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THE UNIFORMIZATION THEOREM

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Almost one hundred years have passed since Felix Klein discovered the uniformization theorem. While such theorems had earlier been proved in specific cases, no one had dared even conjecture that every compact Riemann surface could be parametrized by a variable whose

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HURWITZ: I accept it without reservation.
SCHWARZ: It's false.
POINCARÉ: It's true. I knew it and I have a better way of looking at the problem.

However, the story does not end there. Schwarz recanted shortly thereafter. He made two great contributions to the theory of uniformization. The first was a method for proving the theorem. The second was to establish the relationship between uniformization and the study of conformal representation of Riemann surfaces on $\hat{\mathbb{C}}$. The link is provided by the theory of covering spaces which Schwarz developed specifically to study the uniformization problem. In 1907, Poincaré proved the most general known uniformization theorem for the case where the parameter varies over a simply connected domain.

The result, which is now called the uniformization theorem, is a problem in conformal mapping which is solved by potential theory. It is related to the original problem by algebraic topology. Of the major mathematical disciplines, few have not been enriched by the uniformization theorem or the methods developed for the study of the problem.

Research in uniformization theory has gone through several dormant periods since the concept of a (global) uniformization was introduced by Klein in 1882. Uniformization theorems of power unimaginable to the classical masters have been proved in the past twenty years and no end is in sight.

My original purpose in writing this paper was to put into print a relatively efficient and elementary proof of half of the classical uniformization theorem which I have circulated privately for several years. Ralph Boas suggested that I write it in a more expository form. I have taken this suggestion as an invitation to exercise my personal fascination with the various facets of the uniformization problem. To do historical justice to the problem, I was required to trace the development of the notion of a Riemann surface; for as this notion matured, so did the statements and proofs involved in the solution of the uniformization problem. The discussion in the text contains an outline of this proof with the details placed in an appendix. For completeness I have also included a brief sketch of Maskit's work on the general uniformization problem.

In translating heuristic arguments into mathematics, I shall often refer the reader to Ahlfors's *Conformal Invariants* [**2**], which will be abbreviated as CI.

I would like to express my thanks to Lars Ahlfors, Lipman Bers, Jozef Dodziuk, and Bernard Maskit for their comments, many of which are incorporated in the ensuing text. The figures, which, in a sense, are the soul of Riemann surface theory, were designed and drawn by George Francis. My debt to him is clear.

1. An Example. Consider the variety S in $\mathbb{C}^2$ defined by the equation $X^2 + Y^2 - 1 = 0$. Here X and Y are complex and S is the solution set. There is an obvious method for parametrizing S. For $z \in \mathbb{C}$, set $X(z) = \cos z$ and $Y(z) = \sin z$. S is completely parametrized or, in the language we will use here, S is uniformized by the variable z.

This is usually considered a poor choice of uniformization for the following reasons. $X(z)$ and $Y(z)$ are transcendental functions with essential singularities at ∞. The structure at ∞ is not clearly displayed by the uniformization.

Another choice is to let z vary over $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

$$\text{with } X(z) = 2z/(1+z^2) \quad \text{and} \quad Y(z) = (1-z^2)/(1+z^2).$$

The behavior at ∞ is not exceptional; however, the rational uniformization has poles at $\pm i$, i.e., there is a definite structure at ∞. If we adjoin to S the ideal points $(X(\pm i), Y(\pm i))$, we obtain a compact Riemann surface $\bar{S}$. $\bar{S}$ is homeomorphic to $\hat{\mathbb{C}}$ as may be seen from Riemann's method of branch cuts.

To eliminate the special role of the point at ∞, the variety S may be viewed as lying in the complex projective plane by homogenizing the original polynomial, i.e., we consider the variety in projective space defined by $X^2 + Y^2 - W^2 = 0$ (see, for example, Kendig [**12**]).

2. The Evolution of the Concept of a Riemann Surface. Riemann surfaces were first introduced by Riemann in an attempt to understand multivalued functions of complex variables. The equation $X^2 - Y^2 = 0$ does not define X as a single valued function of Y. However, if X and Y are chosen to have real values, a choice of sign of $\sqrt{Y}$ may be consistently made if Y is not permitted to assume the value zero. At $X = Y = 0$, the differential of $P(X, Y) = X^2 - Y^2$ vanishes, that is, $(0,0)$ is a singularity of the equation $P(X, Y) = 0$. Avoiding the singularities is not sufficient to permit such choice if X and Y are permitted to have complex values.

Using a construction that may be found in any elementary complex variables text, Riemann resolved the problem by constructing sheets above the complex plane. On each sheet, a choice may be made. The sheets are then glued together to form a natural domain S_1 of definition for the function $\sqrt{X^2}$. In Fig. 1, we demonstrate Riemann's solution for real values of X and Y; Fig. 2 gives the local picture in $\mathbb{C}$ near 0. Notice that the modification is made on the domain of the function not on the range. Riemann initially thought of his sheets as lying over the complex plane in Euclidean 3-space. Much of the subsequent development of the notion of Riemann surface was done to remove the artificiality and arbitrary character of Riemann's embedding in $\mathbb{R}^3$. As Weyl noted, S_1 is a natural parameter space for the variety S. In this sense Riemann had a uniformization theorem, but it was not the first.

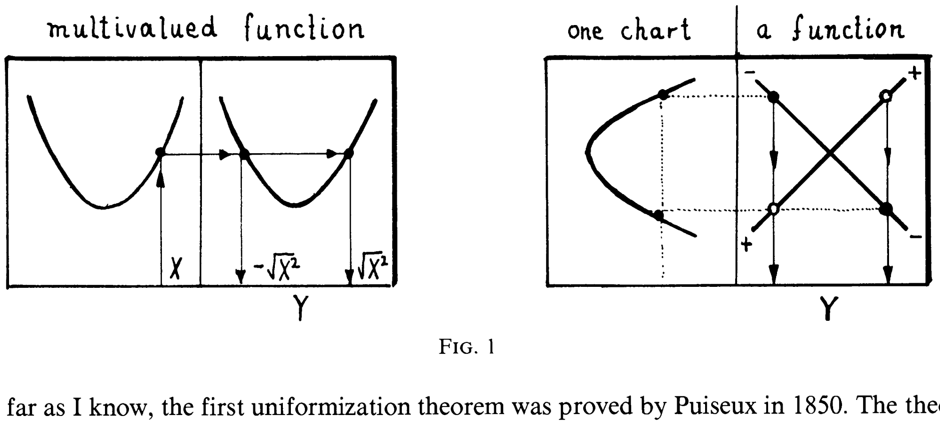

FIG. 1

As far as I know, the first uniformization theorem was proved by Puiseux in 1850. The theorem is a local uniformization theorem for the structure of singularities of plane algebraic curves. It states that if X and Y satisfy a polynomial relation $P(X, Y) = 0$, then, up to a linear change of coordinates, X can be written locally as a holomorphic function of $y = (Y - Y_0)^{1/k}$ for some $k \geqslant 1$; y then becomes a local uniformizing parameter for the variety $P(X, Y) = 0$. A proof may be found in Hille [**11**, vol. 1, p. 265 ff.].

Algebraic equations in two variables are not the only source of multivalued functions. Linear differential equations with regular singular points, such as $XY' + Y = 0$, generally have multivalued solutions.

Algebraic integrals are contour integrals in the complex plane of the form $\int R(z, w)\, dz$ where R

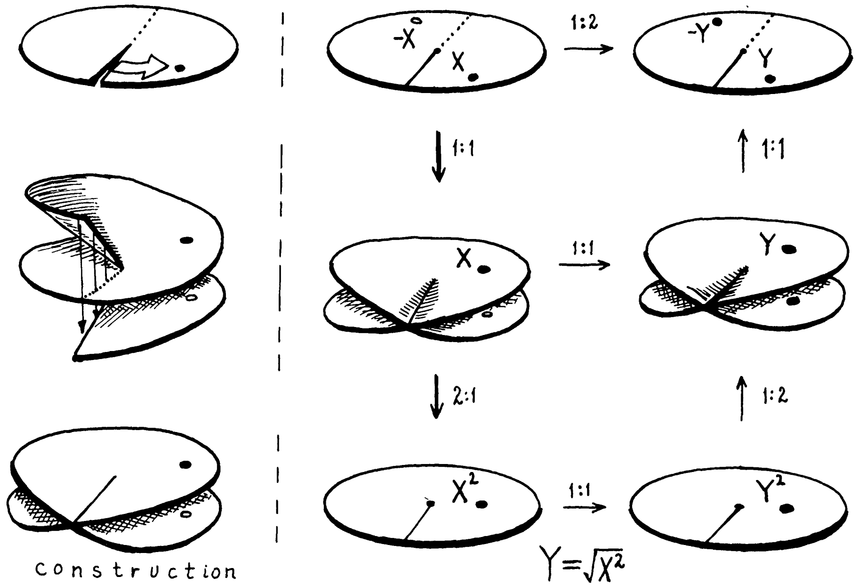

FIG. 2

is rational and z and w satisfy a polynomial identity. As the path of integration varies, we again obtain multivalued functions of the path and upper endpoint of integration. Siegel [**21**, pp. 1 ff.] traces back to Fagnano in 1719 the study of the algebraic properties of these integrals. In the first half of the nineteenth century, these properties were studied by Abel and Jacobi; the integrals are now called abelian integrals.

Riemann later studied these integrals by considering them as defined on the variety S rather than as integrals in the complex plane.

The method discovered by Weierstrass for studying multivalued functions is quite different from Riemann's. It is probably the first step in the development of the modern notions of abstract manifold and sheaf. Weierstrass's original paper on the subject seems to have been written in 1842, although it did not appear in print until 1894. The basic idea is to construct a Riemann surface, in this context called an *analytic configuration*, by starting with a power series $f(z)$ centered at some z_0 of positive radius of convergence. One then considers all meromorphic continuations of f in $\hat{\mathbb{C}}$. Each continuation of f to a point z is given by a chain of discs in which the continuation is possible.

The set of points to which f may be continued is subject to an equivalence relation, and a natural topology may be given to the equivalence classes (see, e.g., Ahlfors [**1**] or Hille [**11**, vol. 2] for details). Singular points admitting uniformizations by Puiseux series are usually added to give *complete analytic configurations*, which are Riemann surfaces in the sense of Weyl (see below). The analytic configurations are Riemann surfaces which lie in the sheaf of germs of holomorphic or meromorphic functions on $\hat{\mathbb{C}}$. Overlapping discs define the topology, a concept which underlies the modern notion of manifold.

In his 1907 paper on uniformization, Poincaré removed the reliance of Weierstrass's construction on the initial power series $f(z)$. He proved the uniformization theorem for collections of overlapping discs on which there exists *some* meromorphic function in the sense of Weierstrass. The geometric entity became paramount; the role of the function was secondary.

The modern era in the study of Riemann surfaces, and, indeed, of all manifolds, opened in 1913 with the publication of Hermann Weyl's book *The Concept of a Riemann Surface*. Here, for the first time, no defining function is assumed. The definition is purely geometric and also independent of any embedding in Euclidean 3-space.

On the justification for abandoning Riemann's embedding of the Riemann surfaces of multivalued functions, Weyl wrote:

> In essence, three-dimensional space has nothing to do with analytic forms [Riemann surfaces of multi-valued functions], and one appeals to it not on logical-mathematical grounds, but because it is closely associated with our sense perception. To satisfy our desire for pictures and analogies in this fashion by forcing inessential representations on objects instead of taking them as they are could be called an anthropomorphism contrary to scientific principles. However, these reproaches of the pure logician are no longer pertinent if we pursue the other approach ... in which the analytic form is a two-dimensional manifold ... To the contrary, not to use this approach is to overlook one of the most essential aspects of the topic.

Weyl's definition, in modern language and without unnecessary hypotheses, follows. Let S be a connected Hausdorff space and $U = \{D_\alpha | D_\alpha \subset S\}$ be an open cover of S. Assume that for all α there is a homeomorphism $z_\alpha^{-1}: D_\alpha \to \mathbb{C}$ which satisfies: $z_\beta^{-1} \cdot z_\alpha$ is holomorphic where defined. The set $A = \{(z_\alpha, D_\alpha)\}$ is called a *holomorphic* or *conformal atlas* and the pair (S, A) is called a *Riemann surface*. Usually, by abuse of language, we speak of the Riemann surface S. z_α is called a *local coordinate* or *uniformizing* variable at the points of D_α. A function f on S is *holomorphic, meromorphic* or *harmonic* if, for each α, $f \circ z_\alpha$ has that property wherever it is defined. A map $f: S_1 \to S_2$ between two surfaces S_1 and S_2 is *holomorphic* if $z_2^{-1} \cdot f \cdot z_1$ is holomorphic whenever defined. Here z_1 and z_2 are local coordinates on S_1 and S_2, respectively.

It is not trivial, but possible, to prove that every Riemann surface in the sense of Weyl carries a pair of nonconstant linearly independent meromorphic functions and may be embedded in $\mathbb{R}^3$. It then follows that, up to holomorphic equivalence, the notions of Riemann surfaces developed by Riemann (suitably generalized), Weierstrass, and Weyl are equivalent.

To return to the uniformization problem, the question changed as the notion of Riemann surfaces changed. Klein's original claim is to have proved the theorem for compact Riemann surfaces, possibly missing a finite number of points, in the sense of Riemann. All proofs until, I believe, the 1920's assumed that the surfaces were first countable; this latter assumption was made by Weyl. The assumption was removed by Radó, but now may be derived as a simple consequence of the uniformization theorem.

We may now state the general uniformization problem. Let S be a Riemann surface. Find *all* domains $D \subset \hat{\mathbb{C}}$ and holomorphic functions $t: D \to S$ so that at each point $p \in S$, t is a local uniformizing variable at p.

Equivalently, there is a topological disc $B \subset S$ with center p so that the restriction of t to each component of $t^{-1}(B)$ is a homeomorphism. The reader should notice that the required conditions on t and D say precisely that the triple (D,S,t) is a (smooth) covering space with base S, total space D, and holomorphic projection t.

The early problem was less ambitious. It was simply to find one uniformization, but where D is simply connected. This theorem was proved independently by Koebe and Poincaré in 1907. Poincaré's solution was somewhat more general but we will ignore that generalization here. Until recently the most common proof was that offered by Hilbert in 1909. We give the modern statement of the Uniformization Theorem in the next section following the preliminaries that tie it to the uniformization problem.

3. Covering Surfaces and Classical Plane Geometries. Let S_1 and S_2 be two Riemann surfaces and $\pi: S_1 \to S_2$ be a local homeomorphism so that each point on S_2 has an evenly covered neighborhood. We then say that (S_1, S_2, π) is a *covering surface* or, more commonly, that S_1 is a *covering surface* of S_2 with *cover map* or *projection* π. The number of points in $\pi^{-1}(p)$ is independent of $p \in S_2$ and is called the *number of sheets* of the covering.

Covering surfaces occur classically in the study of algebraic equations with symmetries. For example, the equation $w^2 - z = 0$ admits the symmetry obtained by replacing w by $-w$. In this context the symmetry is called *sheet interchange*. The Riemann surface S of $w^2 - z = 0$ over $\mathbb{C} \setminus \{0\}$ is a covering surface of $\mathbb{C} \setminus \{0\}$. Notice that the sheet interchange cannot be defined to be a covering surface at $z = 0$. Points where the map topologically looks like $z \mapsto z^n$ are called *ramification points*.

Covering surfaces with isolated singularities, all of which are ramification points, are called *ramified covers*.

A method for constructing a covering surface is to take two copies S' and S'' of a given surface S, slice them along corresponding curves, and glue as indicated in Fig. 3. The resulting surface is a two-sheeted cover of S with the projection taking z' and z'' to z. Notice that there is a simple closed curve that covers β twice. In a sense that may be made quite precise, passing to the covering surface has replaced β by a curve of twice the length of β. If we repeat this process infinitely often, β will have been replaced by an infinitely long, simple curve; i.e., β will no longer be an obstruction to simple connectivity. We may repeat this process for sufficiently many curves so that the resulting covering surface $\tilde{S}$ has only homotopically trivial curves. $\tilde{S}$ is called the *universal covering surface*. This approach is the most classical one; the modern approach (see, e.g., Greenberg [**8**]) is an abstract reformulation of the basic idea of opening up all homotopically nontrivial closed curves. Klein states that the construction above is due to Schwarz.

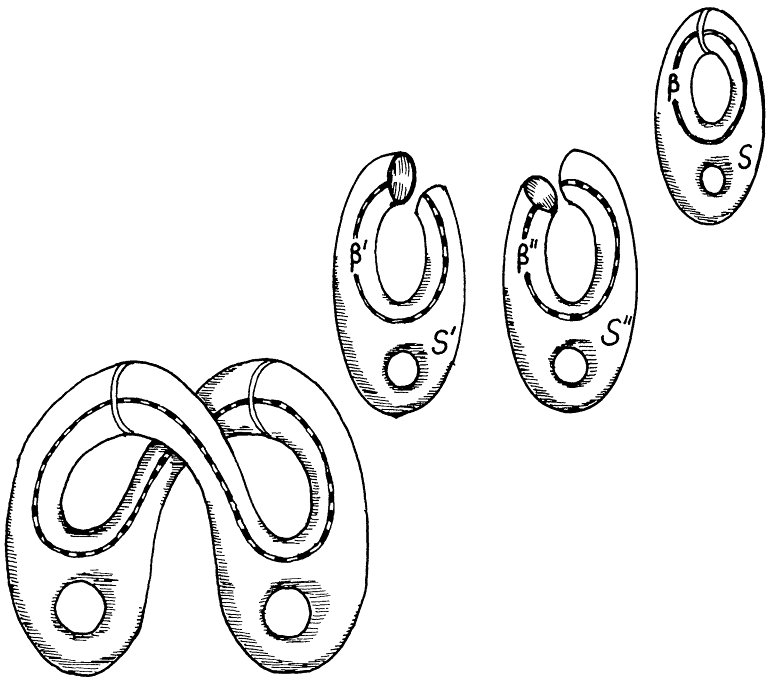

FIG. 3

As we have noted in the previous section, if S is the Riemann surface of some multivalued function Y of X, then both X and Y are functions of ζ if ζ varies over the points of S. If S_1 is a covering surface of S, and ζ_1 varies over S_1, then $X \cdot \pi$ and $Y \cdot \pi$ parametrize the variety in $\mathbb{C}^2$ defined by the functional relationship between X and Y. Thus if S_1 lies in $\hat{\mathbb{C}}$ we obtain a solution to the general uniformization problem. We will return to this question in § 6 but now restrict our attention to $S_1 = \tilde{S}$.

$\tilde{S}$ inherits the structure of a Riemann surface from the conformal structure on S. $\tilde{S}$ is a simply connected Riemann surface because any homotopically nontrivial closed curve has been opened. It follows immediately that we may obtain a solution to the uniformization problem by showing that every simply connected Riemann surface is biholomorphically equivalent to a subdomain of

$\hat{\mathbf{C}}$; this is merely a rephrasing of the Riemann mapping theorem, (see § 4). So a solution to the uniformization problem follows from the statement now known as:

THE UNIFORMIZATION THEOREM. *Every simply connected Riemann surface is biholomorphically equivalent to either* $\mathbf{C}$, $\hat{\mathbf{C}}$, *or the unit disc* Δ.

That the three possibilities are distinct is an immediate consequence of Liouville's theorem.

The universal cover has several very strong properties. Since it is simply connected, every multivalued locally meromorphic function on S lifts to a meromorphic function on $\tilde{S}$; here we are restating the monodromy theorem. The universal cover is the only covering surface with this property.

Another important property is the following. Suppose $\zeta \in S$ and N is an evenly covered neighborhood of ζ. The cover map $\pi: \tilde{S} \to S$ defines a map γ from one component N_1 of $\pi^{-1}(N)$ to another, say N_2, by the rule $\zeta_1 \mapsto \zeta_2$ where ζ_2 is the unique point in N_2 for which $\pi(\zeta_1) = \pi(\zeta_2)$ (see Fig. 4). Using the evenly covered character of N_1, one shows that γ extends to a biholomorphic self-map of $\tilde{S}$. These maps form a group G called the *group of the covering* $(\tilde{S}, S, \pi)$ or *cover group* or *group of deck* (*cover*) *transformations*. Clearly, G acts without fixed points. G also has the stronger property of being *properly discontinuous*, which means that each point $\tilde{\zeta} \in \tilde{S}$ has a neighborhood $\tilde{N}$ so that $\gamma(\tilde{N}) \cap \tilde{N} \neq \varnothing$ if and only if $\gamma =$ identity. Since the sheet interchange maps $\gamma \in G$ are in one-to-one correspondence with the free homotopy classes of closed curves on S, the group G is isomorphic to the fundamental group $\pi_1 S$ of S based at any point. The orbit space $\tilde{S}/G$ is the set of equivalence classes in $\tilde{S}$ where $\zeta_1 \sim \zeta_2$ if there exists $\gamma \in G$ so that $\zeta_2 = \gamma(\zeta_1)$. It is not difficult to show that $\tilde{S}/G$ inherits a conformal structure from $\tilde{S}$ and that S is biholomorphically equivalent to $\tilde{S}/G$.

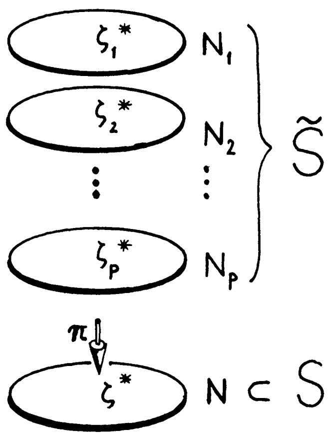

FIG. 4

The line of reasoning developed above is, more or less, historically accurate when $\tilde{S}$ is $\hat{\mathbf{C}}$ or $\mathbf{C}$ (we shall no longer distinguish between biholomorphically equivalent surfaces except when necessary). It shows immediately that $\tilde{S} = \hat{\mathbf{C}}$ only if $S = \hat{\mathbf{C}}$. If S is compact and $\tilde{S} = \mathbf{C}$, π is the inverse to an elliptic integral (see Siegel [**21**, Chapter 1]). The problem of finding inverse functions to algebraic integrals, i.e., cover maps, is called the Jacobi inversion problem, the rich history of which leads directly to Riemann surfaces and uniformization (cf. Weyl [**23**, p. 144]).

When $\tilde{S} = \mathbf{C}$, G must be a properly discontinuous subgroup of the group $\operatorname{Aut}\mathbf{C}$ of biholomorphic self-maps of $\mathbf{C}$. There are precisely three types. The first is when G is trivial. The Riemann surface is then $\mathbf{C}$. The second are the cyclic groups $\{z \mapsto z + nz_0 \mid n \in \mathbf{Z}\}$. All are conjugate in

Aut$\mathbb{C}$ to $G = \{z \mapsto z + n \mid n \in \mathbb{Z}\}$ and conjugate groups determine biholomorphically equivalent surfaces. The surface is $\mathbb{C} \setminus \{0\}$ and $\pi : \mathbb{C} \to S$ is the map $\exp(2\pi iz)$. The last is a class of groups, the lattices, which are, up to conjugation, generated by $\gamma_1 : z \mapsto z + 1$ and $\gamma_2 : z \mapsto z + \tau$ where $\operatorname{Im} \tau > 0$. In this case $\pi_1 S$ is free abelian on two generators and S is topologically a torus. The two generators may be chosen as in Fig. 5.

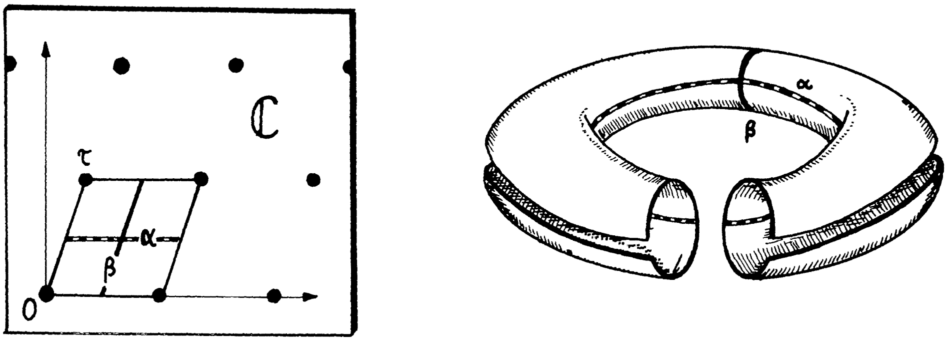

FIG. 5

We have this "found" all Riemann surfaces with $\tilde{S} = \mathbb{C}$ or $\tilde{S} = \hat{\mathbb{C}}$. The uniformization theorem tells us that every other Riemann surface S has $\tilde{S} = \Delta$. Here the history proceeded in the opposite direction. Klein knew in 1882 that if G is a properly discontinuous subgroup of Aut Δ then Δ/G looks like a Riemann surface. However in 1882, a Riemann surface required a defining multivalued function. While it seems that Klein had a method for obtaining such a function, Poincaré claimed two methods, one of which remains the standard method for obtaining automorphic functions for discrete groups acting on bounded domains. The existence of this function shows that all the definitions of Riemann surfaces coincide.

The Riemann sphere has a metric of constant positive curvature induced by the usual embedding in $\mathbb{R}^3$. $\mathbb{C}$, hence by projection $\mathbb{C} \setminus \{0\}$ and tori, have Euclidean or flat metrics, i.e., of zero curvature. Using the uniformization theorem, we see that all other Riemann surfaces inherit any Aut Δ invariant metric from Δ. Up to scale factor, there is only one; it is the Poincaré metric on the hyperbolic plane. It is a metric of constant negative curvature. The universal cover of a surface of constant negative curvature may be conformally and injectively developed on a hyperbolic plane. This is one of many successful approaches to proving the uniformization theorem. The relevant differential equation is $\Delta u = e^{2u}$. Studies of this equation were among the earliest in nonlinear partial differential equations.

We also recall that the hyperbolic plane, in Poincaré's model, is the unit disc Δ. When we map $\tilde{S}$ to Δ, we give natural boundary to $\tilde{S}$, namely $\partial\Delta$. To Klein the edge was a limiting circle and he called his uniformization theorem, the *limit circle theorem*.

To comment on the pre-1907 "proofs" of the uniformization is to tread on the most dangerous ground. One gathers from Hilbert's 1900 lecture at the International Congress [**10**] that he did not accept Klein's proof, for it is not mentioned. Poincaré's uniformization is spoken of, but Hilbert comments that it does not parametrize the whole variety; so it is not a uniformization in the sense considered here (see also the introduction to Poincaré [**20**]).

From the vantage point of 1980, it is not quite so easy to dismiss Klein's argument. One is first presented with a major obstacle, namely, to find the argument. To those of us trained in the Satz-Beweis school of mathematical exposition and discourse, reading Klein is often a mystical experience. In fact, many a mathematician has proved and published a deep and elegant result, later to discover with chagrin that there is a casual and vague reference to the result in Klein. Most of Klein's writing on areas related to uniformization are collected in two books written jointly with Fricke; these comprise over 2,000 pages without an index, and often without definitions or theorem statements. The mathematical insight contained therein is astounding, but it often seems that one can only appreciate a part of it after having independently rediscovered the results. To the modern observer Klein only claimed the proof of the uniformization theorem when S is *conformally finite*, that is, when S is a compact surface missing a finite number of points. A terse modern appraisal of the argument is that it is an excellent outline, but far from a proof; it appears on pages 698–705 of volume 3 of his collected works [**14**].

A brief description follows. First construct one algebraic equation defining $P(z, w) = 0$ as a multivalued function of z. Let S be the Riemann surface of P in the sense of Riemann. On S we find a finite set of piecewise circular closed curves α_i so that $S \setminus \cup\alpha_i$ is a polygon $\Pi \subset S$. On S, z is a well-defined function, $z|\Pi$ immerses Π in $\hat{\mathbb{C}}$, and $\partial z(\Pi)$ consists of circular arcs. P may be chosen in any genus g, so that it is very symmetric. Then for some choice of P and the α_i, $z(\Pi) \subset \Delta$ and $\partial z(\Pi)$ are circular arcs lying on circles orthogonal to $\partial\Delta$. $z^{-1}|z(\Pi)$ may be analytically, but in a multivalued fashion, continued along all paths in S. If the image domain is the unit disc, the continuation is the inverse of the universal cover map. He then considers the corresponding group of cover transformations, and he notes that the space of such (normalized) groups G and the space of dissected Riemann surfaces in genus g both have real dimension $6g - 6$. The local correspondence between them he *assumes* is a local real analytic diffeomorphism. He, more or less, shows that the mapping is injective and proper, hence bijective. This completes Klein's attempt. This technique of proof is called the *continuity method*. Even assuming the uniformization theorem, the last two properties are true but not easily proved. The fundamental difficulty with this proof was recognized quite early and goes as follows. Forget that the correspondence is a local diffeomorphism (or wait some 40 to 80 years until the theorem is proved). You only have a proper, continuous injection f of $\mathbb{R}^n$ into $\mathbb{R}^n$. Brouwer essentially developed dimension theory to prove that f is a homeomorphism and thereby resurrected the continuity method. With Brouwer's proof added, Klein's technique becomes viable; however, before Brouwer's proof appeared, the uniformization theorem had already been proved in complete generality.

Uniformization theory was relatively dormant from 1883 to 1900. In 1900, Hilbert delivered a lecture to the International Congress of Mathematicians in which he stated 23 problems which have had a profound effect on the course of mathematics in the twentieth century. Uniformization was Problem 22. This renewed interest in the question led to the solution in 1907. Both solutions given in 1907 and the argument given in 1909 by Hilbert come from potential theory and it is to that stream of ideas that we now turn.

4. Some Potential Theory. Riemann's thesis is contemporary with many of the great discoveries of nineteenth-century physical science. Klein [**13**, p. x] wrote, "Riemann as we know used Dirichlet's Principle in their place." The physical arguments of which Klein speaks are those associated to a conservative vector field E and its associated potential function V. The classic examples of these fields are electric fields and the flow of an incompressible fluid. I shall not dwell on these concepts save to say that the following equations hold: $\operatorname{div} E = 0$, $E = -\operatorname{grad} V$, and $\Delta V = 0$. In the last equation, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian, and therefore V is a harmonic function.

Among the earliest nontrivial fields to be studied is that induced by a point charge. In the unit disc, the potential defined by a (positive) point charge at $z = 0$ is $V(z) = \log(1/|z|)$ if $\partial\Delta$ is grounded, i.e., $V|\partial\Delta = 0$. Riemann considered the question of whether a point charge could live in an arbitrary simply connected plane domain D whose boundary is a grounded conductor.

The latter condition is that the potential $V|\partial D = 0$. Assume we have a point charge in D. The level curves $L_V(c)$ of V for $0 < c < \infty$ are analytic Jordan curves separating the point charge at z_0 from ∂D. The integral curves C of the gradient field of V, the paths of elections in this field, are again analytic curves from ∂D to z_0. These integral curves may be parametrized by the angle θ at which they enter z_0. We may therefore write $C = C(\theta)$. Each point $z \in D$ lies on a unique level curve $L_V(c)$ and a unique integral curve $C(\theta)$. We form a map

$$f: D \to \Delta$$
$$z \mapsto (e^{-c}, \theta)$$

in polar coordinates. (See Fig. 6.) This map is conformal and proves the Riemann mapping theorem, once we know that a point charge can live in D. To prove existence, Riemann invoked the Dirichlet principle. The principle states that harmonic functions minimize the energy in an electric field and such a minimum exists here if $D \neq \mathbb{C}$. This argument was used by Riemann in several contexts, but was questioned by Weierstrass. The latter noted that even elementary extremum problems need have no solution. The Dirichlet principle fell into disrepute but was later resurrected by Hilbert. It is the key to his 1909 proof of the uniformization theory. Hilbert's proof uses mapping properties associated with electric dipoles.

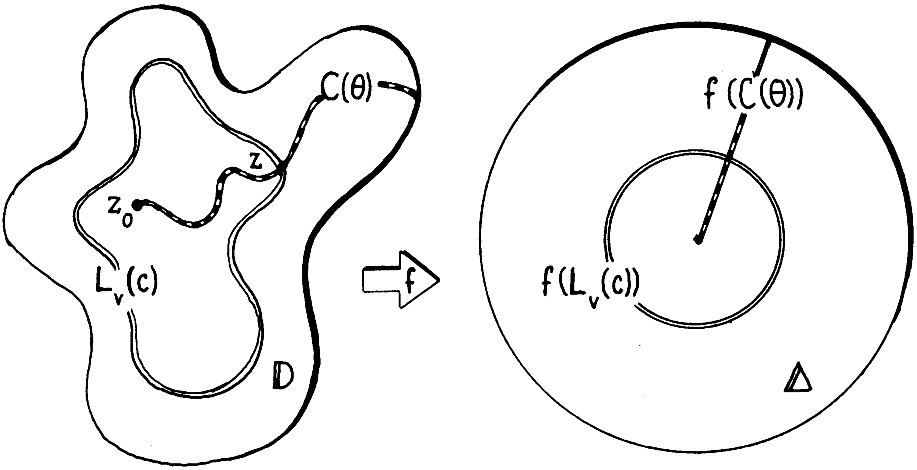

FIG. 6

To continue our intuitive discussion, we next consider incompressible fluid flow. (See Fig. 7.) If we have a point source of water in the plane, for example a faucet, there must be an edge across which the water may flow out of the domain. Otherwise the fluid will compress. If the domain has such an edge, which we call a "thick boundary," then a point source can exist at any point in the

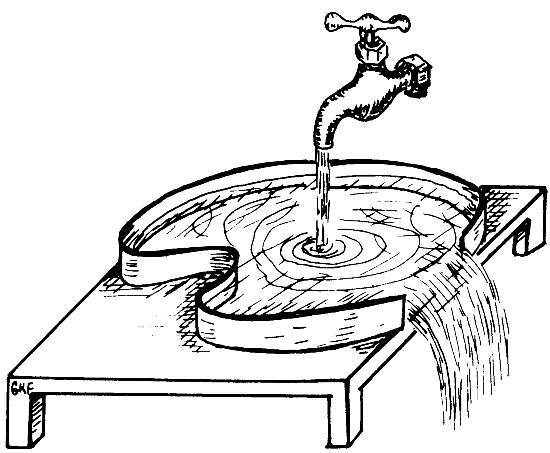

FIG. 7

domain. It is intuitively obvious that any edge save a point for the domain will enable it to support a point source.

There are now many ways to prove the existence of a point charge or source. Perhaps the most elegant is due to Perron. His method may be found in most complex analysis books. Basically the idea is that a harmonic function is the 2 (or n) variable analogue of a linear function $f(x)$ of one variable. $f(x)$ has the property that $f(x) = \sup g(x)$ where $g(x)$ is convex and the boundary values of g are less than or equal to those of f (see Fig. 8). Locally one replaces g by a linear function and g is "bootstrapped" up to f by taking suprema.

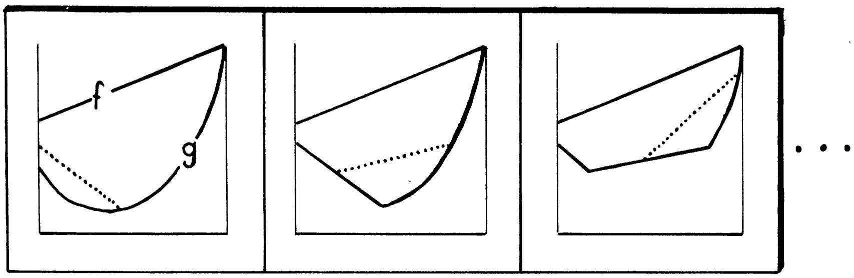

FIG. 8

On a Riemann surface, subharmonic functions assume the role of convex functions. A subharmonic function is a continuous real-valued function whose value at a point $\zeta_0 \in S$ is less than or equal to the average of the values of the function on the boundary of any small symmetric neighborhood of ζ_0. Specifically, let $s\colon S \to \mathbb{R}$ be continuous. s is *subharmonic* if, for each $\zeta_0 \in S$ and each small neighborhood N and local coordinate $z = z(\zeta)$ in N with $z(\zeta_0) = 0$,

$$s(\zeta_0) \leqslant \frac{1}{2\pi}\int_0^{2\pi} s\left(re^{i\theta}\right) d\theta$$

for $r = |z(\zeta)|$ sufficiently small and $\theta = \arg z(\zeta)$.

A *Perron class* $\mathfrak{F}$ is a nonvoid set of subharmonic functions $s\colon S \to \mathbb{R}$ which is closed under the following operations:

(i) taking the maximum of two functions

(ii) local harmonic majorization.

Property (ii) generalizes the local replacement of a convex function by a linear function. To obtain the precise definition, let $\bar{D}$ be a closed disc on S. On $\bar{D}$, let $\hat{s}$ be the harmonic function such that $\hat{s}\,|\,\partial D = s\,|\,\partial D$. Extend $\hat{s}$ to S by setting $\hat{s} = s$ in $S \setminus D$. $\hat{s}$ is called the *local harmonic majorant* of s in D. *Local harmonic majorization* is the process of replacing s by $\hat{s}$ for some disc D.

Perron showed that $\bar{s}(\zeta) = \sup_{s \in \mathfrak{F}} s(\zeta)$ is either harmonic or identically infinite.

We now give a precise definition of the Green's function, or potential of a point charge, using Perron's method. The potential should be the smallest potential which is positive and grows as does $-\log|z|$, near the point charge.

Let $\zeta_0 \in S$ and $\mathfrak{F}$ be the set of subharmonic functions s in $S \setminus \{\zeta_0\}$ having the following properties:

(i) $s(\zeta) \geqslant 0$.

(ii) $\{\zeta \in S \setminus \{\zeta_0\} \,|\, s(\zeta) \neq 0\}$ is compact.

(iii) For any local coordinate z on a neighborhood N of ζ_0 with $z(\zeta_0) = 0$, $s(z) + \log|z|$ is bounded above near ζ_0.

As a consequence of Perron's Theorem (see, e.g., Ahlfors [**1**, p. 240]), $\bar{s}(\zeta) = \sup_{s \in \mathfrak{F}} s(\zeta)$ is either identically infinite or harmonic in $S \setminus \{\zeta_0\}$. In the latter case we call $g(\zeta, \zeta_0) = \bar{s}(\zeta)$ the *Green's function* of S with singularity at ζ_0. In the former case we say that S does not admit a Green's function with singularity at ζ_0. It is known that, if z is a local coordinate at ζ_0 with

$z(\zeta_0) = 0$, $g(\zeta,\zeta_0) + \log|z|$ has a harmonic extension to any small neighborhood of ζ_0 (see CI, p. 141).

Assume that S has a Green's function $g(\zeta,\zeta_0)$. Then, in the sense of fluid flow, it must have a thick boundary—a notion we must also formalize. Asserting that a domain S has a thick boundary is equivalent to stating that the "edge" of S may serve as a source or sink for incompressible fluid flow (with bounded potential). Equivalently the associated potential may have a minimum along the edge. The nonthickness of the boundary is formalized by the following:

DEFINITION. Let S be a Riemann surface and K be a compact subset of S. We say that S satisfies the *maximum principle relative to* K if every bounded harmonic function f: $S \setminus K \to \mathbb{R}$ has the property that

$$\sup_{\zeta \in S \setminus K} f(\zeta) = \overline{\lim_{\zeta \to \partial K}} f(\zeta).$$

Notice that if S has a Green's function with singularity $\zeta_0 \in \operatorname{Int} K$, then $-g|(S \setminus K)$ shows that S does not satisfy the maximum principle relative to K. The converse is also true, namely, the invalidity of the maximum principle relative to K implies the existence of the Green's function with singularity at an arbitrary point $\zeta_0 \in \operatorname{Int} K$ (see CI, p. 139).

Given a Riemann surface or any plane domain, the validity of the relative maximum principle is not directly verifiable. There is, however, a standard technique used to show that it is not valid. Suppose there exists a nonconstant harmonic function h on S. Then h has a maximum on K. If the relative maximum principle were valid, that maximum would be a global maximum contradicting the usual maximum principle. We have proved:

PROPOSITION 4.1. *Let S be a Riemann surface. If there is a bounded harmonic function f: $S \to \mathbb{R}$ which is not constant, then, for all compact $K \subset S$ with nonvoid interior, the maximum principle relative to K is not valid and S has a Green's function with singularity at any point $\zeta_0 \in S$.*

It is not a triviality to produce nonconstant bounded harmonic functions on Riemann surfaces. For example, the usual maximum principle implies that a harmonic function on a compact Riemann surface is constant. $\mathbb{C}$ admits the nonconstant harmonic function $\operatorname{Re} z$, but, by applying the removable singularity theorem to a neighborhood of infinity, $\mathbb{C}$ admits no bounded nonconstant harmonic functions.

Producing nonconstant bounded harmonic functions on a Riemann surface brings us to the venerable Dirichlet problem. The problem is easily stated. Let $S \subset S_1$ be Riemann surfaces and, for simplicity, assume ∂S is a finite collection of piecewise analytic curves. Let $f: \partial S \to \mathbb{R}$ be continuous. The *Dirichlet problem* is to find a continuous function $h: \bar{S} \to \mathbb{R}$ so that $h|S$ is harmonic and $h|\partial S = f$. Notice that ∂S corresponds to our intuitive picture of a thick edge. Further, if we may solve the Dirichlet problem for nonconstant functions f, then, by the proposition above, S will not satisfy the maximum principle relative to a compact subset and our analytic characterization of thickness will have been proved.

The solution of the Dirichlet problem is an application of Perron's method. Assume f is bounded or, more simply, assume ∂S is compact. We form a Perron class $\mathfrak{F}$ of continuous functions $s: \bar{S} \to \mathbb{R}$ which are subharmonic in S and satisfy $s|\partial S \leq f$ and $s \leq \sup f$. If $m = \inf f$, then the constant function $m \in \mathfrak{F}$. Also, the maximum principle, applied to subharmonic functions, implies that for all $\zeta \in S$ and all $s \in \mathfrak{F}$, $s(\zeta) \leq \sup f$. It follows that $h(\zeta) = \sup_{s \in \mathfrak{F}} s(\zeta) < \infty$ and hence is harmonic in S. It remains to show that h extends to ∂S and $h|\partial S = f$. The usual technique for doing so is due to Poincaré (1899) and formalizes the notion of local thickness of the boundary. One aims a microscope with arbitrarily fine resolution at a point $\zeta \in \partial S$, and we see whether it is possible to push water across ∂S near ζ. The potentials of these flows are called *barriers*. The formal definitions and proofs may be found in Bers [4, p. 139 ff.] and Conway [7, p. 265 ff.]. Conway's arguments are stated for plane domains but are equally valid on Riemann surfaces. The precise result that we need is

Proposition 4.2. *Let S_1 and S be Riemann surfaces and $S \subset S_1$. If ∂S is a finite union of closed analytic arcs, then, for all continuous, bounded $f: \partial S \to \mathbb{R}$ there exists $h: \bar{S} \to \mathbb{R}$ so that*

(i) $|h| \leqslant \sup|f|$

(ii) $h|S$ *is harmonic*

(iii) $h|\partial S = f$.

As an immediate consequence of Propositions 4.1 and 4.2, we obtain the first conclusion of

Proposition 4.3. *Let S_1 and S be Riemann surfaces and $S \subset S_1$. If ∂S is a finite union of closed analytic arcs then*

(i) *S has a Green's function $g(\zeta, \zeta_0)$*

and

(ii) $\lim_{\zeta \to \partial S} g(\zeta, \zeta_0) = 0$.

Proof: We prove the second conclusion. Let C be a small circle around ζ_0. Using Proposition 4.2, on S we may solve the Dirichlet problem outside C with boundary data $f|C = g$ and $f|\partial S = 0$. Call the resulting solution h. By the maximum principle for subharmonic functions, h is an upper bound for all functions s lying in the Perron class $\mathfrak{F}$ defining $g(\zeta, \zeta_0)$. It follows that

$$\overline{\lim_{\zeta \to \partial S}}\, g(\zeta, \zeta_0) \leqslant 0.$$

Since the function $s(\zeta) \equiv 0$ lies in $\mathfrak{F}$,

$$\underline{\lim_{\zeta \to \partial S}}\, g(\zeta, \zeta_0) \geqslant 0,$$

which proves the Proposition.

5. The Uniformization Theorem. The uniformization theorem, even today, commands a nontrivial proof. Here we will sketch one style of proof with some details omitted. For the interested reader, we give references either to the accessible literature or to the appendix to this paper.

Let S be a simply connected Riemann surface. We first assume that, for fixed $\zeta_0 \in S$, S admits a point charge or Green's function $g = g(\zeta, \zeta_0)$ with singularity at ζ_0. The complete argument in this case may be found in CI, p. 136 ff. We must define a conformal map $f: S \to \mathbb{C}$. Let $S' = S \setminus \{\zeta_0\}$ and choose a simply connected chart N_ζ in S' near each $\zeta \in S'$. Since g is harmonic in N_ζ, it has a harmonic conjugate h_ζ there and $f_\zeta = \exp[-(g + ih_\zeta)]$ is holomorphic near ζ. Further, f_ζ is unique up to multiplication by a complex number of modulus one. Let N_0 be a simply connected chart near ζ_0 with local coordinate z satisfying $z(\zeta_0) = 0$. $g(\zeta, \zeta_0) + \log|z(\zeta)|$ is harmonic in N_0 and hence has a harmonic conjugate h_0. Set

$$f_0(\zeta) = z(\zeta) \cdot \exp\left[-\left(g + \log|z| + ih_0\right)\right].$$

f_0 is holomorphic in N_0 and vanishes to first order at ζ_0. By adjusting constants of modulus one, f_ζ is an analytic continuation of f_0. Since S is simply connected, the monodromy theorem implies that the analytic continuation defines a homomorphic function $f: S \to \mathbb{C}$. Also $|f(\zeta)| = e^{-g} < 1$ since $g(\zeta, \zeta_0) > 0$. It is possible to show directly that f is a bijection of S with Δ; however, an elegant argument due to Heins [**9**] is far more efficient. We omit the details save to note that the proof makes decisive use of the fact that a Riemann surface admitting a Green's function with singularity at ζ_0 also has a Green's function with singularity at any prescribed point. This completes our sketch of the proof of

Proposition 5.1. *If S is a simply connected Riemann surface which admits a Green's function, then there is a biholomorphic, i.e., conformal, map $f: S \to \Delta$.*

Henceforth we assume that S is simply connected and does not admit a Green's function.

Definition. A *divergent curve* on S is a piecewise analytic simple arc $\phi: [0, \infty) \to S$ so that, for any compact $K \subset S$, $\phi^{-1}(K)$ is compact.

Now assume S admits a divergent curve and set $S_t = S \setminus \phi([t,\infty))$. It should be intuitively clear, but requires proof, that the simple connectivity of S implies that S_t shares that property. Using Proposition 4.3, we then obtain

LEMMA 5.1. *For all $t \geq 0$, S_t is simply connected and for any $\zeta_0 \in S_0$, S_t admits a Green's function with singularity at ζ_0. Further,*

$$\lim_{\zeta \to \partial S_t} g(\zeta,\zeta_0) = 0.$$

Proof: See the appendix.

We shall need the following standard result in function theory.

LEMMA 5.2. *Let $\Delta(r) = \{|z| < r\}$ and $\mathbb{S}_r$ be the set of holomorphic injections $f: \Delta(r) \to \mathbb{C}$ and satisfying*
(i) $f(0) = 0$
(ii) $f'(0) = 1$.
Then $\mathbb{S}_r$ is (sequentially) compact in the topology of uniform convergence on compact subsets.

Proof. The map

$$\mathrm{UC}: \mathbb{S}_r \to \mathbb{S}_1$$
$$f(z) \mapsto F(z) = r^{-1}f(rz)$$

is obviously a homeomorphism. It therefore suffices to show that $\mathbb{S}_1$ is compact. Montel's Theorem (see any graduate-level complex analysis text) states that one must show only that $\mathbb{S}_1$ is closed and bounded. Hurwitz's Theorem states that a limit of holomorphic injections is holomorphic and injective or constant. Condition (ii) rules out a constant limit. Thus $\mathbb{S}_1$ is closed. The estimates necessary to show that $\mathbb{S}_1$ is bounded are given by Koebe's distortion theorem (see CI, p. 84, or Conway [7, p. 351 ff.].

Since S_t is a simply connected Riemann surface with a Green's function, there is a holomorphic bijection $f_t: S_t \to \Delta$. Further we may assume that $f_t(\zeta_0) = 0$ for some fixed $\zeta_0 \in S$. Now choose a sequence (t_i) increasing to infinity and denote S_{t_i} by S_i and f_{t_i} by f_i. Fix the local coordinate $z = f_0(\zeta)$ near ζ_0. We may then compute $c_i = f_i'(z(\zeta))|_{\zeta=\zeta_0}$ and let $F_i(\zeta) = c_i^{-1}f_i(\zeta)$. $F_i: S_i \to \Delta_i = \Delta(c_i^{-1})$ is a holomorphic bijection.

Recursively we define subsequences N_i of $\mathbb{Z}^+$ as follows:
(i) $N_1 = \mathbb{Z}^+$
(ii) If N_i is defined and $j \in N_i$ and $j \geq i$, F_j is defined and injective on S_i and $F_j'(z(\zeta_0)) = 1$. $F_j \circ F_i^{-1}: \Delta_i \to \mathbb{C}$ is injective, maps 0 to 0, and has derivative equal to 1. Thus, by Lemma 5.2, we find that there must be a subsequence $N_{i+1} \subset N_i$ so that, for $j \in N_{i+1}$, $F_j \circ F_i^{-1}$ converges to an injective map $H_i: \Delta_i \to \mathbb{C}$. $H_i(0) = 0$ and $H_i'(0) = 1$. We have defined N_{i+1}.

Choose n_j to be the jth entry in the sequence N_j. For $k > i$, on S_i, $H_k \circ F_k$ is a holomorphic injection and

$$H_k \circ F_k = \left(\lim_{j\to\infty} F_{n_j} \circ F_k^{-1}\right) \circ F_k = \left(\lim_{j\to\infty} F_{n_j} \circ F_i^{-1}\right) \circ F_i \circ F_k^{-1} \circ F_k$$

$$= \left(\lim_{j\to\infty} F_{n_j} \circ F_k^{-1}\right) \circ F_i = H_i \circ F_i.$$

Thus $H_i F_i$ is the restriction to S_i of a globally defined holomorphic map $f: S \to \mathbb{C}$. f is injective since $f \mid S_i$ is injective for all i.

$f(S)$ is simply connected. If $f(S) \neq \mathbb{C}$, then, by the Riemann mapping theorem, $f(S)$ is conformally equivalent to Δ and there is a conformal map $h: S \to \Delta$. $\operatorname{Re} h$ is a bounded nonconstant harmonic function on S. As in Proposition 4.1, S must then have a Green's function which contradicts our original assumption. We have therefore proved

PROPOSITION 5.2. *If S is a simply connected Riemann surface with a divergent curve and admitting no Green's function, then S is conformally equivalent to $\mathbb{C}$.*

We shall need the following

PROPOSITION 5.3. *If S is a simply connected Riemann surface with no divergent curves, then, for all $\zeta_1 \in S$, $\dot{S} = S \setminus \{\zeta_1\}$ is simply connected.*

Proof. Here the reader is offered a choice of two proofs. A proof via potential theory and covering spaces is given in the appendix. The deepest but quickest proof uses the classification of simply connected topological surfaces. A simply connected Riemann surface S is homeomorphic to $\mathbb{C}$ or to $\hat{\mathbb{C}}$ (see Ahlfors and Sario [3, pp. 90–104]). In $\mathbb{C}$ it is easy to find a divergent curve ϕ. The image of ϕ in S may be arbitrarily closely approximated by a divergent curve. Otherwise S is homeomorphic to $\hat{\mathbb{C}}$, $\dot{S}$ is homeomorphic to $\mathbb{C}$ and hence is simply connected.

THE UNIFORMIZATION THEOREM. *If S is a simply connected Riemann surface, then S is conformally equivalent to Δ, $\mathbb{C}$ or $\hat{\mathbb{C}}$.*

Proof. If S has a Green's function then Proposition 5.1 shows that S is equivalent to Δ. If S has no Green's function but has a divergent curve, then S is equivalent to $\mathbb{C}$. In any other case, $\dot{S} = S \setminus \{\zeta_0\}$ is simply connected and obviously has a divergent curve. It follows that $\dot{S} \simeq \Delta$ or $\dot{S} \simeq \mathbb{C}$.

If $f: \dot{S} \to \Delta$ is a conformal equivalence, then f is a bounded holomorpic function on $\dot{S}$, in particular it is bounded near ζ_0. By Riemann's theorem on removable singularities, f extends to a holomorphic map of S into $\bar{\Delta}$. By the maximum principle, $|f(\zeta_0)| < 1$. Since $f(\dot{S}) = \Delta$, there is some $\zeta_1 \in \dot{S}$ so that $f(\zeta_0) = f(\zeta_1)$. By the open mapping theorem, there are points ζ_0', ζ_1' near ζ_0 and ζ_1, respectively, so that $f(\zeta_0') = f(\zeta_1')$. But this contradicts the fact that $f \mid \dot{S}$ is injective. Therefore $\dot{S} \simeq \mathbb{C}$ and $S \simeq \hat{\mathbb{C}}$ which completes the proof.

To illustrate the use of the uniformization theorem, we note

COROLLARY 1. *Every Riemann surface is second countable and separable.*

Proof. These properties project from the universal cover. Δ, $\mathbb{C}$, and $\hat{\mathbb{C}}$ have these properties.

COROLLARY 2 (Picard's Theorem). *Let f be a meromorphic function in $\mathbb{C}$. If $\hat{\mathbb{C}} \setminus f(\mathbb{C})$ contains at least three points, then f is constant.*

Proof. Let $D = f(\mathbb{C})$. By the uniformization theorem, the universal cover $\tilde{D}$ of D is conformally equivalent to Δ. Let π denote the universal cover map, $\pi: \Delta \to D$. Then, locally, π^{-1} exists and $\pi^{-1} \cdot f$ may be continued along all paths in $\mathbb{C}$ to define a map $F: \mathbb{C} \to \Delta$. Since F is holomorphic, Liouville's theorem says F is constant; hence so is f.

6. Maskit's Work on the General Uniformization Problem. In a fifteen-year series of papers, Maskit resolved the general uniformization problem for surfaces of finite conformal type. Weyl noted that the problem has two aspects. One starts with a Riemann surface S. The first part of the problem is topological—namely, find all covering surfaces $D \subset \hat{\mathbb{C}}$ of S. Maskit's planarity theorem [12] classifies them in the following way. On S we find a set of simple closed loops $\{\alpha_i\}$ which are homotopically independent. By this we mean that the α_i are disjoint and not freely homotopic to each other or to the ideal boundary of S. To each α_i we assign a positive integer n_i. The set of pairs $P = \{(\alpha_i, n_i)\}$ determine a planar covering surface S_P which is defined as the largest covering surface on which $\alpha_i^{n_i}$ is a simple loop but α_i^k is not for $k < n_i$. This theorem gives a complete solution to the topological part of the problem.

The second part of the problem is the conformal mapping problem. Here we try to find all conformal maps of all surfaces S_p into $\hat{\mathbf{C}}$. The existence of such a map was proved by Koebe; it is his planarity theorem (see, for example, Tsuji [**18**]). Let f be the map and $D = f(S_p)$. The group of deck transformations G for the covering $D \to S$ lies in the conformal automorphism group Aut D of D. For purposes of classification, it is not important to know all conformal maps of S_p into $\hat{\mathbf{C}}$ but just one good one. All others are obtained by conformal maps of domains D in $\hat{\mathbf{C}}$. Maskit [**13**] found a very good one. Specifically he proved that D may be chosen so that for all $\gamma \in$ Aut D, γ is a Möbius transformation. The group G then becomes a group of Möbius transformations acting properly discontinuously on a domain $D \subset \hat{\mathbf{C}}$. Such objects had first been studied by Schottky and later by Fricke and Klein. They are called function groups. Now assume S has finite conformal type. Maskit further showed [**14**] D may be chosen so that each component of Int$(\hat{\mathbf{C}} \setminus D)$ is a Euclidean disc. Such groups, he called *Koebe groups*. In [**15**], he classified the Koebe groups. This solves the general uniformization for surfaces of finite conformal type.

Appendix

This appendix contains the proofs of Lemma 5.1 and Proposition 5.3. The proof of the former is rather short and we give it first.

Proof of Lemma 5.1. $S_t \subset S$ and ∂S_t is a piecewise analytic arc. As we noted in § 5, Proposition 4.3 implies that S_t has a Green's function for all $t \geq 0$. We claim that S_{t_0} is simply connected for all t_0. Let α be a closed curve in S_{t_0} based at ζ_0. Let

$$A(\alpha) = \{t \in [t_0, \infty) | \alpha \text{ is null homotopic in } S_t\}.$$

Since α is null homotopic in S, and the homotopy takes place in a compact subset of S, $A(\alpha) \neq \varnothing$ and is open. To see that $A(\alpha)$ is closed, observe that if $t_1 \in A(\alpha)$ then so is t for all $t > t_1$. Let $t_2 = \inf\{t | t \in A(\alpha)\}$. Choose a small disc D about $\phi(t_2)$. There is a homotopy F_1 of α to the constant map ζ_0 in $S_{t_2+\varepsilon}$. Choose a homeomorphism h of $\bar{D}$ so that $h | \partial D = \text{id}$ and h moves $\phi(t_2 + \varepsilon)$ to $\phi(t_2 - \varepsilon)$ as shown in Fig. 9. h may be extended by the identity to a homeomorphism of S_0. $h \circ F_1(I^2) \subset S_{t_2}$; hence $t_2 \in A(\alpha)$ and $A(\alpha)$ is closed. From the connectedness of $[t_0, \infty)$, it follows that $t_2 = t_0$ and α is null homotopic in S_{t_0}.

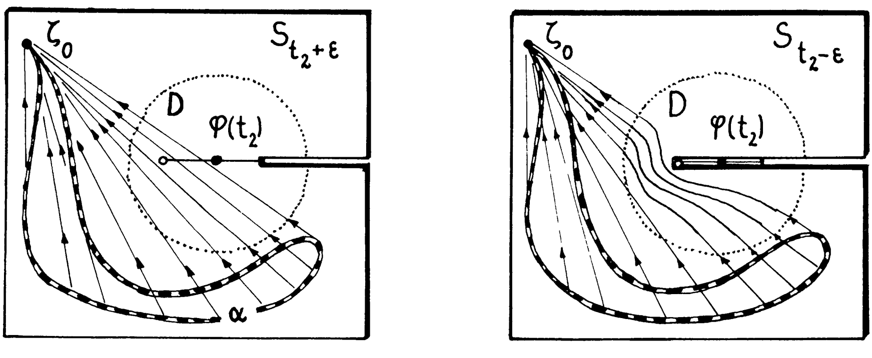

FIG. 9

Proof of Proposition 5.3. The proof of this proposition is modeled after the ideas of Riemann—at least as interpreted by Klein [**13**]. We shall solve a potential problem on a subsurface of S. Then, by studying the level sets and integral curves of the gradient field of that potential, we shall obtain a global topological result. The author begs the indulgence of the knowledgeable reader for presenting several basic results on holomorphic and harmonic functions.

Let f be a function holomorphic in a neighborhood of $0 \in \mathbb{C}$ with $f(0) \neq 0$. Then

$$f(z) = \sum a_n z^n = a_0 + a_k z^k + o(|z|^k)$$

where $o(|z|^k)$ means a function so that

$$\lim_{z \to 0} \frac{o(|z|^k)}{|z|^k} = 0.$$

k is the order of $f(z) - a_0$ at 0.

$\arg(f(z) - a_0) = \arg a_k + k \arg z + o(|z|^k)$. It follows that $\{z | \arg(f(z) - a_0) = 0\}$ consists of k analytic arcs which meet at 0 and whose tangents at 0 are equally spaced as a function of $\arg z$. If $k = 1$, then, near 0, $\{z | \arg(f(z) - a_0) = 0\}$ is a simple analytic arc. The points where $f'(z) = 0$ are discrete. If $u(z)$ is a harmonic function in a simply connected neighborhood of 0, then it has a harmonic conjugate $v(z)$ there.

$f(z) = \exp(u + iv)$ is holomorphic. 0 is called a *critical point* of u if $f'(0) = 0$. $\{z | v(z) = v(0)\}$ consists of k analytic arcs through 0.

The discussion above remains valid for points other than 0 and, indeed, is valid on Riemann surfaces.

Now let S be a simply connected Riemann surface with no divergent curves. Let $\zeta_1 \in S$ and $\bar{N}$ be a closed disc about ζ_1 with analytic boundary. Set $S_1 = S \setminus \bar{N}$. The reader may easily verify that S_1 is homeomorphic to $\dot{S}$, and we must only show that S_1 is simply connected. By Proposition 4.3, S_1 has a Green's function $g(\zeta, \zeta_0)$ for any $\zeta_0 \in S_1$.

As in the proof of Proposition 5.1, we let h_0 be a harmonic conjugate of $g(\zeta, \zeta_0) + \log|z|$ near ζ_0 and let $f_0(\zeta) = z(\zeta)\exp[-(g + \log|z| + ih_0)]$. $f_0(\zeta)$ may be analytically continued along all paths in S_1. Since we do not know *a priori* that S_1 is simply connected, the analytic continuation may not define a function.

Near ζ_0, $f(z(\zeta)) = a_1 z + o(|z|)$ and it follows that f is locally injective near ζ_0 and there is a neighborhood of ζ_0 containing no critical points of g. If g has critical points, let $M = \sup\{g(\zeta, \zeta_0) | \zeta \neq \zeta_0 \text{ and } \zeta \text{ a critical point of } g\}$. Otherwise let $M = 0$. Let $D = \{\zeta \in S_1 | g(\zeta, \zeta_0) > M\} \cup \{\zeta_0\}$ and $B \subset \mathbb{C}$ be the disc of radius e^{-M} about 0.

LEMMA B.1. *Analytic continuation of f_0 in D defines a holomorphic bijection $f: D \to B$.*

Proof. Let $T(\theta)$ be the curve in D starting at ζ_0 along which f_0 continues with constant argument. Since $T(\theta)$ contains no critical points, it is unique and may be parametrized by e^{-g}. Since S has no divergent curves, $T(\theta)$ is a simple arc from ζ_0 to $\partial D = \{\zeta | g(\zeta, \zeta_0) = M\}$.

Let $\zeta \in D \setminus \{\zeta_0\}$ and θ_0 be the argument of some analytic continuation f_1 of f_0 to ζ. Then $|f_1(\zeta)| = \exp(-g(\zeta, \zeta_0)$. Let A be the necessarily unique arc through ζ with $\arg f_1 = \theta_0$. Extend A so that it is maximal with respect to being a curve along which f_1 continues with constant argument in D. Since g is monotone on A and D contains no critical points, A is simple and analytic. Thus it is either divergent or contains an arc $T(\theta_1)$ from ζ to ζ_0. The first possibility is ruled out by our initial assumption. Thus $A = T(\theta_1)$ and we may write $\theta(\zeta) = \theta_1$. Thus each point $\zeta \in D \setminus \{\zeta_0\}$ is parametrized in polar coordinates by $(e^{-g}, \theta(\zeta))$, with ζ_0 being the origin. The parametrization is continuous and injective, hence is a homeomorphism of D with B. It follows that D is simply connected and the parametrization is precisely the analytic continuation of f_0.

If $M = 0$, we are done, since $D = S_1$ and f is a homeomorphism of D with Δ. If $M > 0$, we have

LEMMA B.2. *If $M > 0$, then there is a critical point $\zeta_2 \in \partial D$.*

Proof. First suppose that there is a critical point ζ_2 so that $g(\zeta_2, \zeta_0) = M$. Then arbitrarily close to ζ_2 are points in D and $\zeta_2 \in \partial D$. Otherwise there is a sequence of critical points ζ_n so that $g(\zeta_n, \zeta_0) \to M$. Through each ζ_n there is an arc A_n along which g increases and any continuation of f_0 has constant argument. Since S has no divergent curves, A_n connects ζ_n to ζ_0. For some θ_n, $A_n \cap D = T(\theta_n)$. On a subsequence, $\theta_n \to \theta$ and $T(\theta)$ must have an endpoint ζ_θ. From the local structure of the curves $\arg f = $ constant near ζ_θ, we see that ζ_θ is a limit of critical points. Since critical points of g are discrete in S_1, this is impossible and the Lemma is proved.

LEMMA B.3. *If $M > 0$, then there exists a closed annulus $\bar{A} \subset S$ with piecewise analytic boundary so that $S \setminus A$ is connected. Here $A = \operatorname{Int}\bar{A}$.*

Proof. Using the previous lemma, there is a critical point $\zeta_2 \in \partial D$. Again by the local structure near a critical point, there are at least two curves $T(\theta_1)$ and $T(\theta_2)$ emanating from ζ_2. Choose closed discs $\bar{B}_0$ and $\bar{B}_2$, with centers ζ_0 and ζ_2, respectively, which are defined by $\bar{B}_0 = \{\zeta \mid g(\zeta, \zeta_0) \geq \varepsilon^{-1}\}$ and $\bar{B}_2 = \{\zeta \,\|\, g(\zeta, \zeta_0) - g(\zeta_2, \zeta_0)| \leq \varepsilon\}$. For ε sufficiently small, the $\bar{B}_i$ are disjoint and contain no critical points. For θ sufficiently close to θ_i, $T(\theta)$ is a curve from B_0 to B_2. Let $\Sigma_i = \cup\{T(\theta) \mid |\theta - \theta_i| < \delta\}$. Σ_i is a strip from B_0 to B_2 for $i = 1, 2$. Let $\bar{A} = \Sigma_1 \cup \Sigma_2 \cup B_0 \cup B_2$ as in Fig. 10. For nearly all but finitely many θ_i, $T(\theta_i) \cap \bar{B}_2 = \varnothing$ for ε sufficiently small. Let $\eta \in S \setminus A$ where $A = \operatorname{Int}\bar{A}$. Any analytic continuation f_η of f_0 to a neighborhood of η may be continued along a curve with decreasing modulus and constant argument to ∂S_1. Thus if $g(\eta, \zeta_0) \leq M$, the curve will not meet A and η and ∂S_1 lie in the same path component of $S \setminus A$. If $\eta \in D \setminus A$, then η lies in a complementary sector of $\Sigma_1 \cup \Sigma_2$ in D. Choose $T(\theta)$ in that same sector so that continuation along $T(\theta)$ with decreasing modulus and constant argument does not lead us into B_2. We may then further continue to ∂B. ζ, $T(\theta)$ and ∂B thus lie in the same path component of $S \setminus A$, and $S \setminus A$ is path connected.

PROPOSITION B.1. *If S is any Riemann surface and $\bar{A}$ is a closed annulus on S with piecewise analytic boundary and such that $S \setminus \bar{A}$ is path connected, then S is not simply connected.*

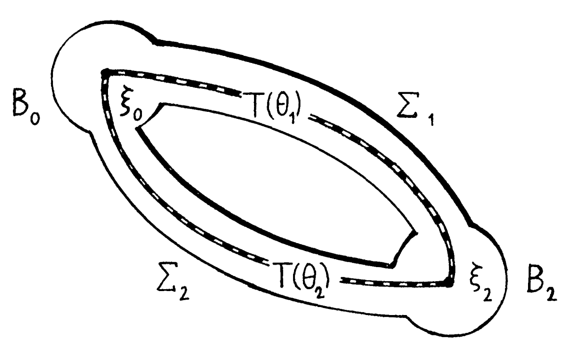

FIG. 10

Proof. $\bar{A}$ has two boundary components F_1 and F_2. Further, there is a simple closed curve α in S so that $\alpha \cap \operatorname{Int}A$ is a simple arc from one boundary component to the other.

Solve the Dirichlet problem in $\bar{A}$ with boundary data $\phi_1 | F_1 = 1$ and $\phi_1 | F_2 = 0$. The Dirichlet problem in $S \setminus \operatorname{Int}A$ may be solved with boundary data $\phi_2 | F_1 = 1$ and $\phi_2 | F_2 = 2$. Set $\phi = \exp i\pi\phi_i$ and notice that ϕ is defined on S and continuous. $\phi \circ \alpha$ has winding number $= 1$ about $z = 0$. If α is null homotopic in S then $\phi \circ \alpha$ is null homotopic in $\partial\Delta$. The latter is impossible; hence S is not simply connected.

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BLACK WOMEN IN MATHEMATICS IN THE UNITED STATES

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Increased attention has been focused on women in mathematics during the past decade, but when I was invited to speak on Black women in mathematics, I could find only two references—a talk by Vivienne Malone Mayes [1] at the Summer Meeting in Kalamazoo in 1975 sponsored by the Association for Women in Mathematics, and the AWM panel I chaired in Atlanta in January, 1978 [2]. Since then I have collected much information, and this article tells about the American Black women holding doctoral degrees in mathematics, all but two of whom I have talked with in the past three years.

The 1970 decennial census revealed more than 1,100 Black women who reported themselves as mathematicians. In that census 244 said that they were college or university teachers, and, of

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Adapted from an invited address given at the annual meeting of the Association of Mathematics Teachers of New England in Springfield, Massachusetts, on November 2, 1979.

Patricia Kenschaft received her Ph.D. with a specialty in functional analysis from the University of Pennsylvania in 1973 under the direction of Edward Effros. Since then she has taught at Montclair State College in New Jersey. She is the author or co-author of three textbooks for nontechnical majors published by Worth Publishers, Inc., and is currently preparing a paper on the life of Charlotte Scott, vice president of the AMS in 1906.—*Editors*