MAT 342 Fall 2016, Midterm 2, 10:00-10:53am, Wed., November 16, 2016

Name	ID	

THIS EXAM IS WORTH 50 POINTS. EACH QUESTION IS WORTH ONE POINT, EXCEPT FOR PROBLEMS 21-30, THAT ARE WORTH TWO POINTS EACH. NO BOOKS, NOTES OR CALCULATORS ARE ALLOWED.

1-5: Write C (for converges) or D (for diverges) for each sequence or series.

(1) D i^n, $n = 1, 2, \ldots$.

(2) D $\sum_{n=1}^{\infty} \frac{1}{n}$

(3) C $\sum_{n=1}^{\infty} (\frac{i}{2})^n$

(4) C $|i^n|$, $n = 1, 2, \ldots$.

(5) C $\sum_{n=1}^{\infty} \frac{i^n}{n}$

6-10: For each function and point, identity the type of singularity: R = removable, P = pole, E= essential singularity.

(6) P $f(z) = \frac{z^2+1}{z^2-1}$, $z = 1$.

(7) E $f(z) = \exp(\frac{1}{z})$, $z = 0$.

(8) R $f(z) = \frac{\sin z}{z}$, $z = 0$.

(9) R $f(z) = \frac{z^2}{1-\cos z}$, $z = 0$.

(10) P $f(z) = \frac{\cos z}{\sin z}$, $z = 0$.

11-20: Write T (for true) or F (for false) in each box.

(11) T If $\sum_{n=0}^{\infty} a_n z^n$ converges at $z = 1$, then if converges for all $|z| < 1$.

(12) T If $\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n z^n$ for all $|z| < 1$, then $a_n = b_n$ for all $n = 0, 1, 2, \ldots$.

(13) F If $\sum_{n=0}^{\infty} a_n z^n$ converges at $z = 1$, then if converges uniformly on $|z| < 1$.

(14) F The power series for $\frac{1}{z^2+1}$ at $z = 1$ has radius of convergence equal to 1.

(15) F If $\sum_{n=0}^{\infty} a_n z^n$ converges at $z = 1$, then if converges for $z = -1$.

(16) T If f has a power series expansion on $|z - 1| < 2$, then f' has a power series expansion on the same disk.

(17) F The function $f(z) = 1/\sin(x)$ has a convergent Laurent series expansion on $|z| > \pi/2$.

(18) F The function $f(z) = \exp(\frac{1}{z})$ takes every complex value in every neighborhood of 0.

(19) F If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are both analytic on $|z| < 1$, then $f(z)g(z) = \sum_{n=0}^{\infty} a_n b_n z^n$ for all $|z| < 1$.

(20) T If $f(z)$ is analytic on $|z| < 1$ and every derivative is zero at $z = 0$, i.e., $f^{(n)}(0) = 0$, then f must be constant.

21-25: Compute the residue of each function at the given point. Each problem is worth TWO points.

(21) $\boxed{-1/2}$ $\frac{\cos z}{z^3}$ at $z = 0$.

$$\frac{\cos z}{z^3} = \frac{1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \cdots}{z^3}$$

$$= \frac{1}{z^3} - \frac{1}{2}\frac{1}{z} + \frac{1}{24}z - \cdots$$

(22) $\boxed{0}$ e^z at $z = 0$.

is analytic at 0.

(23) $\boxed{1/2i = \frac{-i}{2}}$ $\frac{1}{1+z^2}$ at $z = i$.

$$\frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)} \qquad \text{Res} = \frac{1}{i+i} = \frac{1}{2i}$$

(24) $\boxed{-\pi^3/16}$ $\frac{(\log z)^3}{z^2+1}$ at $z = i$.

simple pole at $z = i$, $\text{Res} = \frac{(\text{Log } i)^3}{i+i} = \frac{i^3\pi^3/2^3}{2i} = -\frac{\pi^3}{16}$

(25) $\boxed{3/2}$ $\frac{z^3(1-3z)}{(1+z)(1+2z^4)}$ at $z = \infty$.

$$\operatorname*{Res}_{\infty} = \operatorname*{Res}_{0} -\frac{1}{z^2}f\left(\frac{1}{z}\right) = \operatorname*{Res}_{0} -\frac{1}{z^2}\frac{z^{-3}(1-3z^{-1})}{(1+z^{-1})(1+2z^{-4})}$$

$$= \operatorname{Res} -\frac{1}{z}\frac{(z-3)}{(z+1)(z^4+2)} = -\frac{-3}{1\cdot 2} = \frac{3}{2}$$

26-30: Match each function with its Maclaurin series. Each answer is worth TWO points.

(26) $\boxed{G}$ $\cos(z)$

(27) $\boxed{A}$ $\frac{1}{1-z}$

(28) $\boxed{D}$ $\exp(z^2)$

(29) $\boxed{F}$ $z^2\sin(z)$

(30) $\boxed{I}$ $\frac{1}{(1-z)^2}$

A. $1 + z + z^2 + z^3 + \ldots$

B. $1 + z^4 + z^6 + z^8 + \ldots$

C. $1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \ldots$

D. $1 + z^2 + \frac{1}{2}z^4 + \frac{1}{6}z^6 + \ldots$

E. $z + \frac{1}{6}z^3 + \frac{1}{120}z^5 + \ldots$

F. $z^3 - \frac{1}{6}z^5 + \frac{1}{120}z^7 - \ldots$

G. $1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \ldots$

H. $z^3 - \frac{1}{2}z^5 + \frac{1}{24}z^7 - \ldots$

I. $1+2z+3z^2+4z^3+5z^4+\ldots$

J. none of the above

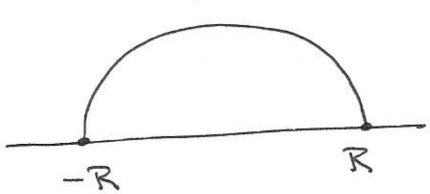
31-35 Evaluate $\int_{-\infty}^{\infty} f(x)dx$ where $f(x) = \frac{x^2}{(x^2+1)(x^2+4)}$, following the steps below.

(31) State the Cauchy residue theorem

Let C be a simple closed contour in the positive direction and suppose f is analytic inside and on C except for finitely many poles at $z_1, \ldots, z_n$. Then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_0} f(z)$$

(32) Draw a closed contour C_R that contains the interval $[-R, R]$ and so that the integral over the rest of the contour tends to zero as $R \nearrow \infty$. Label the points $-R$ and R.



(33) $\boxed{i\ ,\ 2i}$ List all the singularities of f inside the contour.

(34) $\boxed{\frac{1}{3i}\ ,\ -\frac{1}{6i}}$ Compute the residues of f at the singularities inside the contour.

$$\operatorname*{Res}_{i} = \frac{z^2}{(z+i)(z^2+4)}\bigg|_{z=i} = \frac{-1}{2i(-1+4)} = \frac{-1}{6i}$$

$$\operatorname*{Res}_{2i} = \frac{z^2}{(z+2i)(z^2+1)}\bigg|_{z=2i} = \frac{-4}{4i(-3)} = \frac{1}{3i}$$

(35) $\boxed{\pi/3}$ Compute the integral $\int_{-\infty}^{\infty} f(x)dx$.

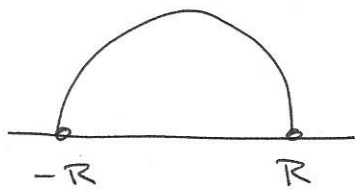
$$\int = 2\pi i \sum$$

$$= 2\pi i \left(\frac{1}{3i} - \frac{1}{6i}\right) = \frac{2\pi}{6} = \pi/3$$

36-40 Evaluate $\int_0^\infty f(x)dx$ where $f(x) = \frac{\cos ax}{x^2+1}$, following the steps below.

(36) $\boxed{e^{iz}/z^2+1}$ Give the analytic function $g(z)$ to which you will apply the Cauchy residue theorem.

(37) Draw a closed contour C_R that contains the interval $[-R, R]$ and so that the integral over the rest of the contour tends to zero as $R \nearrow \infty$. Label the points $-R$ and R.



(38) $\boxed{i}$ List all the singularities of $g(z)$ inside the contour.

(39) $\boxed{e^{-1}/2i}$ Compute the residues of g at all the singularities inside the contour.

$$\left.\frac{e^{iza}}{z+i}\right|_{z=i} = \frac{e^{-a}}{2i}$$

(40) $\boxed{\pi/2e^a}$ Compute the integral $\int_0^\infty f(x)dx$.

$$\int_0^\infty f(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)dx$$

$$= \frac{1}{2}\cdot 2\pi i \cdot \sum_i \operatorname{Res} f$$

$$= \pi i \cdot \frac{1}{e^a}\cdot\frac{1}{2i}$$

$$= \frac{\pi}{2e^a}$$