschrift", which was published in Mathematische Annalen in 1921 [94]. In this article he considered what he denoted as "Kreisbereiche" and what only few years later was called "Reinhardt domains". Assuming for simplicity that their center is the origin O, these are those domains D in $\mathbb{C}^2$ which are invariant under multiplying each coordinate with complex numbers of absolute value 1, i.e., if $z=(z_1,z_2)\in D$ then also $(e^{i\theta_1}z_1, e^{i\theta_2}z_2)\in D$ holds for arbitrary $\theta_1,\theta_2\in\mathbb{R}$. (Of course, Reinhardt's definitions and considerations canonically generalize from the case $n=2$ to n arbitrary, but since most of the articles discussed in this Section restrict in their statements to the two-dimensional case, the following exposition will also be formulated only for this case.) Examples of such Reinhardt domains obviously are B_2 and P_2.

Although Reinhardt wrote down both unit ball and unit polycylinder in their "closed" versions [94, 227], he considered mappings that are just holomorphic in the interior. In particular, he did not demand as Poincaré had done that they holomorphically extend past the closure of the domain [94, 212]. By generalizing results of Schwarz, the Schwarz lemma and the reflection principle ("Spiegelungsprinzip"), to several complex variables [94, II. Teil] he showed that each biholomorphic map between two *convex* bounded Reinhardt domains which preserves the center O is linear [94, 243, Satz 1, also 253, Theorem 1].

For the problem of mapping unit ball B_2 and unit polycylinder P_2 biholomorphically onto one another the condition of mapping O to O is not restrictive since, as Reinhardt did [94, 231, Satz 24 resp. 25], one can easily write down a subgroup of Aut B_2 consisting of fractional linear transformations under which B_2 is *homogeneous*, i.e., for each pair of points in B_2 one can find such a transformation which maps the first point onto the second, analogously for P_2. (The point where one has to work is to show that these are, in fact, *all* automorphisms of B_2 resp. P_2, see [94, 255, Theorem 4].) Hence, if one has a biholomorphic map from B_2 to P_2 one would also have one which maps O to O, so it would have to be linear by the above result which obviously is impossible [94, 253, Theorem 2].

In 1926 Mathematische Annalen dedicated its ninety-seventh volume "Dem Andenken Bernhard Riemanns, geboren am 17. September 1826" (= "To the commemoration of Bernhard Riemann, born on September 17, 1826"). As one of the editors, Carathéodory contributed an article [20] in which he proved anew some of Reinhardt's results using what is nowadays called "Carathéodory metric" (cf. [20, 78–79 resp. 134–135]):

Let D be a domain (or, more generally, a complex manifold). Consider the family $\mathcal{F}$ of all holomorphic functions from D to the unit disc B_1 and define the "Carathéodory (pseudo)distance" of two points $x, y \in D$ to be

$$c_D(x,y) := \sup\{\varrho(f(x), f(y)); f \in \mathcal{F}\}$$

where $\varrho(\xi,\eta) = \log \frac{|1-\xi\overline{\eta}| + |\xi-\eta|}{|1-\xi\overline{\eta}| - |\xi-\eta|}$ denotes the non-Euclidean distance on B_1. (In fact, one could also use (the supremum of) the usual metric on B_1 for this purpose as already Erhard Schmidt (1876–1959) had remarked [20, 79 resp. 135, footnote 5]).

It is, in Carathéodory's own words, "fast evident" (= "almost evident") that each biholomorphic map between two domains D and D' is an isometry with respect to the respective Carathéodory pseudodistances c_D and $c_{D'}$ [20, 80–81 resp. 137, Satz 1]. Furthermore, by relatively straightforward calculation, he could show that those points of B_2 which have a constant distance from the origin O with respect to c_{B_2} form an Euclidean hypersphere [20, 89 resp. 147], whereas those points x of P_2 with $c_{P_2}(O,x)$ constant form the boundary of an Euclidean polycylinder [20, 83 resp. 141, Satz 5]. Hence, by the above isometry statement, each biholomorphic map between B_2 and P_2 which left the origin fixed had to map these two sets onto one another, which is "unmöglich ..., *weil diese letzte Punktmenge eine Kante besitzt, während die Hyperkugel ... überall analytisch regulär ist*" (= "impossible ... *since this last set of points has an edge, whereas the hypersphere ... is analytically regular throughout*") [20, 89 resp. 148].

Following Reinhardt's results – who had mainly concentrated on convex bodies after his "Habilitation" – and Carathéodory's new attack a greater number of publications appeared in the later 1920s and 1930s, e.g., [5], [6], [7], [13], [21], [116], which were authored by mathematicians like Heinrich Behnke (1898–1979), Wilhelm Blaschke (1885–1962), Carathéodory, Ernst Peschl (1906–1986) and Peter Thullen (1907), the last two being doctoral students of Carathéodory and Behnke, respectively. They held the mapping problem in high esteem, for example, Behnke wrote in 1930 [5, 329]:

> "Seit... Henri Poincaré ... ist die Frage nach dem Abbildungssatz das wichtigste ungelöste Problem der Funktionentheorie mehrerer komplexer Veränderlichen." (= "Since ... Henri Poincaré ... the question for the mapping theorem is the most important unsolved problem of the function theory of several complex variables.")

Furthermore, the last – and longest – chapter of the "Ergebnisbericht" [8] on the "Theorie der Funktionen mehrerer komplexer Veränderlichen" (= "Theory of functions of several complex variables") of 1934 by Behnke and Thullen is devoted to "Abbildungstheorie" (= "mapping theory").

These authors generalized Reinhardt domains to "circular domains" – domains D with the property that for each $z=(z_1,z_2) \in D$ and each $\theta \in \mathbb{R}$ also $(e^{i\theta}z_1, e^{i\theta}z_2) \in D$ holds – studied the automorphism groups of and biholomorphic maps between them and showed, e.g., for those bounded circular domains that contain the origin and have a piecewise real analytic boundary that each biholomorphic map between them – in particular, each automorphism – which keeps the origin fixed has to be linear [5, 341, Satz 8]. (It should be noted in this context that, contrary to unit ball and unit polycylinder which are homogeneous under their respective automorphism group, a general circular domain, even a Reinhardt domain may only have automorphisms which map the origin onto itself, e.g., the domain where $|z_1| + |z_2| < 1$ holds as was shown by N. Kritikos [72], [73, 323, Satz 2] verifying a conjecture of Carathéodory. However, in 1977 Robert Braun, Wilhelm Kaup, and Harald Upmeier succeeded in proving that if there is any biholomorphic map between two bounded circular domains containing the origin – which not necessarily keeps the origin fixed! – then there exists a linear bijection between the two domains [15, 100, Theorem 1.2]. Recently, for Reinhardt domains Satoru Shimizu could even get rid of the condition that the domains should contain the origin and showed that if two bounded Reinhardt domains can biholomorphically be mapped onto one another then there exists a biholomorphic mapping between them whose components are Laurent monomials [108, 131, Theorem 1]; for a generalization to compact Lie groups other than S^1 operating on the domain see e.g. [52, esp. 711, Corollary 5].)

Despite of the results of the authors mentioned above, somehow a general line was missing at that time, maybe a criterion on a domain for the existence of a biholomorphic map onto the unit ball: For one complex variable Riemann had stated the theorem for the simply connected case. As discussed in Section 1, the situation became already somewhat tangled in the multiply connected setting, even for one variable. Now, in the case of $n > 1$ complex variables, with Reinhardt's result that the very simple – and homeomorphically equivalent – domains B_n and P_n cannot biholomorphically be mapped onto one another, it seemed difficult even to conjecture the theorems one would like to prove. In [20, 76 resp. 132] Carathéodory brought this to the point by referring to

> "Versuche ..., aus denen man hauptsächlich das eine ersehen kann, daß nämlich die Verhältnisse hier ganz anders liegen, so daß man nicht einmal recht weiß, wo man den Spaten anzusetzen hat, um die vermutete Goldader zu finden." (= "attempts ... from which one can mainly apprehend the one thing that here the situation is totally different so that one does not even really know where to set the spade in order to find the presumed gold mine".)

The new impetus in these years came from another direction, namely from the theory of Lie groups: Already in his 1907 article Poincaré had referred to Sophus Lie's (1842–1899) theory of transformation groups (cf. [90, esp. 188 resp. 248]). Also Élie Cartan made use of this theory in the articles mentioned in connection with the local Riemann mapping problem, esp. [25, Chapitre II]. So it may not seem too surprising that Henri Cartan (1904), his son, attacked the mapping problem for circular domains using these means. It should be remarked here that in those years the (mathematical) relation between Cartan *père* and Cartan *fils* was both intense and by no means one-sided; for example, Élie Cartan wrote in [25] about the first chapter of this article, "la rédaction de ce Chapitre a été beaucoup influencée par des conversations qui j'ai eues avec Henri Cartan" (= "the redaction of this Chapter has much been influenced by the conversations which I have had with Henri Cartan") [25, 18 resp. 1232]; for another collaboration see Section 5.

Having announced his main results in a Comptes Rendus note of 1930 [29, the next year Henri Cartan published his investigations on

the group $\mathrm{Aut}_a D$ of automorphisms of a *bounded* domain D which leave a given point $a \in D$ fixed [30]. In particular, he studied the map σ from the isotropy group $\mathrm{Aut}_a D$ to the group $\mathrm{Gl}(2, \mathbb{C})$ of invertible 2×2-matrices which assigns to each element of $\mathrm{Aut}_a D$ the matrix of its first partial derivatives at a. By the chain rule, the map σ is a group homomorphism, and his first important result is its injectivity: Each $f \in \mathrm{Aut}_a D$ with $\sigma(f)$ the unit matrix has to be the identity [30, 30 resp. 170, Théorème VII]. Since a proof for the case of n complex variables, n arbitrary, instead of 2 was given one year after Cartan's article by Carathéodory [23, 771 resp. 422, Satz 8], this result is sometimes called the "unicity theorem of Cartan and Carathédory". (For one variable it had been shown by Ludwig Bieberbach (1886–1982) in 1913 [11, 556, Eindeutigkeitssatz].)

From this unicity statement one can easily deduce that each biholomorphic map between two bounded circular domains containing the origin which maps the origin onto itself has to be linear, without any condition on the regularity of the boundary of the domains, [30, 30 resp. 170, Théorème VI]. Decisive, however, is the following observation: By the injectivity of σ one can identify $\mathrm{Aut}_a D$ with a subgroup of the Lie group $\mathrm{Gl}(2, \mathbb{C})$. But since this subgroup is closed [30, 58 resp. 198, Théorème XXI], the group $\mathrm{Aut}_a D$ itself is a Lie group [30, 63 resp. 203, Corollaire].

So the machinery of Lie theory can start to work and gives an explict list of the possible groups that can appear as $\mathrm{Aut}_a D$ [30, 67–68 resp. 207–208, Théorème XXIII] (at least for $n = 2$ complex variables). By a linearization of the action of a compact subgroup Henri Cartan proved the

Cartan Mapping Theorem [30, 57–58 resp. 197–198, Théorème XX]. *Let D be a bounded domain in $\mathbb{C}^2$. Assume that there is an $a \in D$ such that $\mathrm{Aut}_a D$ is infinite. Then there are mutual prime integer numbers m and p and a domain D' over $\mathbb{C}^2$ which is invariant under each map $(z_1, z_2) \mapsto (z_1 e^{im\theta}, z_2 e^{ip\theta})$ with $\theta \in \mathbb{R}$ such that one can biholomorphically map D onto D'.*

It is a little bit annoying that in the statement of this theorem the given domain D *in* $\mathbb{C}^2$ is biholomorphically mapped only onto a domain *over* $\mathbb{C}^2$ – in this case, a finite covering of a domain in

$\mathbb{C}^2$ – and not onto a circular domain *in* $\mathbb{C}^2$. In fact, Henri Cartan had announced the better result in the Comptes Rendus note [29, 356 resp. 134, Théorème VII], but had to correct the statement to the above version later on [30, 3, 57 resp. 143, 197]. However, if one is willing to invest a little bit more in the assumptions one is also better paid off:

Theorem [30, 80–81 resp. 220–221, Théorème XXV]. *Let again D be a bounded domain in $\mathbb{C}^2$. Assume that there is an $a \in D$ such that $\mathrm{Aut}_a D$ has at least two independent real parameters, i.e., is a Lie group of (real!) dimension at least two. Then D can biholomorphically be mapped onto a Reinhardt domain in $\mathbb{C}^2$.*

Just as a closing aside, the analogous result for the one variable case was first enunciated and proved by Carlo Bruné [16], one year *after* Cartan.

5. Homogeneous bounded domains.

Now, one could be content with Henri Cartan's above result and its obvious generalization to an arbitrary finite number of complex variables. But one might also insist on looking for conditions on a domain for the existence of a biholomorphic map onto the unit ball B_n and remember a theorem for the one variable situation:

In 1875 Schwarz showed that an algebraic Riemann surface with a continuous group of automorphisms must have genus 0 or 1 [104, 139 resp. 285] (for a more "algebraic" proof than the original one of Schwarz see the letter of Weierstraß to Schwarz of October 3, 1875, also containing Weierstraß' famous "Glaubensbekenntnis" (= "confession of faith") [121]). Via a footnote of Klein in his booklet on Riemann's theory of algebraic functions [57, 67, footnote ** resp. 560, footnote 53] and a letter from Klein to Poincaré on April 3, 1882 [59, 3:608–610, esp. 610] this led to a proof of Poincaré [89, 16–19 resp. 17–20] and a final polishing of the statement by Hermann Weyl (1885–1955) in the last section of [123] so that one has the theorem: The only Riemann surfaces which have a non-discontinuous group of automorphisms can biholomorphically be mapped either onto

the Riemann sphere or onto the the complex plane or onto the unit disc B_1 or onto a (possibly degenerate) annulus or onto an elliptic curve.

For the mapping problem this result implies, if necessary by straightforward case checking, that each bounded homogeneous domain in $\mathbb{C}$ can biholomorphically be mapped onto the unit disc B_1.

Of course, even for only two variables this cannot hold any more since, as mentioned above, both the unit ball B_2 and the unit polycylinder P_2 are homogeneous. But this is the worst thing that can happen for two variables by reasons of a result Élie and Henri Cartan proved by joint efforts:

In 1933 Henri Cartan announced in the Comptes Rendus [31] that, in generalization of his results from 1931 [30], for each bounded domain D in $\mathbb{C}^n$ with n arbitrary the group $\mathrm{Aut}\, D$ is a Lie group. The proof followed in 1935 [32]. (For a modern presentation of the proof the reader may consult [78, Chapter 9].)

So one gets the following access to the Riemann mapping problem due to Henri Cartan [27, 116 resp. 1259]: Since this Lie group can at most have the real dimension $n(n+2)$ – at most n^2 parameters for the isotropy groups $\mathrm{Aut}_a D$ and at most $2n$ parameters for moving around points in D – for fixed n one only has a finite list of Lie groups which can appear as automorphism groups and, therefore, one can find out all homogeneous domains in $\mathbb{C}^n$ by case by case-analysis.

And Élie Cartan was the mathematician who knew enough about Lie groups to be successful with this task. As one learns from the Comptes Rendus note [31, 994 resp. 461] of his son, he had completed the analysis for $n = 2$ already in 1933 and so Henri Cartan could announce the

Theorem (É. & H. Cartan)[31, 995 resp. 462], [27, 160 resp. 1303, Conclusion générale]. *Each homogeneous bounded domain in $\mathbb{C}^2$ can biholomorphically be mapped either onto the unit ball B_2 or onto the unit polycylinder P_2.*

Élie Cartan published the proof of this result in 1935 [27]. Meanwhile he had also coped with larger n and so his article also contains the analogous list for domains in $\mathbb{C}^3$, namely

B_3, P_3, $B_2 \times B_1$ and a biholomorphic image of the (unbounded) domain $\{(z_1, z_2, z_3) \in \mathbb{C}^3; \operatorname{Im} z_3 > \sqrt{(\operatorname{Im} z_1)^2 + (\operatorname{Im} z_2)^2}\}$.

[27, 116–117 resp. 1259–1260].

For higher n, alas, the situation becomes rather involved. If one restricts to the case of *symmetric* bounded homogeneous domains – which, in fact, are already all for $n = 2$ and $n = 3$, whereas the situation was not clear to Élie Cartan for $n \geq 4$ [27, 117–118 resp. 1260–1261] – one gets for each fixed n still a finite number of different types, but, e.g., for $n = 12$ this number equals 300 [27, 153 resp. 1296]. Even worse, in 1959 Ilya I. Piatetski-Shapiro constructed non-symmetric bounded homogeneous domains for $n = 4$ and $n = 5$ [83] and found out later on that for $n \geq 7$ there is a continuum of different types of bounded homogeneous domains [84, 1461], almost all of which are non-symmetric, of course. So, despite its theoretical elegance the Cartans' program to treat the Riemann mapping problem cannot lead to a list of the different types. Due to a result of Èrnest B. Vinberg, Simon G. Gindikin, and Piatetski-Shapiro [117] there is, however, a classification theory even for non-symmetric bounded homogeneous domains, cf., e.g., [85].

Referring the interested reader to the surveys [10, Section 4], [39], and [45] for further recent results on the Riemann mapping problem, the present account of this classical mathematical problem shall end with just a little bit of the sparkling gold that Carathéodory was in search of: In 1979 Jean-Pierre Rosay, improving a result of Bun Wong [124] of 1977, showed [100] that each bounded homogeneous domain in $\mathbb{C}^n$ with C^2-smooth boundary can biholomorphically be mapped onto the unit ball B_n. And, in fact, his proof does not use Lie theory at all but a maximalizing procedure for families of functions like the classical proofs of the one variable Riemann mapping theorem and the Carathéodory metric discussed above.

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Peter Ullrich
Westfälische Wilhelms-Universität,
Mathematisches Institut,
Einsteinstraße 62,
D-48149 Münster

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FRIEDRICH PRYM (1841 - 1915) — AND HIS INVESTIGATIONS ON THE DIRICHLET PROBLEM*

MICHAEL VON RENTELN

1. Introduction.

In this article we should like to report on the paper [11] of Friedrich Prym on harmonic functions in the plane which appeared 1871 in Crelle's Journal. This paper contains two important results, which felt later into oblivion.

The first result concerns the Dirichlet boundary value problem for the disc with a non–continuous boundary function. It consists of a formula for the limit of the Poisson integral in a boundary point along a straight line. This formula is important and can be found in several standard texts and papers, sometimes in a slightly different setting, but all without mentioning the name of Prym.

The second result concerns the Dirichlet principle. Weierstraß' criticism on this principle is well known. Much less is known that there is another reason why Dirichlet's principle does not work in general. This was discovered first by Prym. He presented a counterexample in his article. Some thirty years later it was rediscovered by Hadamard [4] and subsequently the result was attributed to him (see [8], 41-42).

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