PROBLEM SET 4

1. Compute the integral of Lebesgue's function $\int_0^1 F(x)dm$ from Chapter 2.
2. If $f_n(x) = \sin^2(nx)$ find both $\int_0^{2\pi} \liminf_{n\to\infty} f_n(x)dm$ and $\liminf_{n\to\infty} \int_0^{2\pi} f_n(x)dm$.
3. What is $\lim_{n\to\infty} \int_{-\infty}^{\infty} x^n e^{-n|x|}dm$? Find the limit and prove it is correct.
4. Suppose $\{f_n\}$ is a sequence of functions that converges almost everywhere to a function f and define $F_n = \sup_{k=1,\dots,n} |f_n|$. Show that if the integrals of F_n remain bounded as $n \to \infty$ then $\lim_n \int f_n dm = \int f dm$.
5. Given a measureable function f define its "maximal function" as

$$Mf(x) = \sup_{t>0} \frac{1}{2t} \int_{x-t}^{x+t} |f(y)|dm(y).$$

   If f is integrable on the reals, does Mf have to be integrable?
6. Show that $\sum_{n=1}^{\infty} \cos^n(2^n x)$ converges for a.e. x, but diverges on a dense set of x's.