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ON THE HISTORY OF THE RIEMANN MAPPING THEOREM

JEREMY GRAY

Introduction.

This paper traces the history of two closely related theorems in complex function theory: the Riemann mapping theorem and the uniformisation theorem. They are taken from their origins in the work of Riemann, Poincaré, and Klein to their first rigorous proofs, in a slew of papers by several mathematicians around 1910, among whom Koebe was perhaps the most influential. Ahlfors has hailed the first of these results as "one of the most important theorems of complex analysis" [1953, 172] and the second as "perhaps the single most important theorem in the whole theory of analytic functions of one variable" [1973, 136]. As he went on to remark: the uniformisation theorem "does for Riemann surfaces what Riemann's mapping theorem does for plane regions". Despite their importance, it remains the case, as Lipman Bers commented, that "a scholarly history of that period is yet to be written" [Bers, 1974, 559].[1]

[1] The best accounts are the remarkably informative essays by Lichtenstein [1921] and Bieberbach [1921] in the *Enzyklopädie der Mathematischen Wissenschaften.*

In the course of these developments it became clear that several aspects of each problem needed greater clarification than the original proponents had supposed. Essentially new topological ideas were introduced to explore the intuitive notion of curve, and steadily more rigorous existence theorems were developed in the theory of harmonic functions. The new ideas also highlighted a running debate, among the largely German mathematical community that pursued these developments, about the relative merits of ideas drawn from outside function theory (such as potential theory) and those that came strictly from within it.

1. Riemann to Prym.

In his paper of 1851, Riemann claimed that

> Two given simply connected plane surfaces can always be mapped onto one another in such a way that each point of the one corresponds to a unique point of the other in a continuous way and the correspondence is conformal; moreover, the correspondence between an arbitrary interior point and an arbitrary boundary point of the one and the other may be given arbitrarily, but when this is done the correspondence is determined completely. [1851, 40].

This assertion is called the Riemann mapping theorem. It is usually given a modern interpretation along these lines: every simply connected domain with at least two boundary points can be mapped conformally, one-to-one and onto the interior of the unit disc, in such a way that the map extends to a map on the boundaries. The map, f say, is uniquely specified by the requirements that at some point z_0 in the interior of the domain, $f(z_0) = 0$ and $f'(z_0) > 0$. All of the changes between Riemann's version and a modern one are interesting and will be explored in this paper. The explicit requirement that the boundary has at least two points is made to rule out the cases where the domain is either the plane or the sphere, counter-examples that would have been known to Riemann himself. The observation that it

is enough to establish the indicated equivalence between any given domain of the stated kind and the disc was first made by Riemann himself. But the careful specification of what a domain is was only made in this century, when it was realised how delicate the question can be. Similarly, awareness of the truly problematic nature of the boundary came only with the work of Osgood and Carathéodory. It was under Carathéodory's influence that the uniqueness of the map was first specified by conditions on it only at an interior point.

As Riemann explained in his later paper [1857] on Abelian functions, his reason for proclaiming the mapping theorem was to extend Dirichlet's general existence theorem to functions with singularities. The consequences of such a theorem for his theory would be profound, for as he was already clear in 1851, the theorem established that every such domain admits complex functions with prescribed boundary values and prescribed points at which it becomes infinite in allowable ways. Therefore, "[these] principles open the way to the study of definite functions of a complex variable independent of an expression for it" [1851, §19]. As Riemann argued, for example, his results greatly simplified such tasks as deciding when two expressions represented the same function. It is immediately clear, he said (§20), that if a function is defined everywhere on a region that covers the entire plane once or several times, and has singularities only of finite orders and only at finitely many points, then it is an algebraic function.

The Riemann mapping theorem itself also establishes that any two simply connected regions (with boundaries) are equivalent for the purposes of Riemannian complex function theory. This would be an important step for anyone seeking to base a theory of complex functions on topological ideas rather than algebraic ones. Nonetheless, it is clear that the gap between Riemann's own formulation and a modern one is a large one. As Ahlfors wrote a century later, Riemann wrote "almost cryptic messages to the future" and stated his mapping theorem in a form that "would defy any attempt at proof, even with modern methods"[2]. One of the purposes of this paper is to attempt

[2] Ahlfors [1953b], pp. 3, 4; quoted in Bottazzini [1986], p. 234.

to trace the journey across this rocky traverse from Riemann's time to the first fully acceptable modern versions.

It is often said that Riemann's proof rested at a crucial point on an appeal to Dirichlet's principle, and that for this reason it was not widely accepted [3]. Such, influentially, seems to have been Hilbert's view in his first paper on Dirichlet's principle, [1905]. There Hilbert characterised the problem in this way: a boundary curve and a function on this curve are given. Let T be the part of the plane bounded by this curve. Then the function $f(x,y)$ is taken for which the value of the integral $L(f)$ (defined below) is a minimum. This function is necessarily harmonic. Hilbert claimed that considerations of this nature had led Riemann to his proof of the existence of functions with given boundary values, but that Weierstrass was the first to show that this approach was not reliable. According to Hilbert, Dirichlet's principle had then fallen into disrepute, and only Brill and Noether continued to hope that it could be resurrected, perhaps in a modified form.

In fact Riemann's approach was rather different and, more to the point, was soon shown to be fatally flawed. Riemann did not naively apply something called Dirichlet's principle, if this principle is taken to be the claim that a continuous function defined on the boundary of a simply connected region extends to a harmonic function defined on the whole region. Rather, Riemann first asserted that if a certain integral over a surface T,

$$L(\alpha,\beta) = \int \left[\left(\frac{\partial\alpha}{\partial x} - \frac{\partial\beta}{\partial y}\right)^2 + \left(\frac{\partial\alpha}{\partial y} + f\frac{\partial\beta}{\partial x}\right)^2\right] dT,$$

is finite, where α and β are two arbitrary real functions of x and y, then by varying α by a continuous function (or one discontinuous only at single points) which is zero everywhere on the boundary of T , the integral attains a minimal value and moreover this minimum is attained by a unique function if one excludes the points of discontinuity. This unique minimizing function is harmonic. The

[3] But see the discussion in Bottazzini [1986].

condition on the integral $L(\alpha,\beta)$ is certainly not naive, even if, as we shall see, it is inadequate to ensure the purpose.

Riemann then considered a function λ that vanished on the boundary, could be discontinuous at isolated points, and for which the integral (later called the Dirichlet integral by Hilbert)

$$L(\lambda) = \int \left(\left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2\right) dT$$

is finite. He let $\alpha + \lambda = \omega$, and considered the integral

$$\Omega = \int \left[\left(\frac{\partial\omega}{\partial x} - \frac{\partial\beta}{\partial y}\right)^2 + \left(\frac{\partial\omega}{\partial y} + \frac{\partial\beta}{\partial x}\right)^2\right] dT.$$

"The totality of these functions λ," he wrote, "represents a connected domain closed in itself, in which each function can be transformed continuously into every other, and a function cannot approach indefinitely closely to one which is discontinuous along a curve without $L(\lambda)$ becoming infinite". So for each λ, Ω only becomes infinite with L, which depends continuously on λ and can never be less than zero; consequently Ω has at least one minimum. The uniqueness followed more straight-forwardly from looking at functions of the form $u + h\lambda$ near to a minimum u. So Riemann's use of the Dirichlet's principle rests on a claim that certain boundary behaviour is sufficient to guarantee that a certain integral, L, is always finite, and that therefore the integral Ω is also finite and attains its minimum.

It was at this point, and as an illustration of his ideas, that Riemann proposed his mapping theorem. Riemann argued that it was enough to show that any such region T could be mapped conformally onto the unit disc, and outlined a proof in two stages. First, and more generally, he let T be a Riemann surface and T' that surface rendered simply connected by suitable cuts. Suppose that α and β are functions mapping T to $\mathbb{R}$ for which the Dirichlet integral is finite:

$$L(\alpha,\beta) < +\infty.$$

Then, Riemann claimed, there are unique functions μ and ν such that:

1) μ is continuous and vanishes on the boundary of T', and
2) $L(\mu, 0) < +\infty$, and $u + vi := (\alpha + \beta i) - (\mu + \nu i)$ is a function of a complex variable.

The function $u + vi$ therefore has prescribed real part, α, on the boundary of T', and prescribed singularities inside T (those of α and β).

For the second stage of his proof, Riemann restricted his attention to a simply connected region T of the z-plane. To obtain the conformal map of T to the unit disc, and to show that it extended to the boundary, Riemann took local coordinates $z - z_0$ defined on a suitably small disc around an arbitrary interior point of T, and looked at the map $\log(z - z_0) = \log\rho + \phi i$ defined on that small disc, which he imagined cut along a radius. He then extended this cut to a specified point on the boundary of T. He extended the function defined on the small disc by a continuous function to a function $\alpha + \beta i$ on the whole of T with the properties that:

1) α vanished on the boundary of T,
2) $\alpha + \beta i$ jumped across the cut by $2\pi i$ (like log), and
3) the new function agreed with the old one on the boundary of the small disc.

The theorem established in the first stage showed that there was then a function $u + vi$ with suitable jumps. Clearly the function u took every value from $-\infty$ (at z_0) to 0 (on the boundary of T). Riemann showed that for every real value of a the inverse images $u^{-1}(a)$ were single simple closed curves. This followed from the facts that T was simply connected and the function u was harmonic. Consequently, he concluded, the sought-for function was

$$e^{u+vi} : T \to \mathbf{D}.$$

There can be no doubt that Riemann sought to give his theory of complex functions the greatest degree of generality, whence the role of the mapping theorem in his published papers. But he took a more prudent view in his lecture courses. Some of these contain arguments along the lines of the one just described, while others omit it entirely[4]. This throws extra light on Riemann's intentions. When the aim is to get started, Riemann was content to rely on the method of power series, analytic continuation, and the Cauchy integral theorem. These methods guarantee the existence of a large class of analytic functions, including all the familiar ones. The purpose of the mapping theorem is not to be the sole source of functions, but rather to give the theorist a better conceptual grasp, and to set limits on how far the theory can be taken. Nonetheless, it presented an approach to the definition of complex functions that others, notably Weierstrass, were to find uncomfortable.

Historians of mathematics have described the way that other mathematicians gradually came to distrust Riemann's approach, and perhaps even the results he reached as well [5]. The central case of Schwarz is described elsewhere in this volume by Rossana Tazzioli. Here it is enough to note that Riemann's outline of a proof was shown to be flawed by Riemann's former student Prym, in a short paper [1871]. His reasoning was endorsed by Schwarz, who proceeded to give a thorough description of an alternative, potential-theoretic approach, and knowledge of Prym's work seems to have faded from history. It seems worth stressing that the Riemannian approach did not lapse because of vague misgivings but because it was known to be deficient.

Prym took the case of a disc and an arbitrary continuous function, u, defined on the boundary, for which Dirichlet's principle implied that there was an extension to a finite and continuous harmonic function defined on the whole of the disc. He pointed out that all known proofs of this result relied on the claim that the function u was identical with its Fourier series, but that this result was only known for functions having only finitely many maxima and minima. In particular, he said, it was a misapprehension to think,

[4] See Bottazzini and Gray, forthcoming.

[5] See Bottazzini [1986].

as Hankel did, that Riemann had proved that an arbitrary function is representable by its Fourier series. But in any case, he said, even when Dirichlet's principle was true, Riemann's approach to it might be in error.

He considered a branch of the complex function

$$u + iv = i\sqrt{-\log(R + x + iy)}$$

defined on a disc of radius $R < \frac{1}{2}$, and introduced polar coordinates ρ and τ centred on the point $(-R, 0)$. In the disc ρ took every value from 0 to $2R < 1$ and τ every value from $-\pi$ to π. The branch of logarithm taken was to satisfy $-\log(R + x + iy) = -\log\rho - i\tau$. From the explicit form for u and v in terms of polar coordinates it then followed that the functions u and v are everywhere defined and single-valued, even on the boundary of the disc. Since the function u is the real part of a complex function it is certainly harmonic. A closer examination showed that the function u was zero at the point $\rho = 0$. Prym then considered the Dirichlet's integral $L(u, 0)$ and showed that it was infinite. The reason, as his formulae make clear, is that the function u oscillates infinitely often in any neighbourhood of the point $\rho = 0$. Consequently there is no hope that step 2 of Riemann's argument ($L(u, 0) < +\infty$) can be made to work.

With Prym's criticism, and more weightily the criticisms coming from Berlin (see the paper in this volume by Tazzioli) Riemann's approach using a version of Dirichlet's principle lapsed into disrepute. Other approaches were developed. Christoffel and Schwarz took up the problem of representing the interior of a polygonal region on a disc or half-plane, and showed that the conformal mapping can be written down explicitly as an integral. Schwarz and C.A. Neumann showed how the methods of potential theory could be developed to cope with a wide variety of regions bounded by analytic arcs. But it was seemingly not felt that a general principle existed which was adequate, even appropriate, to deal with all the cases at hand. As late as 1894 Brill and Noether, in their survey of the history of algebraic functions, were to dismiss Riemann's approach as confusing. Speaking of the application of Dirichlet's principle, in the generality with which it underlies Riemann's work, they said that "The function idea, in such generality, incomprehensible and evaporating before one's eyes, no longer leads to reliable conclusions" [1894, 265]. Even so, as Hilbert remarked, they went on to express the hope that the simplicity of the ideas, standing as they did in organic connection with problems of mathematical physics, might be revived, perhaps in a modified form.

It is also worth noting that the whole problem looked different if one took a firmly Weierstrassian view of function theory. From that perspective, as Hurwitz explained in his address to the first International Congress of Mathematicians [1897] an analytic function is something defined on a set of perhaps overlapping discs, on each of which there is given a convergent power series. So the question is to determine which domains (in the sense of Harnack, see below) are the domains of definition of an analytic function. Affirmative answers were given by Mittag-Leffler [1884], and more simply by Runge [1885] and Stäckel [1893]. The question then becomes the nature of the points on the natural boundary of a function with a given domain [6].

Hurwitz then turned to the approach to complex function theory pioneered by Cauchy and Riemann, which he found less elementary. He pointed out that if the Cauchy integral theorem is to play a central role then one must have a clear understanding of the possible nature of a closed curve. One should not forget, he said, that this class included the space-filling curves of Peano and Hilbert. Even simple closed curves posed problems, and he referred to Schoenflies's attempts to prove the Jordan curve theorem. We shall see in Section 3 how perceptive Hurwitz's intuition was.

2. Poincaré and uniformisation.

The theory of conformal mappings took a novel turn with the arrival of Poincaré and problems to do with the theory of algebraic

[6] See Bottazzini [1986].

curves, as we shall now. In 1882 Poincaré, and independently Klein, were led to proclaim a remarkable result called the uniformisation theorem. This asserts that every algebraic curve of genus greater than one can be obtained as the quotient of the unit disc by the action of a suitable Fuchsian group, and therefore that there is a map from the unit disc to the algebraic curve that parameterises the curve. The parameterising functions are Fuchsian functions automorphic with respect to the group [7]. This astonishing and completely unexpected result gave such curves for the first time an intrinsic geometry in the sense of Gauss, because it makes every such curve locally isomorphic to a patch of non-Euclidean geometry [8].

The original proclamation was based on little more than counting constants: the number of parameters that determined a Fuchsian group and an algebraic curve were the same: $3g-3$ complex numbers, where g is the genus of the algebraic curve. Although Klein attempted to go further and argue that the one set of parameters varies continuously with the other, Poincaré replied that unless the $(3g-3)$-dimensional "manifolds" in question were closed, not enough could be said to yield the theorem even if the correspondence were continuous.

But in 1883 Poincaré stated a remarkable generalisation of the uniformisation theorem, and gave it the outlines of a proof. He let y be any analytic function of x which is not single-valued, and claimed that one can always express x and y as single-valued functions of a complex variable z. To prove this, Poincaré considered m functions $y_1, y_2, \dots, y_m$ of x. The value of each y_i would be known at a point x when its value was known at some initial point x_0 and a path from x_0 to x was specified. He supposed that the point x was one coordinate of a point moving on a Riemann surface having infinitely many leaves. To study this surface he proposed to construct what today is called its universal cover, which topologically is a disc. His method was to construct a new Riemann surface by opening out the loops on the surface which corresponded to non-trivial analytic continuations of any of the m functions $y_1, y_2, \dots, y_m$. To obtain this

7 See Gray [1986, Ch 6].
8 See Gray [1992 LNP 432].

surface he took the arbitrary point x_0 as the starting point for all loops drawn on the surface. As one then traced a loop starting and finishing at x_0, the values of the m functions $y_1, y_2, \dots, y_m$ varied. For each loop he asked whether the values of any of the functions were different at the start and the end of the loop. If any were, he said the loop was of the first sort (*la première sorte*), else it was of the second sort. Among the loops of the second sort, some could be continuously shrunk to a point without ever losing their defining property; they were said to be of the first type (*la première espèce*). Those that could not be shrunk in this manner he said were of the second type. Poincaré then said that the initial and final points of a Riemann surface corresponding to these functions had a point for each loop of the first sort and for each loop of the second type and the second sort. Each loop of the first type and the second sort yields only one point on the new Riemann surface. Because all the non-trivial loops have been opened out, the new Riemann surface is simply connected, and is topologically a disc, as Poincaré remarked.

He then showed how to draw infinitely many non-intersecting circles C_n in the surface such that each C_n spanned a disc containing the circle C_{n-1} and every point of the surface was contained in such a disc. Poincaré then looked for a suitable Green's function. He took the elliptic modular function ϕ which is holomorphic except at 0, 1, and ∞ and which maps the plane with the points 0, 1, and ∞ removed onto the upper half plane. He then defined a new function ψ by the formula

$$\psi(x) = \frac{\phi\left(\dfrac{\alpha x+\beta}{\chi x+\delta}\right) - \sqrt{-1}}{\phi\left(\dfrac{\alpha x+\beta}{\chi x+\delta}\right) + \sqrt{-1}}.$$

where β, χ, and δ are constants, which is defined and holomorphic except at

$$x = -\frac{\beta}{\alpha}, \quad x = -\frac{\delta}{\gamma} \text{ and } x = \frac{\delta-\beta}{\alpha-\gamma}.$$

Moreover, β and δ are chosen so that $\phi\left(\dfrac{\beta}{\delta}\right) = \sqrt{-1}$ and $\psi(0) = 0$.

Note that $\psi(x) = \dfrac{x - \sqrt{-1}}{x + \sqrt{-1}}$ maps the upper half plane to the interior of the unit disc. Without loss of generality, $y_m = y$. Then defining the function t to be $\log\left|\dfrac{1}{\psi}\right|$, Poincaré obtained a function t which was essentially positive, logarithmically infinite at certain points, and was harmonic.

He then introduced functions u_n satisfying $\Delta u_n = 0$, which vanished along C_n, and which were holomorphic inside C_n except at the point O where they became logarithmically infinite. It followed that each function u_n was positive everywhere inside its boundary contour C_n . Consequently, so were

$$u_n + 1 - u_n, \text{ and } t - u_n,$$

and so he deduced that the series

$$u = u_1 + (u_2 - u_1) + \ldots + (u_{n+1} - u_n) + \ldots$$

converges, because the u_n's increase with n and are bounded above by the function t. The demonstration fails when the function t is infinite, and Poincaré gave a separate argument to show how it can be modified.

Poincaré then showed that the function u is continuous, and that the series defining it converges uniformly. Moreover, it satisfies the equation $\Delta u = 0$ away from the point O (where it is logarithmically infinite). To prove this, Poincaré assumed that there was a function U which solved the Dirichlet problem for any contour c in a small region of the Riemann surface, in that U agreed with the function u on the contour c and was harmonic inside c. But U was the limit of the functions u_n, so $U = u$ and therefore the function is harmonic. A separate argument ensured that u was harmonic at the points O_i. So Poincaré assumed that the Dirichlet problem was always solvable, i.e. that there always exists an harmonic function on a simply connected domain having prescribed continuously varying values on an arbitrary boundary. It is remarkable that this part of his proof was not to be questioned.

Because the function u is harmonic it has an harmonic conjugate v making $u + iv$ a holomorphic function. The functions z and z_n defined by the equations

$$z = e^{-(u+iv)}, \quad z_n = e^{-(u_n+v_n)}$$

are then well-defined and Poincaré showed that z_n is one-to-one inside C_n and that z is one-to-one everywhere. It followed that the functions y_k are uniformised by z, because the surface S was constructed in such a way that they have a unique value at each point of S.

Poincaré added two notes which show that he was aware of the central weakness in his approach. He excluded singular points from the Riemann surface, banishing them to its boundary. In particular, he noted that the points where the modular function is not holomorphic are singular. They spawn infinitely many points on the boundary of the Riemann surface S. For this reason his proof was to be criticised by Hilbert, in the course of presenting the topic as the 22nd of his 23 mathematical problems. Hilbert stressed that it was extremely desirable to check that the uniformising map was indeed surjective. Poincaré was to endorse this criticism when he took up the question again in 1907, and give two ways of dealing with it. He added that his original method left it uncertain whether the conformal map of S mapped it onto the unit circle or merely onto a part of it. "The problem," he said, "is none other than the Dirichlet problem applied to a Riemann surface with infinitely many leaves".

The connection between the Riemann mapping theorem and the uniformisation theorem is a subtle one, and was well described by Bieberbach [1915, 127 ff]. An algebraic curve such as $z^2 + w^2 = 1$ may be uniformised for example by

$$z = \frac{2t}{1+t^2} \text{ and } w = \frac{1-t^2}{1+t^2},$$

The Riemann surface in w and z is mapped by t onto the schlicht t-plane, and w and z are single-valued functions of t. But other uniformisations are possible, for example, $z = \sin\phi$ and $w = \cos\phi$, mapping an infinitely branched Riemann surface onto another schlicht plane. Poincaré established a general uniformisation theorem for domains branched over the points 0, 1, and ∞. Each solution of a uniformisation problem is necessarily a solution of the Riemann mapping problem. Moreover, given two solutions of a uniformisation problem, the uniformising functions in the one case may themselves be uniformised by the solutions from the other case. (In the above example, one may set $t = \sin\phi$.) The ultimate goal of the theory of uniformisation was to show that over any Riemann surface there sits exactly one of three simply connected surfaces (the sphere, the plane, and the disc) branched in a way described by the uniformising parameters. In this sense, the Riemann mapping theorem is naturally a special case of the uniformisation theorem.

3. Harnack to Osgood.

The year after Poincaré's paper appeared, C.A. Neumann published the second edition of his book of lectures on Riemann's theory of Abelian integrals, [1884]. This book directed attention back to one of Riemann's own principal uses of his mapping theorem: to establish the existence of Abelian integrals on a Riemann surface. In it, Neumann revised Riemann's account of how a simply connected surface is obtained from a given Riemann surface of genus g by cutting it open along $2g$ curves. He then showed how his own earlier approach to the Riemann mapping theorem in terms of an iterative process based on potential theory could be used to establish the existence of a complex funcq on which jumped by a constant across each cut and for which the real part of each jump was prescribed [9]. Because the curves may be chosen at will, this extension of Riemann's theorem did not have to confront difficult questions about the boundary, and later readers, even if accepting its validity rather too readily, were right to locate it along side Schwarz's work. For

[9] Neumann's treatment did not specify any condition on the normal derivative along the cuts.

Schwarz's alternating method left the nature of the boundary vague. It was to be made up piecewise by analytic arcs (the situation at the vertices was only dealt with by Picard) but Schwarz did not investigate the kind of curves that could arise in this way. On the other hand, Carathéodory was to praise Schwarz in Schwarz's *Festschrift* volume [1914, 20] for separating out the interior part and the boundary part of the Riemann mapping theorem. Poincaré's *méthode de balayage* [1890] similarly made certain simplifying assumptions about the boundary but left the extent of the method unresolved. In the next few years attention was to be directed first to finding more rigorous proofs of the Riemann mapping theorem valid for any boundary, and second to the nature of the boundary itself. So for example, in a note [1891] Painlevé showed that it was enough to insist that the boundary have an everywhere continuously varying tangent.

Throughout the 1880s and 1890s, the running was made by those developing the methods of potential theory. The mathematician who first published a satisfactory proof for solving a suitable version of Dirichlet's problem was Harnack [1887]. In this book Harnack first reviewed the attempts by Schwarz and Neumann to solve the Dirichlet problem for a variety of domains (including Neumann's new account [1887]) and noted the limitations that these authors had been unable to remove on the nature of the boundary. He admitted that he, too, had been unable to extend their methods, and so he had turned to a different approach using Green's functions. While this method could be applied to two- and three-dimensional problems alike, nonetheless Harnack found that some planar questions could be resolved by the theory of conformal mappings, and so in his book he confined his attention to potential theory in two dimensions. He gave a thorough account of the existence theory for functions with prescribed singularities, from which it was possible to derive the general theorems in Riemann's paper on Abelian functions and also Poincaré's uniformisation theorem. In the final section of the book he showed how his ideas led to a proof of the Riemann mapping theorem.

Harnack's book was generally cited favourably by subsequent mathematicians, usually for what has become known as Harnack's theorem [1887, 67]. This involves a sequence of harmonic functions u_n defined on a surface F. The function u_n restricts to a continuous function U_n on the boundary of F. Harnack assumed that a uniform continuity held: for every arbitrarily small δ there was, at each point s of the boundary, a finite domain which contained interior points of F and was partly bounded by a piece of the boundary of F containing s, such that the values each function u_n took on this domain (including its boundary) varied by less than δ. For this, he remarked, it is necessary that the functions U_n be continuous. A preliminary theorem then asserted that: if the sum $U = \Sigma U_n$ converges uniformly, then the sum $u = \Sigma u_n$ converges at every interior point of the surface F to a harmonic function. From this what became known as Harnack's theorem followed: if a sequence of harmonic functions u_n all have the same sign (say, positive) and the sum $u = \Sigma u_n$ converges at an interior point of the surface F, then it converges at every interior point of the surface F to a harmonic function. Alternatively, if a sequence of harmonic functions u_n tend from below to the values of a function u, then u is an harmonic function.

Harnack explained more carefully than any previous author what it is for a domain to be connected: any two points can be joined by a finite polygonal arc which can be covered by overlapping discs all lying in the interior of the domain. Later authors were to tease apart the concepts of domain (every point has a disc-like neighbourhood lying entirely in the domain) and connectedness (here, path-connectedness). Boundary points were then defined as those points every neighbourhood of which contained some points belonging to the domain and some that did not. Harnack then claimed that the boundary of a simply connected domain has a continuous boundary, although the boundary may be nowhere differentiable and may have corners and cusps. Implicitly it need not even be rectifiable. One could even add to a boundary an arbitrary number of incisions, lines drawn inwards from boundary points, which would be traversed twice by any circuit of the boundary.

On the basis of his theorem and this analysis of the boundary Harnack then established the existence of a Green's function for

every bounded region with an arbitrary boundary. He accepted that Neumann's approach established the existence of a unique harmonic function for polygonal regions with no re-entrant angles agreeing with a given continuous function on the boundary. If the given function on the boundary is always finite but has isolated jump discontinuities then there is still a harmonic function agreeing with the given one at points on the boundary where the given function is continuous. Harnack then used an approximative argument which he attributed ultimately to Schwarz to deal with the general simply connected domain, in which the domain is steadily approximated by polygonal regions. He used his theorem from p. 67 (see p. 62 above) to establish that the sequence of harmonic functions converged to a harmonic function on the given domain. He then patched together arbitrary bounded domains out of simply connected pieces. To prove the Riemann mapping theorem (and its extension to non-simply connected domains) Harnack then used his Green's function approach to establish the existence of a suitable harmonic function and thence a complex function mapping the given bounded domain onto a circle (if simply connected) or a domain bounded by several circles (if not).

Harnack's work was read carefully by several authors and his theorem widely accepted. His attention to the nature of the boundary, in particular his idea of incisions was also perceptive, and surfaced in the work of his most prominent successor, the American mathematician W.F. Osgood. Osgood did not accept all of Harnack's approach, finding that the boundary behaviour was even subtler than had been suspected. One might reconstruct Osgood's initial problem this way. Harnack had seemingly shown that the harmonic function on the interior extends to a continuous map of the boundaries. When the boundary is not a simple closed Jordan curve Harnack's approximation argument might fail, if the incisions cluster together awkwardly. The claim the Riemann mapping theorem made about boundary behaviour has perhaps to be restricted to simple closed Jordan curves. Osgood was aware that Jordan had shown that the straight-forward claim that a simple closed curve (defined as the image of interval under a continuous function from $\mathbb{R}$ to $\mathbb{R}^2$ divides the plane into two different regions, is something that has to be proved. When he proved that if such a curve is rectifiable it has zero area, he implicitly at least opened the way to showing than non-rectifiable Jordan curves might be very strange indeed. Thus primed, Osgood set to work.

In 1900 while "on the Atlantic" he wrote a short paper for the first issue of the *Transactions of the AMS* in which he established the existence of a Green's function for any simply connected plane domain, T, other than the entire plane of complex numbers. By the term "Green's function" he meant a single-valued function vanishing on the boundary and harmonic in the interior except for one point where it had a logarithmic pole (like $\log(1/r)$ as $r \to 0$). Finally, on the boundary the value of the function u was to be zero. The novelty of Osgood's paper resides in its insight into the possible nature of the boundary. He was at pains to point out that his proof did not require that the boundary curve be a Jordan curve. He pointed out that his proof was valid even for the region of the upper half-plane bounded by the real axis to which has been added unit verticals at every point of a perfect nowhere dense set. (Osgood had given an example of such a set in an earlier paper, [1898, lecture VI]). Let T be the region of the upper half plane from which these lines have been removed. As Osgood noted, if the point set on the real axis has positive content (as it does in his example) then the boundary of T likewise has positive content. If this is not awkward enough, there are points, A , of the boundary which have the property that every neighbourhood of such a point A contains points of T, yet there is no continuous curve approaching the point A that lies entirely in the interior of T. Osgood therefore defined the condition that as a point (x, y) approaches the boundary the sequence of function values $f(x, y)$ tends to zero with care. He interpreted it to mean that if the sequence (p_n) of points in the interior of T has the point A as its unique limiting point, then $\lim_{n\to\infty} u(p_n) = 0$. Osgood's boundary is an example of a set with prime ends, and it is original with him. It does not seem to have been discussed previously by those working on Cantor's problem of characterising the continuum. Carathéodory, writing in 1912-13, attributed the term and perhaps the concept to Schoenflies in 1908, seemingly unaware of its appearance in the American's paper.

Osgood's proof was succinct. He divided the plane into suitably small squares of width $1/n$, let C_n be the union of such squares lying inside T , and let u_n be a Green's function on C_n. Then he showed by Harnack's theorem that u_n converged to an harmonic function on T. If T was finite, then the sequence u_n was dominated by a Green's function for a region C containing T. If T was infinite, Osgood had an argument modelled on Poincaré's use of the modular function. Osgood remarked (p. 314) that his argument made it "probable" that the uniformisation theorem extended to the singular points on the boundary, but evidently he was still uncertain. He also indicated that his theorem was still true when T and its boundary together were mapped by stereographic projection to the entire Riemann sphere.

So Osgood established the existence of Green's functions on arbitrary domains, and thereby resolved the Riemann mapping theorem on the interior of any simply connected domain. The next year, 1901, Osgood finished his article for Klein's *Enzyklopädie der Mathematischen Wissenschaften.* There [1901, 56] he distinguished between two types of simply connected domains. The first type had boundaries that were (simple) Jordan curves (they may be called Jordan domains); the second did not. He observed that if the boundary was a curve with a continuously varying tangent then it was certain that the conformal map on the interior extended to a continuous map on the boundary. However, he had established the existence of a Green's function for all Jordan domains so he said it was probable that there would still be an affirmative solution to the Riemann mapping problem. For domains of the second type, as he remarked, it did not make sense to ask about behaviour on the boundary.

That Osgood was fully aware of the problems posed by the boundary of even a Jordan domain is shown by one of his most remarkable discoveries, made in 1902. This is the existence of simple closed Jordan curves of finite area. Jordan had discussed the result that now carries his name (the Jordan curve theorem) in his *Traité d"analyse* (1st ed. 1887, 2nd ed. 1893). Jordan had outlined a proof, which began by passing from the curve to an arbitrarily close polygonal approximation, but this was justly criticised by Schoenflies on the grounds that Jordan had assumed the truth of the theorem for polygonal curves (which is not obvious if the number of sides is not known to be finite) and had gone on to omit many details. As a result, others offered proofs: Veblen, who gave the first, Schoenflies himself, and most lucidly Brouwer in 1909 [10].

Osgood gave his instructive example of a simple closed Jordan curve of finite area by means of successive approximations. To obtain the first curve he took a diagonal AB of the unit square, S_1, and extended it outside the square to divide the complement of the square into two regions (coloured blue and yellow for water and land) [11]. The square S_1 was itself divided into three regions: canals of water, dykes of land, and nine squares (numbered 1 to 9) of undecided matter, coloured white. There were eight short segments where water and land met, they were coloured red by Osood and formed part of the Jordan curve he was constructing. The boundary of the water from A to B was then traversed by a moving point in a uniform way such that the eight red arcs were parameterised by values of t such that

$$\frac{2n-1}{17} \leq t \leq \frac{2n}{17} \qquad n = 1, 2, \ldots, 8.$$

The widths of the canals and dykes was chosen so that their total area was some value $\alpha_1 < 1$.

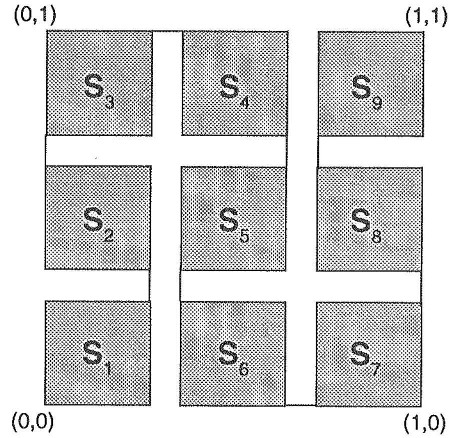

Fig. 1

[10] See D.M. Johnson [1979, 163-177].

[11] The figures are taken from Sagan's informative discussion, [1993].

The construction was then repeated in each square of undecided matter, so that the eight red arcs in square 1 were parameterised by values of t for which

$$\frac{2n-1}{17^2} \leq t \leq \frac{2n}{17^2} \qquad n = 1, 2, \ldots, 8.$$

The whole of the water still forms a single connected region, as does the land. The width of the new canals and dykes was chosen as follows: Osgood let $\varepsilon_1 + \varepsilon_2 + \ldots$ be a convergent series with value $\lambda < \frac{1}{2}$. At the first stage the area of the canals was $\frac{\alpha_1}{2} = \varepsilon_1$. In each of the nine squares the area of the canals was $\frac{\varepsilon_2}{9}$, so that the total area of new canal was ε_2.

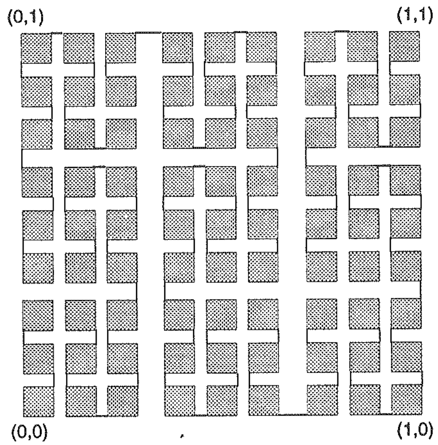

Fig. 2

This construction was then repeated indefinitely. The curve was completed by adding all of its limit points, and observing that such points always lie in the interior of white squares. It follows that they can each be assigned unique parameter values. A little more work then showed that the curve so obtained is continuous, and from its construction the curve is a one-to-one map of the open unit interval. The curve is therefore a Jordan curve. Its exterior area is $1 - 2\lambda$, being the difference between the area of the unit square and that of the final collection of canals and dykes.

The curve so obtained enjoys a number of other unusual properties. It is not rectifiable, for rectifiable Jordan curves had been shown by Jordan himself to have zero area [Cours[2], 107]. In any neighbourhood of any point that is not an interior point of a red segment, the curve has infinitely many points around which it wraps infinitely often. It would be possible, Osgood observed, to adapt the construction so that the red segments shrank to zero and the resulting curve nowhere had a tangent. Modifying the construction still further by letting the canals and dykes have breadths tending to zero, and suitably changing the parameterisation, Peano's space-filling curve would be obtained. By starting with a region bounded by the unit circle and whose boundary lies inside an annulus, the above construction could be made to yield a closed Jordan curve of positive exterior area, bounding completely a simply connected region contained within the unit circle. The interior of the region so bounded has different interior and exterior areas.

4. Poincaré and Koebe.

In 1907 Poincaré and Koebe independently proved the uniformisation theorem by rigorous methods. Poincaré published his proof in Acta Mathematica. There he began by reviewing his earlier paper and the attempts by others to overcome or else avoid the problems which it raised. Then he outlined his new approach. He characterised the problem, as he had done earlier, as a Dirichlet problem for a Riemann surface with infinitely many leaves. Then he set out, first to make more precise and more supple the concept of a Riemann surface, by enlarging Weierstrass's notion of an analytic element to include ramification points. Then he constructed a Green's function for his surface, using his *méthode de balayage* and simplifying it using Harnack's theorem. From the Green's function he then deduced the existence of a function that mapped the Riemann surface conformally into the unit disc. He then compared the different

functions that could be used to this end, and showed that they are all linearly related. He could then show, following earlier work of Osgood, that the conformal representation was indeed onto the interior of the disc.

His definition of a Riemann surface, D, implied that it was covered by a countable number of open discs, D_n, each the domain of a convergent power series. For such a surface he proceeded as follows. He defined a function that was positive at all but one point, O, of the first disc, D_0, where it was logarithmically infinite, and which was zero on the boundary of D_0. He extended this function to the rest of D by defining it to be zero outside D_0, so it is continuous away from O, but its derivatives are not continuous on the boundary of D_0. He then defined a function u_{n+1} inductively, using his *méthode de balayage*. In this way he obtained at stage $n+1$ a function that was generally positive on the domain consisting of all the discs contiguous with the discs encountered at stage n. The points of discontinuity in the derivatives are the boundary points of the domain of definition of u_{n+1}. The function was harmonic everywhere it was positive except at O. To apply Harnack's theorem it was therefore enough to ensure that there was a point, P, of the domain D at which the sequence of function values $u_n(P)$ did not increase indefinitely. The sought-for convergence was ensured by introducing a suitable majorising function, which Poincaré found amongst the classes of Fuchsian functions he had studied at the start of his career. It followed that the sequence (u_n) tended to a Green's function on D.

From the Green's function, which is harmonic inside D except at the point O, Poincaré could obtain by repeated use of Harnack's theorem the function called v in his memoir of 1883, and thence the function $z = e^{-(u+iv)}$ that mapped the domain D into the unit disc. He also showed that if instead of the arbitrarily chosen point O he had begun with another, O' say, and had been led to a function z', say, then the functions z and z' would be connected by a linear relation of the form

$$z' = \frac{a'}{\rho}\,\frac{z - \rho e^{i\phi}}{\rho z - e^{i\phi}}$$

where $a = \phi e^{i\phi}$ is the value of z at the point O', and a' is the value of the function z' at the point O.

It remained to make sure what the memoir of 1883 had left obscure: the function z mapped the domain D onto the interior of the unit disc. Poincaré gave two proofs of this result, one following Osgood's and one of his own; they will not be discussed here. The memoir then proceeded to analyse the cases when a given simply connected region is to be represented conformally not on a circle but on the whole plane, and showed how these cases could be distinguished.

Poincaré's paper came out at the same time that Koebe began his work. Koebe was a student of Schwarz's in Berlin, and wrote his doctoral dissertation in 1905. He then embarked on a lengthy series of papers which quite deliberately and successfully brought him to the attention of the leading German mathematicians. He published at length in the *Göttingen Nachrichten*, making sure his papers were presented by Hilbert and Klein, in the *Journal für Mathematik*, in *Mathematische Annalen*, and the *Comptes Rendus*. In these papers he solved the problem of uniformising algebraic and analytic curves, the Riemann mapping theorem and its generalisation to non-simply connected domains, and then turned to rescue the old continuity method of Klein and Poincaré [12]. He responded rapidly to the work of anyone else who strayed into the area: Poincaré, the Finnish mathematician Johansson, Hilbert, and others. Doubtless as a result of all this activity he was invited to speak to the International Congress of Mathematicians in Rome on the subject, quite an honour for a young man.

In the second of these papers, [1907b] he observed that Poincaré's original method had the defect that "certain points of the domain must be excluded which cannot necessarily be excluded in the nature of the problem". He then outlined his own approach to the uniformisation theorem for analytic curves. The main result he established is that the interior of any Riemann surface over a simply connected domain in the plane may be mapped one-to-one and conformally onto one of three regions on the sphere: the entire

[12] For an account of the implications for the theory of manifolds, see Scholz [1980].

sphere, the sphere minus a point, or a disc. So his proof of the uniformisation theorem includes the Riemann mapping theorem as a special case. The paper concludes with a very interesting comment. After noting the work of Runge on the representation of a function by a series of rational functions, Koebe wrote

> The idea of an analytic domain *(analytisches Gebilde)* with an independent variable is connected by Weierstrass with the representation of infinitely many uniformly convergent series with rational terms. Each series represents only one part of the domain, and among the series there is always at least one defined on the neighbourhood of any given point of the interior. The fundamental problem is to find a selection of these uniformly convergent series of rational functions which represents the whole domain. Here it is shown that this can always be done. In this way a problem which one might say belongs to the Weierstrassian mode of analysis is solved by principles which belong to the Riemannian circle of ideas. (1907b, 210)

Koebe also took note of some other entrants in what he plainly saw as a competition to establish the uniformisation theorem. One was T. Brodén, who had published a book on the subject in 1905. It is doubtful if Koebe had seen this book. Another, more serious, contender, was the Finnish mathematician Severin Johansson, who had published a version of his doctoral thesis [1905] and then reworked parts of it for two papers in the *Mathematische Annalen* [1906a, b]. Koebe observed that the chain of reasoning in the [1906b] was flawed.

In his [1907c] Koebe compared his approach with that of Poincaré. He noted that he avoided the use of modular functions entirely, unlike Poincaré, Johansson, and Osgood, which he felt had led to a notable simplification of the argument. Like Poincaré, he had relied on Schwarz's methods, and he had made a modest use of Harnack's theorem, upon which Poincaré had relied heavily. The comparison inspired Koebe to give a new proof of his theorems the next year, completely avoiding Harnack's theorem. Johansson for his part observed that he had been unaware of Brodén's work until Klein had shown him a copy, when he found it unoriginal. It contained little, he remarked, other than a proof along the lines of Poincaré's [1883] together with a proof that the map thus obtained is indeed onto the unit disc, which he (Johansson) had proved in his thesis.

Koebe's energy drove him to give several proofs of his results. When Hilbert revived Dirichlet's principle by establishing a rigorous, and somewhat different, minimising principle, Koebe responded with an explanation of how these ideas lay close to his own. When others gave simpler proofs using techniques drawn only from complex function theory, Koebe showed that he too could operate in that way. Knowing that Fricke was at work on improving the original approach to the uniformisation theorem due to Poincaré and Klein (the continuity method) Koebe showed how that too could be rigorised, thus entering Brouwer's territory and seemingly extending Brouwer's proof of the invariance of dimension. It would therefore be difficult to summarise his work, and it seems better to be selective.

No-one disputed the rigour of Koebe's methods. His division of the uniformisation theorem into two parts was also accepted. The first part is topological and asserts the existence of a simply connected covering surface for any Riemann surface. The second part is analytic and asserts that the map from this covering surface to exactly one of three surfaces (the sphere, plane, or disc) is analytic. Indeed, his proofs of the uniformisation theorem were usually taken as definitive; Fricke acknowledged "with the greatest thanks" not only Koebe's papers but "the many hours of conversations on a whole series of points" that had helped him write his [1912]. The chief response of several authors despite, or perhaps because of Koebe's work, was the desire to give short direct proofs of what seemed buried under the torrent of his papers, many of them very long. In particular, it seemed worthwhile to rescue the Riemann mapping theorem, as we shall see below.

As a sample of Koebe's methods, one could do worse than select Koebe's distortion theorem *(Verzerrungssatz)* proved by him in his [1909] and again in his [1910]. There he stated it this way:

If $f(z)$ is a one-to-one analytic function of z on $|z| < 1$, and z_1 and z_2 are two points in $|z| \leq q < 1$, then there is a real constant $Q > 0$ independent of f such that

$$\frac{1}{Q} < \left|\frac{f'(z_1)}{f'(z_2)}\right| < Q.$$

Koebe proved it by a series of lemmas concerning functions $f_1(z)$ of z on $|z| < 1$ which are of the form $f_1(z) = \frac{1}{z} + f(z)$, where $f(z)$ is an analytic function of z on $|z| < 1$. The first asserted that the maximum modulus M_ρ of f_1 on the circle $|z| = \rho$ is less than some quantity that depends on ρ but not the function f_1. From this it followed that if the function $f(z)$ was one-to-one and $0 < q < 1$, then there were two real constants g_1 and g_2 independent of the function $f(z)$ such that $|z| = q$ implies $g_1 < |f(z)| < g_2$. The distortion theorem itself followed (not completely directly) from this lemma and the Cauchy integral theorem applied to the derived function $f'(z)$. A special case of the distortion theorem would be when $z_2 = 0$ and $f'(0) = 1$, when it asserts the existence, for all such functions f, of a real non-zero constant Q such that $\frac{1}{Q} < |f'(z)| < Q$. Just as the Schwarz lemma gives information about the function $f(z)$, Koebe's distortion theorem is informative about the derivative of a large class of functions.

A more artificial example would be Koebe's proof that the simply connected cover of a Riemann surface (with a Fuchsian group of covering transformations) can mapped conformally and one-to-one onto the unit disc and that the map so obtained extends to the boundaries. His proof was based on an exhaustive argument using Green's functions. Each partial domain was mapped by a function of the form $u_n = \log \frac{1}{r} + c_n +$ a regular function vanishing at the origin. The difference $u_{n+1} - u_n$ is everywhere analytic , and from the fact that the boundary values are non-negative it follows that

$$c_n < c_{n+1}.$$

To obtain an upper bound for the sequence of c_n's Koebe employed a majorising argument. He then set $\lim_n c_n = c$. Knowing that this sequence was bounded he could then deduce the uniform convergence of the sequence of functions u_n to a function u. After this, the theorem (a special case of the Riemann mapping theorem) followed in the by-then standard way. The point of this manoeuvre was spelled out in a footnote (p. 208): it enabled one to avoid Harnack's theorem "which I had used in earlier work . . . . but which seemed to me with the benefit of a certain hindsight to be an essential completion if not, perhaps a conceptual simplification." This is true, but later workers were to object more radically to the continued presence of potential-theoretic ideas in a fundamental theorem of complex function theory.

As for people's response to Koebe's work, Osgood's response to the the proof of the uniformisation theorem may stand for many among that generation. Osgood (like Fubini [1908]) adopted Koebe's new argument about Green's functions. He summarised Koebe's papers ("a task", he drily noted, "of some labor"), in second edition of his *Funktionentheorie*, vol 1, and in a paper [1913]. To prove the algebraic case, Osgood explained, Koebe began by taking the Riemann surface F corresponding to the algebraic function, cutting it up to obtain a simply connected region F_1, and the successively joining copies of F_1 along the cuts in the usual way to obtain a sequence of surfaces Φ_n, among which $\Phi_1 = F_1$. Each domain Φ_1 can be mapped onto a plane region by a one-to-one, continuous, and conformal map $F_n(z)$ onto the extended t-plane from which some slits have been removed corresponding to the boundary curves of Φ_n. An arbitrary point of Φ_1 may be mapped to $t = \infty$ in the extended t-plane. The existence of such a map was shown by standard Green's function arguments.

The extension of this process to the region Φ which is the limit of the regions Φ_n is naturally much more delicate and profound. The functions

$$f_n(z) = \frac{1}{F_n(z)} \quad z \neq 0 \quad f_n(0) = 0$$

also map the interiors of the domains F_n in a one-to-one, continuous, and generally conformal way onto the finite part of the t-plane.

Moreover,

$$f_n(0) = 0, \quad f_n'(0) = 1.$$

Koebe showed that it was possible for each n to find a sequence of functions $f_{n_j}(z)$ which converged uniformly on Φ_n. Osgood simply regarded this as a consequence of Montel's theorem (Montel, [1907]). Koebe gave a whole series of names, but not references, going back via Arzelà and Montel to Ascoli (with the date this time, 1883) while still indicating some originality for himself. The indices n_j are independent of n, and the limiting function

$$f(z) = \lim_{j \to \infty} f_{n_j}(z)$$

is uniquely defined at each point of Φ and maps Φ onto a single-leaved region of the t-plane with a discrete boundary. The proof hinged on the distortion theorem, which Osgood stated in this form: a function which maps a circle $|z| < \rho$ one-to-one onto a region of the extended plane not containing ∞ in its interior and which satisfies $f(0) = 0$ and $f'(0) = 1$, remains finite in the circle $|z| \leq \frac{\rho}{k^2}$, where k is a constant independent of the choice of function f. The constant was later called Koebe's constant (the name is due to Osgood, *Funktionentheorie*[2], 727), and shown to be greater than or equal to $\frac{1}{4}$ by Bieberbach [1916]. More precisely,

$$|z| \leq \frac{\rho}{k^2}, \text{ implies } |f(z)| \leq \frac{\rho}{k},$$

and

$$|z| \leq \frac{\rho}{2k^2}, \text{ implies } |f'(z)| \leq 4k.$$

The distortion theorem enabled Osgood, following Koebe, to prove that the functions $f_{n_j}(z)$ remain finite.

Such a summary is entirely fair to Koebe, and indicative of the importance attached to his work. It is also true, as others pointed out, that Koebe's distortion theorem is of independent interest. A further proof of it was given by Study and Blaschke, following Osgood and using the Cauchy integral theorem, in their [1912]. We shall see below that the theorem was particularly appreciated by Bieberbach. But it is indicative of the stir occasioned by Koebe's work that a special meeting of the Deutsche Mathematiker Vereinigung was held (in Karlsruhe, 27 September 1911) to discuss recent work on automorphic functions, and a report published the next year in the *Jahresbericht*, **21**, 153-166. Brouwer spoke on his proof of the invariance of dimension under a one-to-one, continuous map, with reference to the Fuchsian case. Koebe replied that he had been able to extend this to other cases, and gave a Schottky-type example, where, he said, Poincaré's methods could not work. He then gave his own report on the uniformisation theorem for both analytic and algebraic curves. Bieberbach reported briefly on single-valued automorphic functions, and Hilb on many-valued ones. Klein summed up, expressing the view that "one must learn to calculate with single-valued automorphic functions as easily as one can calculate with elliptic functions", and referred in the spirit to his old paper on *Primformen* [1889].

5. Hilbert and Courant.

There can be little doubt that an important stimulus was Hilbert's mention of the uniformisation problem among the 23 problems he singled out on the occasion of the International Congress of mathematicians in Paris in 1900 and his own related work in the years immediately following. In the course of his address on mathematical problems at the Paris ICM, 1900, Hilbert held out the hope that his approach to Dirichlet's principle could be extended to more general boundary value problems, such as those where a condition on the derivative is specified, or where it is not a potential function that is involved, but, say, the minimal surface equation. He also reminded his audience of another problem intimately connected to the Riemann mapping theorem: the uniformisation problem. His comments formed part of the tenth and last of his commentaries in the oral address, and the 22$^{\text{nd}}$ of the 23 in the written paper. The problem may well have been this prominent because Hilbert was then actively at work on the problem himself.

In his [1904] Hilbert first restored Dirichlet's principle, at least for boundaries that had smoothly varying tangents and curvature and where the function given on the boundary is itself differentiable. In his [1905] Hilbert gave the details of the proof for any region that could be approximated by a net of rectangles. As he said, the main advantage of the proof was that it used only the minimum property, thus rescuing not the conclusions so much as the original transparency of the proof. This was, as he pointed out, further evidence for the point raised in the 19$^{\text{th}}$ of his Paris problems that certain problems in the calculus of variations seemed to have solutions which were much more differentiable (even to the point of being analytic) then was to be expected.

In his [1909] Hilbert took the occasion of Poincaré's visit to Göttingen in April (where he gave a series of six lectures) to apply his new methods directly to the Riemann mapping theorem for any domain. The domain could have finitely or infinitely many leaves, finite or infinitely many branch points, and arbitrary boundary curves and boundary points. He modified the approach of Harnack and Osgood by looking at functions which were infinite at a single interior point of a given domain like $\frac{x}{x^2+y^2}$, and he modified the Dirichlet integral slightly (in a way not to be described here). He then showed that among all functions defined on a domain that are continuously differentiable except at one interior point, where they are infinite in the specified way, there is one minimising the value of the (Hilbert)-Dirichlet integral. This function is harmonic except at the origin and can then taken as the real part of a complex function mapping the domain conformally onto a canonical domain in the (x, y) plane. Which domain depends, in a way Hilbert described, on the original domain.

Hilbert published his lecture in the *Göttingen Nachrichten* on 17 July 1909. Koebe, who had heard the lectures, could not let such an opportunity of his own slip by. The very next paper in the *Göttingen Nachrichten* after Hilbert's is one by Koebe, dated 31 July, [1909b], in which he showed how to pass directly from Hilbert's construction to one that yields a Green's function (one with a logarithmic infinity).

Although they lie askew to the subject of this paper, mention should briefly be made of Hilbert's student Courant, who published two papers closely related to his mentor's work. The first [1912] was based on his inaugural dissertation at Göttingen in 1910 and showed how to achieve the conformal representation of certain multiply connected regions on disc-like domains and how to solve the uniformisation theorem for algebraic functions by automorphic functions of Schottky type. The second [1913] simplified Hilbert's proof of the Hilbert-Dirichlet principle by showing that it was enough to use rectangles because the sought-for minimising function is also a minimum among all piecewise twice differentiable functions.

6. Carathéodory, Bieberbach, and others.

Courant was not the only doctoral student at Göttingen to take up the Riemann mapping theorem around 1910. So did Bieberbach and Carathéodory. It was also discussed in Study and Blaschke's book [1912]. All sought to give proofs of the Riemann mapping theorem entirely within the spirit of complex function theory. They did not dispute that the results had been proved, by Koebe and others. But while Carathéodory discretely noted that the role played by Harnack's theorem could be played by Schwarz's lemma [1912, 109], Bieberbach more forcefully commented [1915, 95] that he would not discuss Hilbert's work, "Nor shall we describe the potential-theoretic methods that Riemann's successors developed as a substitute for Dirichlet's principle. We shall rather deal entirely with purely function theoretic methods."

Carathéodory was a former student of Klein's who was encouraged by Klein and Hilbert to work on questions in complex function theory. The result was his series of several papers on the Riemann mapping theorem. In his paper of 1912 Carathéodory divided the Riemann mapping problem into two parts: the interior problem; and the existence of a continuous extension of that map to the boundaries. In the first of the papers of 1913, Carathéodory gave two reasons why the Riemann mapping theorem had once again become worth studying: there had been an important realisation that

the sensible way to specify the sought-for conformal map so that (were it to exist) it would be unique, was to specify the both value it took at an internal point and also the sign of the derivative there. Riemann had spoken about points on the boundary, but it is exactly the behaviour of the map on the boundary that is hard to understand. This realisation led various mathematicians to contemplate a process of pushing out the conformal map from the interior to the boundary (as we saw with Osgood and Poincaré, and as was still more the case with Koebe's and Bieberbach's approach). Caratheodory's second reason was the advent of Lebesgue's theory of integration, and in particular consequences drawn from it by Fatou in a paper of 1906. As we shall see, Koebe disputed the significance of this point.

In the first of his papers, Carathéodory solved the interior problem, i.e. the Riemann mapping problem for an arbitrary simply connected domain having at least two boundary points. His is the first truly function theoretic proof. He eliminated Osgood's use of Harnack's theorem by an appeal to Schwarz's lemma, which he was the first to call by this name, and which he located in Schwarz's *Gesammelte Abhandlungen*, vol 2, p. 109. He stated it this way: if $f(z)$ is an analytic function, regular on $|z| < 1$, for which $f(0) = 0$, and $|f(z)| \leq 1$ then $|f(z)| < |z|$ for all $|z| < 1$ unless $f(z) = e^{i\theta}z$, in which case $|f(z)| = |z|$. He gave this result the simplest proof he said he knew: the function $f(z)/z$ is regular and analytic on $|z| \leq 1$, so it takes its maximum value on $|z| \leq \rho \leq 1$ on the boundary, so

$$|f(z)/z| \leq 1/\rho$$

whence

$$|f(z)| \leq |z|/\rho$$

whence, in particular

$$|f(z)| \leq |z|$$

and equality can only hold when $f(z) = e^{i\theta}z$.

Carathéodory based his solution on the resolution of the following question: Let (G_n) be a sequence of infinitely many domains in the u-plane that all contain $u = 0$ as an interior point, and all lie in the disc $|u| < M$, and let $f_n(z)$ be a sequence of analytic functions that represent the domains G_n conformally on the interior of the unit disc in such a way that the points $u = 0$ and $z = 0$ always correspond and $f'(0)$ is always real and positive. What necessary and sufficient conditions must the domains G_n satisfy for the functions $f_n(z)$ to converge with increasing n to a limit function and what then will be the properties of this function? He was proud of the fact that his solution relied entirely on purely function-theoretic methods, rather than those of potential theory.

Carathéodory began by following Poincaré in reducing the proof of the claim that the interior problem has a unique solution satisfying the constraints on $f(0)$ and $f'(0)$ to a simple application of Schwarz's lemma. He then turned to the above question about G_n and f_n. Using Montel's theorem, Carathéodory showed that the limit function was everywhere a conformal and one-to-one map of some domain G onto the interior of the unit disc.

What could be said about Γ? Carathéodory showed that any closed domain H lying inside Γ and containing the point $u = 0$ necessarily lay inside all G_n for some suitably large n. Moreover, if Γ_n was another domain such that any closed set it contained also necessarily lay inside all G_n for some suitably large n, then Γ_1 lay inside Γ, so Γ was the largest domain with this property. Carathéodory called it the kernel [*Kern*] of the sequence of domains (G_n) and observed that its definition was entirely set-theoretical (we should say topological). Plainly, the limit function $f(z)$ of the previous paragraph provides a conformal representation of the interior of the unit disc onto the kernel K. This concludes his solution of the Riemann mapping problem by purely function-theoretic means. Carathéodory then extended the theorem to cover domains given as branched coverings, to domains bounded, for example, by simple closed Jordan curves, and to domains with much worse boundaries, such as one described by Brouwer [13].

In the first paper of 1913 he proved the conjecture of Osgood's,

[13] In which one originally disc-shaped domain is wrapped infinitely many times around a fixed disc.

that the conformal map extends to a homeomorphism of the boundaries if and only if the boundary is a simple Jordan curve. In Caratheodory's opinion, when Osgood made his conjecture a proof might have seemed unattainable. That a proof could now be given, he went on, was due to the far-reaching discovery of the Lebesgue integral and the theorems that Lebesgue had been able to prove with its help.[14] Carathéodory began proving two lemmas. The first relied on this theorem of Fatou's: given a single-valued analytic function defined and bounded on the interior of the unit disc, there is an everywhere dense set of points p on the boundary of the disc with the property that as z tends to p along a radius, $f(z)$ converges to a definite value.[15] Carathéodory deduced the corollary that if f is not constant then it takes at least three values on any boundary arc. He assumed first of all that the function is not constant. On the circle of radius ρ the function

$$F(\rho, \theta) = \int_0^\theta f(\rho e^{i\theta}) d\theta = \int_\rho^{\rho e^{i\theta}} \frac{f(z)}{iz} dz$$

is well-defined because $\dfrac{f(z)}{iz}$ is analytic. Moreover, by Schwarz's theorem, this quantity is also bounded by M, say, for $|z| < 1$. Consequently,

(i) $$|F(\rho + \Delta\rho, \theta) - F(\rho, \theta)| \leq 2M\Delta\rho,$$

and

(i) $$|F(\rho\theta + \Delta\theta) - F(\rho, \theta)| \leq |M\Delta\rho|,$$

Consequently, $\lim_{\rho \to 1} F(\rho, \theta) = F(\theta)$ exists for every θ and the convergence is uniform, so $F(\theta)$ is continuous and (ii) implies that it even satisfies a Lipschitz condition (it has a bounded differential quotient everywhere).

[14] Lebesgue [1902].

[15] Fatou [1906].

A Poisson integral argument then showed that at every point θ_0 at which $F(\theta)$ was differentiable, as $z = re^{i\theta_0}$ tended to $e^{i\theta_0}$ along the radius, $f(z)$ tended to $F'(\theta_0)$. It is a theorem of Lebesgue's that a real function with bounded differential quotient is differentiable almost everywhere and (up to a constant) equals the indefinite integral of its derivative. Evidently the function $F(\theta)$ is such a function, so the function $f(z)$ converges almost everywhere to a definite value on the boundary, thus proving the first half of the theorem. To prove the corollary, Carathéodory assumed that along the arc $\theta_1 < \theta < \theta_2$ the function $f(z)$ took only two values, α and β. Then the function

$$g(z) = \frac{if(z) - \alpha}{\beta - \alpha}$$

takes the values 0 and i only. A Poisson integral argument then showed that the real part of $g(z)$ is analytic along the arc $\theta_1 < \theta < \theta_2$ and so, by the Schwarz reflection principle, the functions $g(z)$ and $f(z)$ are analytic, which contradicts the assumption on the number of values of $f(z)$.

Caratheodory's second lemma was that if the boundary of the domain contains a free arc, then a conformal map of this domain onto a disc necessarily maps this arc continuously onto a piece of the unit circle and leaves the end-points distinct. (A free arc is one for which a sufficiently small circle around any point of the arc is divided by the arc into two sectors one of which lies wholly in the domain). He derived this result from the Schwarz reflection principle.

Carathéodory then considered regions bounded by a simple closed Jordan curve. He referred to Brouwer for the proof that such a curve divides the plane into two parts, of which one (called the interior) is finite.[16] He thought of the finite part as lying in the plane of the complex variable u, let O be an arbitrary point of the interior, and let f be the unique conformal map of the interior onto the interior of the unit disc, sending the point O to the point $z = 0$ and having a real positive derivative there. He then showed (by a Fatou-type argument) that the conformal map associates to each point

16 Brouwer [1909].

A of the Jordan curve a point A_1 on the boundary of the unit disc with the property that to any sequence (u_n) of points tending to A the corresponding sequence of points (z_n) tends to A_1. It follows that the map of the boundaries is continuous. Next Carathéodory showed that the map on the boundaries is one-to-one. This followed from his observation about free arcs. Thus he established that a conformal map between a type one domain and the disc extends to an invertible continuous map of the boundaries. (Such a map is necessarily a homeomorphism).

In his third paper [1913b] he discussed what can happen when the boundary is not such a curve. This paper is often taken to inaugurate the theory of prime ends, although, as we have seen, the origins of such a theory are to be found in the work of Osgood and, as Carathéodory was happy to acknowledge, related ideas were to be found in the book by Study and Blaschke. Carathéodory began by following Harnack in defining carefully what a simply connected domain is (one in which every two points can be joined by a finite polygonal arc lying entirely in the domain and such that every closed polygon lying in the domain encloses only points of the domain). Again like Harnack he restricted his attention to bounded domains. A boundary cut (*Querschnitt*) in the domain was a Jordan arc joining any two boundary points; an incision (*Einschnitt*) a Jordan arc from a boundary point and otherwise lying in the domain. A careful discussion of connectedness followed. Then Carathéodory defined a chain of boundary cuts q_n as cuts with no points in common such that q_n separates q_{n-1} from q_{n+1}. The domain bounded by q_n that contains q_{n+1} he called g_n. This chain of domains g_n defined an *end*, a concept that he defined axiomatically (following the suggestion, he said, of E. Schmidt). Informally, the end E_g defined by a chain of domains g_n is the intersection of every domain H that contains the interior of some g_n. A sequence of points was said to converge to an end when only finitely many points of the sequence lie outside the domains g_n that define the end.

It followed that an end E_g is a subset of an end E_h if every series of points converging to the end E_g also converges to the end E_h.[17] Ends were said to lie outside one another when there was no sequence of points converging to both of them. So ends that lie outside one another have no common subset. On the other hand a sequence of ends E_n with the property that each E_{n+1} is a subset of E_n has a non-trivial intersection. So an end consists either of isolated points or else is a perfect, connected set. Finally, Carathéodory defined a prime end to be an end which has no end as a subset. He showed that every end defined by a chain of boundary cuts converging to a point is a prime end, that two distinct prime ends lie outside one another, and that two ends which do not lie outside one another either have a common point in the interior of the domain or have at least one prime end in common. It follows that prime ends cannot contain interior points of the domain.

Thus prepared for every topological subtlety, Carathéodory proceeded to give the first thorough investigation of the boundaries of two conformally equivalent domains, one of which, for simplicity, is the unit circle. He showed that any sequence of subdomains converging to a prime end is mapped to a sequence of discs converging to a point on the unit circle, and that the converse is also true. So the map of the interiors provided by the Riemann mapping theorem extends to a one-to-one map of prime ends on the boundaries. This was the final resolution of a problem originally raised by Riemann. In its mixture of ideas drawn from topology and complex function theory it is in many ways closer to the spirit of the original than had been many of the intervening papers.

Such success rapidly drew a response from others in the area. Koebe wrote a short, three-page, note [1913] disputing the need for Lebesgue's theory. Instead he showed how to generalise a theorem of Schwarz to the same effect. This theorem is the result that an analytic function defined on some domain and constant on an arc in the interior of the domain is necessarily constant. The generalisation was to an arc on the boundary of the domain, and asserted that

[17] Carathéodory used the word divisor *(Teiler)* for subset, perhaps because *Teil* means part, perhaps because it means divisor, and in Riemann surface theory divisors are subsets.

if f is an analytic function and such that f tended to a constant as z tended to an arc in the boundary then f was a constant. A similar result was claimed independently by Osgood and Taylor in their [1913]. In Koebe's view, the matter was very simple and the introduction of Lebesgue's ideas appeared to be a significant complication. Carathéodory himself published a short note [1913] showing that a special case of Fatou's theorem could be proved without Lebesgue's theory, but he generally remained of the opinion that progress required it.

Bieberbach in Berlin wrote a short paper [1913] to show how Carathéodory's reliance on Schwarz's lemma was in his view excessive, and how the theory could be simplified and extended by using only Montel's theorem. The next year he wrote a paper [1914] simplifying Carathéodory's work by invoking another maximum principle. This principle picked out among all (suitably normalised) regular mappings of a given simply connected domain the one that minimised the area of the image and asserted that this minimising function was also the one that mapped the domain conformally onto a disc. He argued that the minimising function could be shown to exist by invoking the solution to the Riemann mapping theorem, and said that an independent proof of this fact would be worth having. But, granting its truth for the moment, the principle could be used to simplify Carathéodory's work, as he proceeded to show. Reversing his criticisms of only a year before, he now showed how the theory could be freed of any reliance on Montel's theorem. One claim that he made for this work was that it showed one way in which the ideas of Ritz could be made of practical use.

Koebe found once again that he could not resist the opportunity provided by Carathéodory's papers to go back to some old ideas of his own and extend them [1912]. He was led in this way to offer what he called his 'squeezing method' (*Schmiegungsverfahren*) for solving the Riemann mapping theorem by nothing more than the repeated taking of square roots. This was to prove his most acceptable presentation, being in some sense entirely elementary. Carathéodory incorporated these criticisms into his paper for the Schwarz *Festschrift* [1914], which was to remain his final account until the newer methods of

Perron were introduced.

The heart of Bieberbach's monograph on the subject of conformal mappings was also closely based on Koebe's *Schmiegungsverfahren*, which he described this way. The domain G to be mapped conformally onto the unit disc D may be assumed to have at least two boundary points. It can therefore be mapped onto a 2-leaved Riemann surface with these two points as branch points, which can in turn be mapped onto a schlicht plane in such a way that a piece of the plane does not lie in the image. By a further map if need be it can therefore be assumed that the initial domain G lies inside the unit disc and has the origin, $z = 0$ as an interior point. The method then proceeds iteratively, squeezing out the image until it covers the interior of D.

Let the point $z = a$ be one of the boundary points of G nearest to the origin. Consider a piece of a 2-leaved Riemann surface R_1 that has $z = a$ as its sole branch point, that is spread out over D and whose boundary covers the unit circle twice. The domain G will be thought of as lying entirely in one leaf of this surface. The quadratic map

$$z = \sqrt{\frac{\bar{A}}{A} w \frac{A - w}{1 - \bar{A}w}}$$

maps the domain G onto a new domain G_1 that also lies in D. By the Schwarz lemma applied to the inverse function, every boundary point of G_1 lies further out than the corresponding boundary point of G.

This process, continued indefinitely, maps the domain G into (the interior of) the unit disc. To show that the boundary points of G are mapped onto the unit circle, Bieberbach observed that the sequence of radii a_n such that $|z| < a_n$ is the largest disc lying entirely in G_n is certainly increasing. Indeed, it followed from his use of the Schwarz lemma that $\frac{a_{n+1}}{a_n} > 1$. On the other hand, because G_n lies in $D, a_n < 1$. So $\lim a_n = 1$, and the limiting map is onto D.

That limiting map of G onto D is the pointwise limit of the maps $f_n : G_n \to D_n$ followed from a more delicate convergence argument using Bieberbach's observation about areas. The map is not a constant because the limit of the sequence $\phi'(0)$ was shown to be

non-zero. Another delicate argument showed that the functions $f_n(z)$ and therefore $f(z)$ are one-to-one and analytic; indeed, they are not far from being constant functions.

The last thing to check is that the map of G to D is onto, or equivalently that the inverse map from D to G is onto. Bieberbach remarked that it would be possible to show this by repeated application of the Schwarz lemma, but that it seemed preferable to use Koebe's distortion theorem. That theorem establishes upper and lower bounds on the quotient of the derivative of a map at two points. Since integrating a derivative yields arc length, Bieberbach, by fixing one point as 0 and letting the other vary, could interpret the distortion theorem as giving upper and lower bounds on the ratio of the lengths of an arc and its image. So if the inverse map from D to G is not onto then some path can be found that shrinks indefinitely as n increases while its image is bounded. This contradicts the consequence of the distortion theorem, and so the Riemann mapping theorem is proved.

Further evidence of the nature of Koebe's impact and the intense discussions that were evidently going on around this time is provided by the book Study wrote on the subject with Blaschke in 1912. They circulated the manuscript to Carathéodory, Schmidt, and Koebe, who approved it, and it contains ideas that these authors were often themselves only in the process of putting into print. Study and Blaschke fixed their attention on domains (defined very carefully to be point sets such that every point of the set lay in a disc lying entirely in the domain) which had only finitely many branch points, all of finite order. Their approach was essentially Koebe's, and so potential-theoretic but avoiding Harnack's theorem. But where they were most innovative was in the profusion of boundary curves they indicated any theory would have to account for. Here they mentioned von Koch curves, nowhere differentiable Jordan curves, various types of spirals, and, for example, a rectangle from which the left-hand edge has been removed together with two disjoint sets of vertical lines as follows.

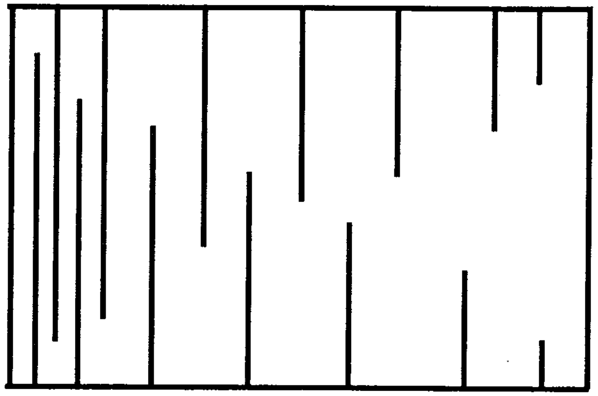

Fig. 3

One set hangs downwards, and is attached at a discrete set of points accumulating at the left-hand end. The segments grow in length from right to left. The other set is similar but is attached on the bottom edge and points upwards. The result is that the "missing" left hand edge is in the closure of the set of all these incisions. Thus motivated, they discussed Carathéodory's theory of ends. Finally, towards the end of the book they returned to what they could have discussed earlier and gave what they described as essentially Osgood's proof of Koebe's distortion theorem.

Independent of these German authors, Osgood returned to the scene in a paper he wrote with E.H. Taylor [1913]. For the interior Riemann mapping theorem they relied on the theory of logarithmic potential functions and claimed to offer little novel except in the rigour of their proofs, to which, they said, they had been led by intuition, "an harmonic function being thought of as the temperature in a flow of heat or the potential in a flow of electricity". The chief result concerned the nature of the mapping at accessible points on the boundary. They called a point "accessible" if it could be approached by a curve lying entirely in the domain. They showed that if A was an accessible point of the boundary that was approached by a curve C lying in a domain S and this domain was mapped conformally and one-to-one onto a domain S', then the image of the curve C is a curve with the image of the point A as its unique boundary point, and that if a point P travels along the curve C to the point A, then the image

of P travels along the image of C to the image of A. It follows that distinct accessible points are mapped to distinct accessible points and that if the boundaries are simple Jordan curves then the conformal map of the domains extends to a continuous one-to-one map of the boundaries. However, even if there are inaccessible boundary points, the map of the boundary may be one-to-one on the accessible points, as Osgood had noted before [*Funktionentheorie*^{2}, 154].

4. Conclusion.

While the study of the Riemann mapping theorem and the uniformisation theorem did not end here, and simpler proofs were to be given by Perron and Heims (see Ahlfors [1973]), the coincidence of the complete resolution of the problems using only the techniques of complex function theory and the outbreak of the First World War make the 1910s a natural terminus. The importance of all this work for the development of complex function theory is two-fold. On the one hand, it led to the first rigorous proofs of two of the subject's most important theorems. As it unfolds one can see the gradual acceptance of Riemann surfaces as the natural domain of definition of a complex function, rather than (parts of) the plane. This represents a victory, one might say, for Riemann's stand-point over that of Weierstrass's. Furthermore, it enabled mathematicians to explore the relative merits of methods drawn from potential theory and from pure function theory. In this way they came to a deeper appreciation of some of the early lemmas of Schwarz. On the other hand this work was also a proving ground for potential theory in its own right. Most importantly, the work was a focus for refining ideas in the topology of sets of points. That the boundary of a domain (a simply connected open set) could be as strange as Osgood and Carathéodory had discovered was a powerful stimulus to precision in these matters. It would be reasonable to claim that in proving the Riemann mapping theorem and the uniformisation theorem mathematicians came to unite geometric function theory with topology, to the mutual advantage of both subjects.

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