MAT 342 Fall 2016, Sample Midterm 2,
Actual Midterm is 10:00-10:53am, Wed., November 16, 2016

Name	ID	

THIS EXAM IS WORTH 50 POINTS. EACH QUESTION IS WORTH ONE POINT. NO BOOKS, NOTES OR CALCULATORS ARE ALLOWED.

1-10: Write C (for converges) or D (for diverges) for each sequence or series.

(1) C $(\frac{i}{2})^n$, $n = 1, 2, \ldots$.

(2) D $1 + n$, $n = 1, 2, \ldots$.

(3) C $\sum_{n=0}^{\infty} \frac{1}{n^2}$

(4) C $\sum_{n=0}^{\infty} (\frac{i}{3})^n$

(5) C $\sum_{n=[illegible]}^{\infty} \frac{i^n}{n^3}$.

(6) D $i^n + (-i)^n$, $n = 1, 2, \ldots$.

(7) C $|\exp(i\pi n/4)|$, $n = 1, 2, \ldots$.

(8) C $\sum_{n=-\infty}^{\infty} \frac{i^n}{1+n^2}$

(9) D $\sum_{n=-\infty}^{\infty} 2^{-n}$

(10) C $\frac{n^2+n-1}{2n^2-3}$, $n = 1, 2, \ldots$.

11-20: Match each function with its Maclaurin series.

(11) Q $z^2 \cos(z)$

(12) N $\sinh(z)$

(13) D $\frac{1}{(1-z)^2}$

(14) H $\log(1 - z)$

(15) Q $\cosh(2z)$

(16) K e^{-z}

(17) B $\cos(z)$

(18) E $\frac{1}{1-z}$

(19) Q $\sin(z)$

(20) M $\exp(z^2)$

A. $z^3 - \frac{1}{6}z^5 + \frac{1}{120}z^7 - \ldots$

B. $1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \ldots$

C. $z^3 - \frac{1}{2}z^5 + \frac{1}{24}z^7 - \ldots$

D. $1+2z+3z^2+4z^3+5z^4+\ldots$

E. $1 + z + z^2 + z^3 + \ldots$

F. $1 + z^4 + z^6 + z^8 + \ldots$

G. $1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \ldots$

H. $z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \frac{1}{4}z^4 + \ldots$

I. $z + z^2 + z^3 + z^4 + \ldots$

J. $1 - \frac{1}{2}z^4 + \frac{1}{24}z^6 - \ldots$

K. $1 - z + \frac{1}{2}z^2 - \frac{1}{6}z^3 + \ldots$

L. $1 + 2z^3 + \frac{2}{3}z^4 + \ldots$.

M. $1 + z^2 + \frac{1}{2}z^4 + \frac{1}{6}z^6 + \ldots$

N. $z + \frac{1}{6}z^3 + + \frac{1}{120}z^5 + \ldots$

P. $1 + \frac{1}{2}z^2 + \frac{1}{24}z^4 + \ldots$

Q. none of the above

21-30: Write T (for true) or F (for false) in each box.

(21) $\boxed{\text{F}}$ If $\sum_{n=0}^{\infty} a_n z^n$ converges at $z = 1$, then if converges for all $|z| \leq 1$.

(22) $\boxed{\text{T}}$ If $\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n z^n$ for all $|z| < 1$, then $a_n = b_n$ for all $n = 0, 1, 2, \ldots$.

(23) $\boxed{\text{T}}$ If $\sum_{n=0}^{\infty} a_n z^n$ converges at $z = 1$, then the $\{a_n\}$ are a bounded sequence.

(24) $\boxed{\text{F}}$ The power series for $\frac{1}{z^2+1}$ at $z = 2$ has radius of convergence equal to 1.

(25) $\boxed{\text{F}}$ If $\sum_{n=0}^{\infty} a_n z^n$ converges at $z = 1$, then if converges for all z with $|z| = 1$.

(26) $\boxed{\text{T}}$ If f has a power series expansion on $|z - 1| < 2$, then f is analytic in that disk.

(27) $\boxed{\text{T}}$ The function $f(z) = \sin(x)$ has a convergent Laurent series expansion on $|z| > \pi/2$.

(28) $\boxed{\text{F}}$ If f has an essential singularity at 0 it takes every complex value in every neighborhood of 0.

(29) $\boxed{\text{F}}$ If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are both analytic on $|z| < 1$, then $f(z)/g(z) = \sum_{n=0}^{\infty} \frac{a_n}{b_n} z^n$ for all $|z| < 1$.

(30) $\boxed{\text{T}}$ If $f(z)$ has a pole of order m at $z = 0$, then $g(z) = z^m f(z)$ has removable singularity at 0.

31-35: Compute the residue of each function at the given point.

(31) $\boxed{0}$ $\frac{\cos z}{z^4}$ at $z = 0$.

$$\frac{\cos z}{z^4} = \frac{1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \cdots}{z^4} = \frac{1}{z^4} - \frac{1}{2}\frac{1}{z^2} + \frac{1}{24} - \cdots$$

NO z^{-1} term

(32) $\boxed{0}$ $\exp(z + z^2)$ at $z = 0$.

analytic at $z=0$

(33) $\boxed{-i/6}$ $\frac{1}{1+z^6}$ at $z = i$.

$$\operatorname{Res}_{z_0} \frac{P(z)}{g(z)} = \frac{P(z_0)}{g'(z_0)} \quad \text{if simple pole}$$

(34) $\boxed{\pi/4}$ $\frac{\mathrm{Log} z}{z^2+1}$ at $z = i$.

$$\frac{\log z}{(z-i)(z+i)}, \quad \operatorname{Res} = \frac{\log i}{i+i} = \frac{i\pi/2}{2i} = \pi/4$$

(35) $\boxed{-4}$ $\frac{4z-5}{z(z-1)}$ at $z = \infty$.

$$\operatorname{Res}_{\infty} = \operatorname{Res}_{0} -\frac{1}{z^2} f\left(\frac{1}{z}\right) = \operatorname{Res}_{0} -\frac{1}{z^2}\frac{\frac{4}{z}-5}{\frac{1}{z}\left(\frac{1}{z}-1\right)} = -\frac{1}{z}\frac{4-5z}{(1-z)}$$

36-40: For each function and point, identity the type of singularity: R = removable, P = pole, E = essential singularity.

(36) $\boxed{E}$ $f(z) = \sin(\frac{1}{1-z})$, $z = 1$.

(37) $\boxed{P}$ $f(z) = \exp(1 + z + z^2)/\sin(z)$, $z = 0$.

(38) $\boxed{R}$ $f(z) = \frac{1-\cos z}{z^2}$, $z = 0$.

(39) $\boxed{P}$ $f(z) = \frac{1}{1-\cos z}$, $z = 0$.

(40) $\boxed{R}$ $f(z) = \frac{z}{\sin z}$, $z = 0$.

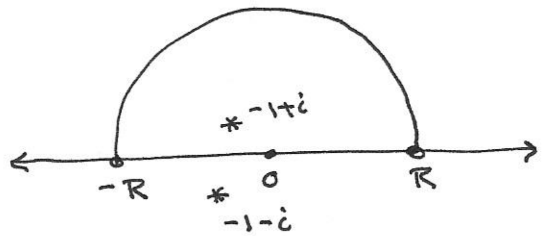
41-46 Evaluate $\int_{-\infty}^{\infty} f(x)dx$ where $f(x) = \frac{1}{x^2+2x+2}$, following the steps below.

(41) State the Cauchy residue theorem

Let C be a simple closed contour described in positive sense. If a function f is analytic inside and on C except for a finite number of singular points $z_1, \dots, z_n$ inside C, Then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z_k} f(z)$$

(42) Draw a closed contour C_R that contains the interval $[-R, R]$ and so that the integral over the rest of the contour tends to zero as $R \nearrow \infty$. Label the points $-R$ and R.



(43) $\boxed{-1+i}$ List all the singularities of f inside the contour.

By quadratic formula roots of z^2+2z+2 are

$$\frac{-2 \pm \sqrt{4-8}}{2} = \frac{-1 \pm \sqrt{-4}}{2} = -1 \pm i$$

(44) $\boxed{1/2i}$ Compute the residues of f at the singularities inside the contour.

simple pole

$$\operatorname{Res}_{z=z_1} \frac{1}{(z-z_1)(z-z_2)} = \frac{1}{z_1 - z_2} = \frac{1}{(-1+i)-(-1-i)} = \frac{1}{2i}$$

(45) $\boxed{\pi}$ Compute the integral $\int_{-\infty}^{\infty} f(x)dx$.

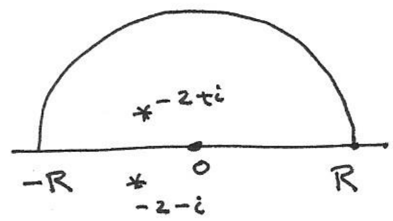
$$\int = 2\pi i \cdot \operatorname{Res} = 2\pi i \cdot \frac{1}{2i} = \pi$$

41-46 Evaluate $\int_{-\infty}^{\infty} f(x)dx$ where $f(x) = \frac{\sin x}{x^2+4x+5}$, following the steps below.

(46) $\boxed{e^{iz}/(z^2+4z+5)}$ Give the analytic function $f(z)$ that you will apply the Cauchy residue theorem to evaluate this integral.

For $y=0$, $\frac{e^{iz}}{z^2+4z+5} = \frac{e^{ix}}{x^2+4x+5} = \frac{\cos x + i\sin x}{x^2+4x+5}$, So we want imaginary part of integral

(47) Draw a closed contour C_R that contains the interval $[-R, R]$ and so that the integral over the rest of the contour tends to zero as $R \nearrow \infty$. Label the points $-R$ and R.



(48) $\boxed{-2+i}$ List all the singularities of $f(z)$ inside the contour.

By Quadratic formula roots of z^2+4z+5 are

$$\frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$$

(49) $\boxed{e^{-1-2i}/2i}$ Compute the residues of f at all the singularities inside the contour.

Simple pole, $\frac{\phi(z)}{(z-z_0)} \Rightarrow \text{Res} = \phi(z_0)$

$$\Rightarrow \quad \text{Res} = \frac{e^{i(-2+i)}}{(-2+i)-(-2-i)} = \frac{e^{-1-2i}}{2i}$$

(50) $\boxed{-\frac{\pi}{e}\sin(2)}$ Compute the integral $\int_0^{\infty} f(x)dx$.

$$\int \frac{e^{iz}}{z^2+4z+5}\,dz = 2\pi i \cdot \frac{e^{-1-2i}}{2i} = \pi \cdot \tfrac{1}{e} \cdot [\cos(-2) + i\sin(-2)]$$

$$\int \frac{\sin x}{x^2+4x+5}\,dx = \text{Im}\left(\int \frac{e^{iz}}{z^2+4z+5}\,dz\right) = \text{Im}\left(\pi \tfrac{1}{e}(\cos(-2) + i\sin(-2))\right)$$

$$= \frac{\pi}{e}\sin(-2)$$

$$= -\frac{\pi}{e}\sin(2)$$