MAT 342 Fall 2016, Midterm 1, 10:00-10:53am, Friday, October 14, 2016

Name	ID	

THIS EXAM IS WORTH 50 POINTS. EACH QUESTION IS WORTH ONE POINT. NO BOOKS, NOTES OR CALCULATORS ARE ALLOWED.

1-10: Write T (for true) or F (for false) in each box.

(1) T $(1+i)+i+(i-2)=-1+3i$

(2) F $(2+i)(3-i)=6-i$

(3) F $(1+i)^3=-1-i.$

(4) T $i^{2016}=1$

(5) F $e^{-\pi i/4}=1-i$

(6) T $e^{3\pi i}=-1$

(7) T $\frac{i}{1+i}=\frac{1+i}{2}$

(8) T $\mathrm{Arg}(-1)=\pi$

(9) F $\log(i)=\{i(\pi+2\pi n): n\in\mathbb{Z}\}$

(10) F $\sin(i)=\frac{1}{2}(e+\frac{1}{e}).$

11-15: Place the letter of the corresponding point in the box.

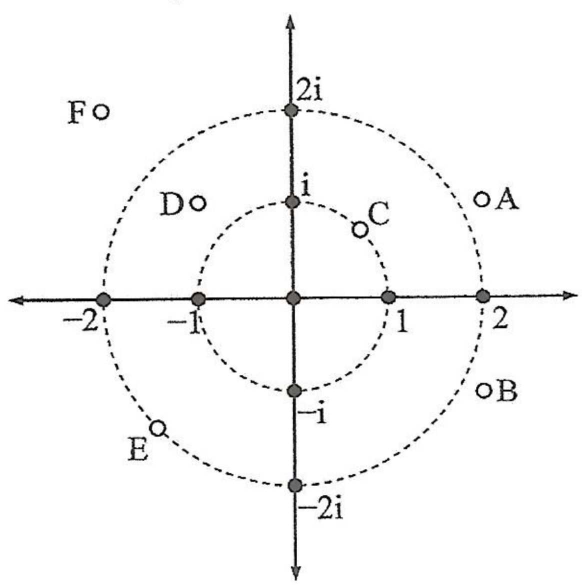
(11) C $|z|=1$

(12) A $z=\overline{B}.$

(13) C $z^2=i$

(14) D $|z|=\sqrt{2}$

(15) E $\mathrm{Re}(z^2)=0, \mathrm{Im}(z)<0.$



16-20: Match each function with its definition. Assume $z = x + iy$.

(16) $\boxed{E}$ $\cosh(z)$

(17) $\boxed{I}$ e^z

(18) $\boxed{B}$ $\cos(z)$

(19) $\boxed{M}$ z^i

(20) $\boxed{G}$ $\exp(\overline{z})$

A. $\frac{1}{2i}(e^{iz} - e^{-iz})$

B. $\frac{1}{2}(e^{iz} + e^{-iz})$

C. $(-i)\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}$

D. $\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}$

E. $\frac{1}{2}(e^z + e^{-z})$

F. $e^y(\cos x + i \sin x)$

G. $e^x(\cos y - i \sin y)$

H. $(i)\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}$

I. $e^x \cos(y) + ie^x \sin(y)$

J. $e^{z \log i}$

K. $\frac{1}{2}(e^z - e^{-z})$

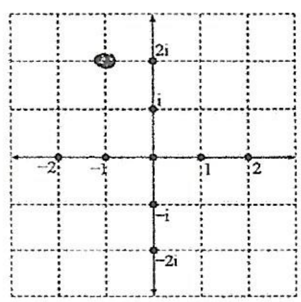
L. $\frac{1}{2} \log \frac{1+z}{1-z}$

M. $e^{i \log z}$

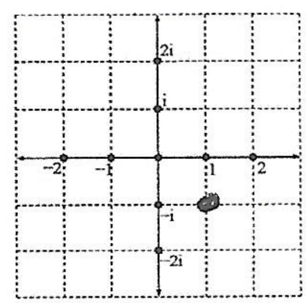
N. none of the above

21-25: Draw the following points or regions as accurately as you can.

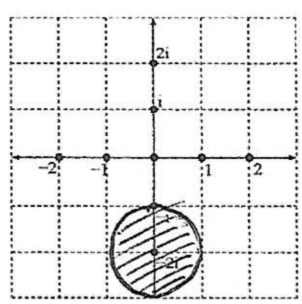
(21) Draw the point $z = -1 + 2i$.



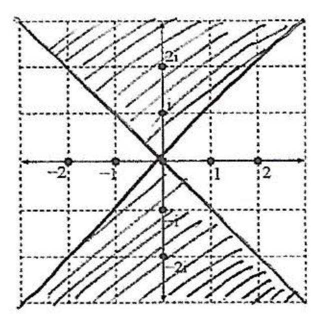
(22) Draw the point $\overline{z}$, where $z = -\sqrt{2} \cdot e^{3\pi/4}$.



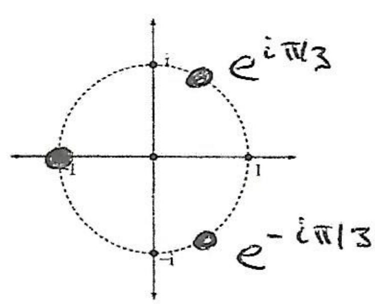
(23) Draw the region $|z + 2i| \leq 1$.



(24) Draw the region $\text{Re}(z^2) \leq 0$.



(25) Draw all solutions of $z^3 = -1$



26-35: Write T (for true) or F (for false) in each box.

(26) $\boxed{\text{T}}$ $p(z) = z^2 + 2z + 17$ has roots at $-1 \pm 4i$.

(27) $\boxed{\text{T}}$ If $f = u + iv$ and u, v have continuous partial derivatives that satisfy the Cauchy-Riemann equations on a domain D, then f is analytic on D.

(28) $\boxed{\text{T}}$ If f is analytic on a disk D, then f must have an anti-derivative on D.

(29) $\boxed{\text{F}}$ The function $1/z$ has an antiderivative on $\mathbb{C} \setminus \{0\}$.

(30) $\boxed{\text{F}}$ A polynomial of degree n must be zero at n distinct complex values.

(31) $\boxed{\text{T}}$ If f is analytic on a domain D, then f' must also be analytic on D.

(32) $\boxed{\text{T}}$ If f is entire, then $\int_C f(z)dz = 0$ for any closed contour C.

(33) $\boxed{\text{T}}$ The polar form of the Cauchy-Riemann equations are $ru_r = v_\theta$, $u_\theta = -rv_r$.

(34) $\boxed{\text{T}}$ If $f = u + iv$ is analytic and $u = v$ everywhere, then f must be constant.

(35) $\boxed{\text{F}}$ If $f(z)$ is a branch of $\sqrt{z}$ on a domain D containing both 1 and -1, then $f(1) = 1$ implies $f(-1) = i$.

36-40: Give a precise statement of each definition or result (1pt each):

(36) State the fundamental theorem of algebra.

Any complex polynomial of degree at least one has a zero.

(37) Define "entire function".

f is entire if it analytic on the whole plane (or is differentiable on the whole plane)

(38) State Cauchy's formula (include the assumptions).

If f is analytic everywhere inside and on a closed contour G and z_0 is a point interior to G, then $f(z_0) = \frac{1}{2\pi i} \int_G \frac{f(z)}{z - z_0}\, dz$.

(39) State the Cauchy-Riemann equations for $f = u + iv$.

$$u_x = v_y$$
$$u_y = -v_x$$

(40) State Cauchy's inequality for the nth derivative of f (include the assumptions).

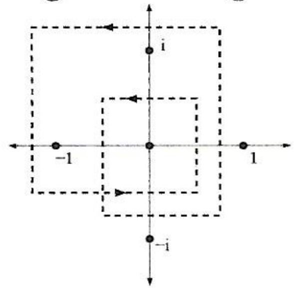
Suppose f is analytic inside and on a circle C_R of radius R and $|f(z)| \leq M$ on C_R. Then

$$|f^{(n)}(z_0)| \leq \frac{n!\, M}{R^n}$$

for $n = 0, 1, 2, \ldots$ where z_0 is center of C_R.

41-45 Evaluate each integral for the given contour; put answers in the boxes.

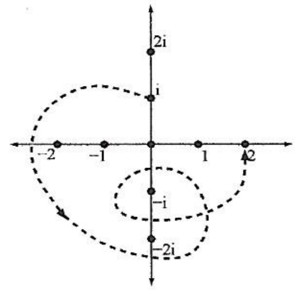
(41) $\int_C e^{\sin(z^2)}dz$



$\boxed{0}$

Integrand is entire function

(42) $\int_C \frac{dz}{z^2}$



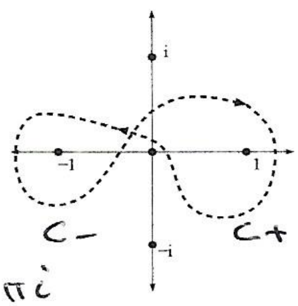
$\boxed{-\frac{1}{2} - i}$

$$= -\frac{1}{z}\Big|_i^2 = -\frac{1}{2} + \frac{1}{i}$$

$$= -\frac{1}{2} - i$$

Since $\left(-\frac{1}{z}\right)' = \frac{1}{z^2}$

(43) $\int_C \frac{dz}{z^2-1}$



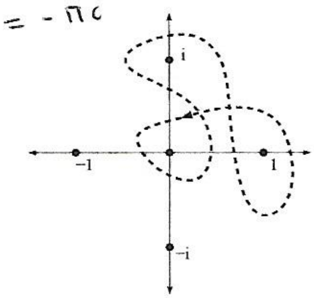
$\boxed{2\pi i}$

$C = C_+ + C_-$

$$\int_{C_-} \frac{dz}{(z-1)(z+1)} = \frac{2\pi i}{-2} = -\pi i$$

$$\int_{C_+} \frac{dz}{(z-1)(z+1)} = -\frac{2\pi i}{2} = -\pi i$$

(44) $\int_C \frac{e^z}{z^2+1}dz$

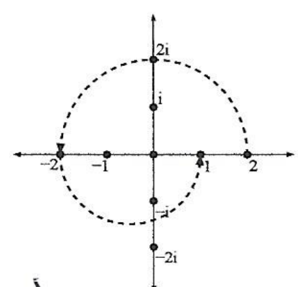


$\boxed{-\pi e^i}$

$$= \int_{C_1} \frac{e^z}{(z+i)(z-i)}dz$$

$$= -2\pi i \frac{e^i}{i+i}$$

(45) $\int_C \frac{1}{z}dz$



$\boxed{2\pi i - \ln 2}$

$$2\pi i = \int_{C_1} \frac{1}{z}dz + \int_1^2 \frac{1}{x}dx$$

$$\Rightarrow \int_{C_1} \frac{1}{z}dz = 2\pi i - \int_1^2 \frac{1}{x}dx = 2\pi i - \ln 2$$

46-50: Answer each question.

(46) Write the function $f(z) = z \cdot e^z$ in the form $u(x,y) + iv(x,y)$ with u, v real-valued.

$$f(z) = (x+iy)\, e^x (\cos y + i \sin y)$$

$$= \underbrace{e^x (x\cos y - y \sin y)}_{u} + i\, \underbrace{e^x (y \cos y + x \sin y)}_{v}$$

(47) Give an example of an entire function that is bounded on the real line, but unbounded on the plane.

$\sin(z)$

(48) If f is entire and $|f(z)| \le 1$ for $|z| \le 4$, how large can $|f''(0)|$ be? Justify your answer.

By the Cauchy estimate with $M=1,\ n=2,\ R=4$

$$|f''(0)| \le \frac{n!\, M}{R^2} = \frac{2 \cdot 1}{4^2} = \frac{1}{8}$$

(49) Is $u(x,y) = x^2 \cdot e^y$ the real part of an analytic function? Explain why or why not.

No. $u_{xx} = 2 \cdot e^y$ and $u_{yy} = x^2 e^y$, so

$u_{xx} + u_{yy} \neq 0$, so u is not harmonic.

(50) If $|z| = 1$, show that $1/z = \overline{z}$.

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{\overline{z}}{1} = \overline{z}$$