3.2 The Divider-Chooser Method

- **divider-chooser method:** two players; one cuts the assets into two shares, and the other one chooses one of the shares, **72**

3.3 The Lone-Divider Method

- **lone-divider method:** N players $(N \geq 2)$; the lone divider cuts the assets into N shares; the others (choosers) declare which shares they consider to be fair, **74**

3.4 The Lone-Chooser Method

- **lone-chooser method:** N players $(N \geq 2)$; all but one player (the dividers) divide the assets fairly among themselves, and each then divides his or her share into N sub-shares; the remaining player (chooser) picks one sub-share from each divider, **78**

3.5 The Method of Sealed Bids

- **method of sealed bids:** each player bids a dollar value for each item with the item going to the highest bidder; the other players get cash from the winning bid for their equity on the item, **81**
- **reverse auction:** an auction in which the item being auctioned is a job or a chore; the lowest bidder gets the amount of his or her bid as payment for doing that job or chore, **83**

3.6 The Method of Markers

- **method of markers:** each player bids on how to split an array of the items into N sections; the player with the "smallest" bid for the first section gets it; among the remaining players, the player with the "smallest" bid for the second section gets it, and so on until every player gets one of his or her sections, **85**

EXERCISES

WALKING

3.1 Fair-Division Games

1. Henry, Tom, and Fred are dividing among themselves a set of common assets equally owned by the three of them. The assets are divided into three shares (s_1, s_2, and s_3). Table 3-12 shows the values of the shares to each player expressed as a percent of the total value of the assets.

   (a) Which of the shares are fair shares to Henry?

   (b) Which of the shares are fair shares to Tom?

   (c) Which of the shares are fair shares to Fred?

   (d) Find all possible fair divisions of the assets using s_1, s_2, and s_3 as shares.

   (e) Of the fair divisions found in (d), which one is the best?

	s_1	s_2	s_3
Henry	25%	40%	35%
Tom	28%	35%	37%
Fred	$33\frac{1}{3}$%	$33\frac{1}{3}$%	$33\frac{1}{3}$%

■ TABLE 3-12

2. Alice, Bob, and Carlos are dividing among themselves a set of common assets equally owned by the three of them. The assets are divided into three shares (s_1, s_2, and s_3). Table 3-13

shows the values of the shares to each player expressed as a percent of the total value of the assets.

(a) Which of the shares are fair shares to Alice?

(b) Which of the shares are fair shares to Bob?

(c) Which of the shares are fair shares to Carlos?

(d) Find all possible fair divisions of the assets using s_1, s_2, and s_3 as shares.

(e) Of the fair divisions found in (d), which one is the best?

	s_1	s_2	s_3
Alice	38%	28%	34%
Bob	$33\frac{1}{3}$%	$33\frac{1}{3}$%	$33\frac{1}{3}$%
Carlos	34%	40%	26%

■ TABLE 3-13

3. Angie, Bev, Ceci, and Dina are dividing among themselves a set of common assets equally owned by the four of them. The assets are divided into four shares (s_1, s_2, s_3, and s_4). Table 3-14 shows the values of the shares to each player expressed as a percent of the total value of the assets.

(a) Which of the shares are fair shares to Angie?

(b) Which of the shares are fair shares to Bev?

(c) Which of the shares are fair shares to Ceci?

(d) Which of the shares are fair shares to Dina?

(e) Find all possible fair divisions of the assets using s_1, s_2, s_3, and s_4 as shares.

	s_1	s_2	s_3	s_4
Angie	22%	26%	28%	24%
Bev	25%	26%	22%	27%
Ceci	20%	30%	27%	23%
Dina	25%	25%	25%	25%

■ TABLE 3-14

4. Mark, Tim, Maia, and Kelly are dividing among themselves a set of common assets equally owned by the four of them. The assets are divided into four shares (s_1, s_2, s_3, and s_4). Table 3-15 shows the values of the shares to each player expressed as a percent of the total value of the assets.

(a) Which of the shares are fair shares to Mark?

(b) Which of the shares are fair shares to Tim?

(c) Which of the shares are fair shares to Maia?

(d) Which of the shares are fair shares to Kelly?

(e) Find all possible fair divisions of the assets using s_1, s_2, s_3, and s_4 as shares.

	s_1	s_2	s_3	s_4
Mark	20%	32%	28%	20%
Tim	25%	25%	25%	25%
Maia	15%	15%	30%	40%
Kelly	24%	26%	24%	26%

■ TABLE 3-15

5. Allen, Brady, Cody, and Diane are sharing a cake. The cake had previously been divided into four slices (s_1, s_2, s_3, and s_4). Table 3-16 shows the values of the slices in the eyes of each player.

(a) Which of the slices are fair shares to Allen?

(b) Which of the slices are fair shares to Brady?

(c) Which of the slices are fair shares to Cody?

(d) Which of the slices are fair shares to Diane?

(e) Find all possible fair divisions of the cake using s_1, s_2, s_3, and s_4 as shares.

	s_1	s_2	s_3	s_4
Allen	$4.00	$5.00	$6.00	$5.00
Brady	$3.00	$3.50	$4.00	$5.50
Cody	$6.00	$4.50	$3.50	$4.00
Diane	$7.00	$4.00	$4.00	$5.00

■ TABLE 3-16

6. Carlos, Sonya, Tanner, and Wen are sharing a cake. The cake had previously been divided into four slices (s_1, s_2, s_3, and s_4). Table 3-17 shows the values of the slices in the eyes of each player.

(a) Which of the slices are fair shares to Carlos?

(b) Which of the slices are fair shares to Sonya?

(c) Which of the slices are fair shares to Tanner?

(d) Which of the slices are fair shares to Wen?

(e) Find all possible fair divisions of the cake using s_1, s_2, s_3, and s_4 as shares.

	s_1	s_2	s_3	s_4
Carlos	$3.00	$5.00	$5.00	$3.00
Sonya	$4.50	$3.50	$4.50	$5.50
Tanner	$4.25	$4.50	$3.50	$3.75
Wen	$5.50	$4.00	$4.50	$6.00

■ TABLE 3-17

7. Alex, Betty, and Cindy are sharing a cake. The cake had previously been divided into three slices (s_1, s_2, and s_3). Table 3-18 shows the values of s_1 and s_2 to each player expressed as a percent of the total value of the cake.

   (a) Which of the slices are fair shares to Alex?

   (b) Which of the slices are fair shares to Betty?

   (c) Which of the slices are fair shares to Cindy?

   (d) Find a fair division of the cake using s_1, s_2, and s_3 as shares. If no such fair division is possible, explain why.

	s_1	s_2
Alex	30%	40%
Betty	31%	35%
Cindy	30%	35%

■ TABLE 3-18

8. Alex, Betty, and Cindy are sharing a cake. The cake had previously been divided into three slices (s_1, s_2, and s_3). Table 3-19 shows the values of s_1 and s_2 to each player expressed as a percent of the total value of the cake.

   (a) Which of the slices are fair shares to Alex?

   (b) Which of the slices are fair shares to Betty?

   (c) Which of the slices are fair shares to Cindy?

   (d) Find a fair division of the cake using s_1, s_2, and s_3 as shares. If no such fair division is possible, explain why.

	s_1	s_2
Alex	30%	34%
Betty	28%	36%
Cindy	30%	$33\frac{1}{3}$%

■ TABLE 3-19

9. Four partners (Adams, Benson, Cagle, and Duncan) jointly own a piece of land with a market value of $400,000. Suppose that the land is subdivided into four parcels s_1, s_2, s_3, and s_4. The partners are planning to split up, with each partner getting one of the four parcels.

   (a) To Adams, s_1 is worth $40,000 more than s_2, s_2 and s_3 are equal in value, and s_4 is worth $20,000 more than s_1. Determine which of the four parcels are fair shares to Adams.

   (b) To Benson, s_1 is worth $40,000 more than s_2, s_4 is $8,000 more than s_3, and together s_4 and s_3 have a combined value equal to 40% of the value of the land. Determine which of the four parcels are fair shares to Benson.

   (c) To Cagle, s_1 is worth $40,000 more than s_2 and $20,000 more than s_4, and s_3 is worth twice as much as s_4. Determine which of the four parcels are fair shares to Cagle.

   (d) To Duncan, s_1 is worth $4,000 more than s_2; s_2 and s_3 have equal value; and s_1, s_2, and s_3 have a combined value equal to 70% of the value of the land. Determine which of the four parcels are fair shares to Duncan.

   (e) Find a fair division of the land using the parcels s_1, s_2, s_3, and s_4 as fair shares.

10. Four players (Abe, Betty, Cory, and Dana) are sharing a cake. Suppose that the cake is divided into four slices s_1, s_2, s_3, and s_4.

   (a) To Abe, s_1 is worth $3.60, s_4 is worth $3.50, s_2 and s_3 have equal value, and the entire cake is worth $15.00. Determine which of the four slices are fair shares to Abe.

   (b) To Betty, s_2 is worth twice as much as s_1, s_3 is worth three times as much as s_1, and s_4 is worth four times as much as s_1. Determine which of the four slices are fair shares to Betty.

   (c) To Cory, s_1, s_2, and s_4 have equal value, and s_3 is worth as much as s_1, s_2, and s_4 combined. Determine which of the four slices are fair shares to Cory.

   (d) To Dana, s_1 is worth $1.00 more than s_2, s_3 is worth $1.00 more than s_1, s_4 is worth $3.00, and the entire cake is worth $18.00. Determine which of the four slices are fair shares to Dana.

   (e) Find a fair division of the cake using s_1, s_2, s_3, and s_4 as fair shares.

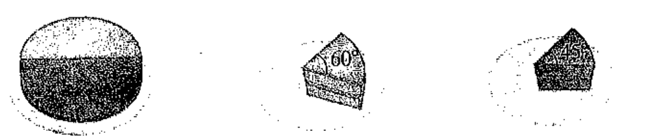
11. Angelina and Brad jointly buy the chocolate-strawberry mousse cake shown in Fig. 3-18(a) for $36. Suppose that Angelina values chocolate cake *twice* as much as she values strawberry cake. Find the dollar value to Angelina of each of the following pieces of cake:

   (a) the strawberry half of the cake

   (b) the chocolate half of the cake

   (c) the slice of strawberry cake shown in Fig. 3-18(b)

   (d) the slice of chocolate cake shown in Fig. 3-18(c)



(a) (b) (c)

FIGURE 3-18

12. Brad and Angelina jointly buy the chocolate-strawberry mousse cake shown in Fig. 3-18(a) for $36. Suppose that Brad values strawberry cake *three* times as much as he values chocolate cake. Find the dollar value to Brad of each of the following pieces of cake:

   (a) the strawberry half of the cake

   (b) the chocolate half of the cake

   (c) the slice of strawberry cake shown in Fig. 3-18(b)

   (d) the slice of chocolate cake shown in Fig. 3-18(c)

13. Karla and five other friends jointly buy the chocolate-strawberry-vanilla cake shown in Fig. 3-19(a) for \$30 and plan to divide the cake fairly among themselves. After much discussion, the cake is divided into the six equal-sized slices $s_1, s_2, \ldots, s_6$ shown in Fig. 3-19(b). Suppose that Karla values strawberry cake *twice* as much as vanilla cake and chocolate cake *three* times as much as vanilla cake.

(a) Find the dollar value to Karla of each of the slices s_1 through s_6.

(b) Which of the slices s_1 through s_6 are fair shares to Karla?

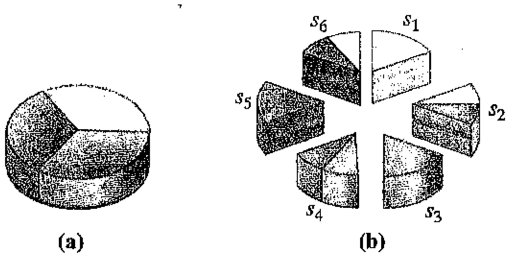


FIGURE 3-19

14. Marla and five other friends jointly buy the chocolate-strawberry-vanilla cake shown in Fig. 3-19(a) for \$30 and plan to divide the cake fairly among themselves. After much discussion, the cake is divided into the six equal-sized slices $s_1, s_2, \ldots, s_6$ shown in Fig. 3-19(b). Suppose that Marla values vanilla cake *twice* as much as chocolate cake and chocolate cake *three* times as much as strawberry cake.

(a) Find the dollar value to Marla of each of the slices s_1 through s_6.

(b) Which of the slices s_1 through s_6 are fair shares to Marla?

3.2 The Divider-Chooser Method

Exercises 15 and 16 refer to the following situation: Jared and Karla jointly bought the half meatball–half vegetarian foot-long sub shown in Fig. 3-20 for \$8.00. They plan to divide the sandwich fairly using the divider-chooser method. Jared likes meatball subs three times as much as vegetarian subs; Karla is a strict vegetarian and does not eat meat at all. Assume that Jared just met Karla and has no idea that she is a vegetarian. Assume also that when the sandwich is cut, the cut is made perpendicular to the length of the sandwich. (You can describe different shares of the sandwich using the ruler and interval notation. For example, [0, 6] *describes the vegetarian half,* [6, 8] *describes one-third of the meatball half, etc.).*

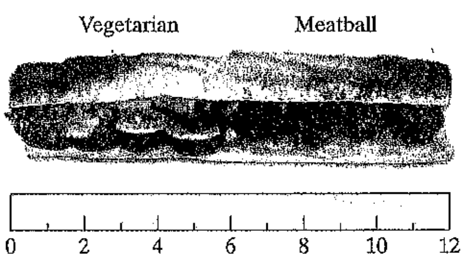


FIGURE 3-20

15. Suppose that they flip a coin and Jared ends up being the divider.

(a) Describe how Jared should cut the sandwich into two shares s_1 and s_2.

(b) After Jared cuts, Karla gets to choose. Specify which of the two shares Karla should choose and give the value of the share to Karla.

16. Suppose they flip a coin and Karla ends up being the divider.

(a) Describe how Karla should cut the sandwich into two shares s_1 and s_2.

(b) After Karla cuts, Jared gets to choose. Specify which of the two shares Jared should choose and give the value of the share to Jared.

Exercises 17 and 18 refer to the following situation: Martha and Nick jointly bought the giant 28-in. sub sandwich shown in Fig. 3-21 for \$9. They plan to divide the sandwich fairly using the divider-chooser method. Martha likes ham subs twice as much as she likes turkey subs, and she likes turkey and roast beef subs the same. Nick likes roast beef subs twice as much as he likes ham subs, and he likes ham and turkey subs the same. Assume that Nick and Martha just met and know nothing of each other's likes and dislikes. Assume also that when the sandwich is cut, the cut is made perpendicular to the length of the sandwich. (You can describe different shares of the sandwich using the ruler and interval notation. For example, [0, 8] *describes the ham part,* [8, 12] *describes one-third of the turkey part, etc.).*

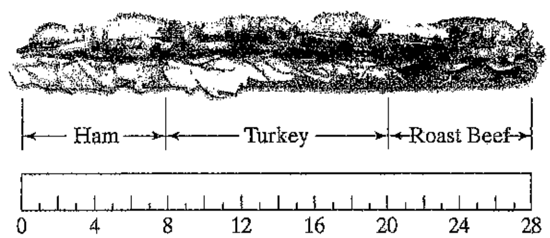


FIGURE 3-21

17. Suppose that they flip a coin and Martha ends up being the divider.

(a) Describe how Martha would cut the sandwich into two shares s_1 and s_2.

(b) After Martha cuts, Nick gets to choose. Specify which of the two shares Nick should choose, and give the value of the share to Nick.

18. Suppose that they flip a coin and Nick ends up being the divider.

(a) Describe how Nick would cut the sandwich into two shares s_1 and s_2.

(b) After Nick cuts, Martha gets to choose. Specify which of the two shares Martha should choose and give the value of the share to Martha.

Exercises 19 and 20 refer to the following situation: David and Paula are planning to divide the pizza shown in Fig. 3-22(a) using the divider-chooser method. David likes pepperoni and sausage pizza equally well, and he likes sausage pizza twice as much as mushroom pizza. Paula likes sausage and mushroom pizza equally well, but she hates pepperoni pizza.

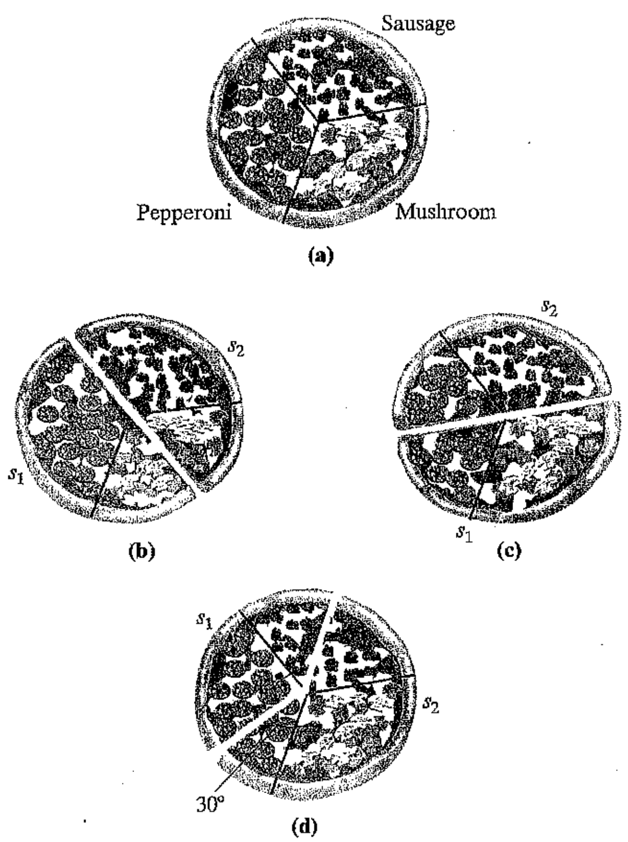


FIGURE 3-22

19. Suppose that they flip a coin and David ends up being the divider. (Assume that David is playing the game by the rules and knows nothing about Paula's likes and dislikes.)

(a) Is the cut shown in Fig. 3-22(b) a possible 50-50 cut that David might have made as the divider? If so, describe the share Paula should choose and give the value (as a percent) of that share to Paula. If the cut is not a 50-50 cut, give the values of the two shares to David.

(b) Is the cut shown in Fig. 3-22(c) a possible 50-50 cut that David might have made as the divider? If so, describe the share Paula should choose and give the value (as a percent) of that share to Paula. If the cut is not a 50-50 cut, give the values of the two shares to David.

(c) Is the cut shown in Fig. 3-22(d) a possible 50-50 cut that David might have made as the divider? If so, describe the share Paula should choose and give the value (as a percent) of that share to Paula. If the cut is not a 50-50 cut, give the values of the two shares to David.

20. Suppose that they flip a coin and Paula ends up being the divider. (Assume that Paula is playing the game by the rules and knows nothing about David's likes and dislikes.)

(a) Is the cut shown in Fig. 3-22(b) a possible 50-50 cut that Paula might have made as the divider? If so, describe the share David should choose and give the value (as a percent) of that share to David. If the cut is not a 50-50 cut, give the values of the two shares to Paula.

(b) Is the cut shown in Fig. 3-22(c) a possible 50-50 cut that Paula might have made as the divider? If so, describe the share David should choose and give the value (as a percent) of that share to David. If the cut is not a 50-50 cut, give the values of the two shares to Paula.

(c) Is the cut shown in Fig. 3-22(d) a possible 50-50 cut that Paula might have made as the divider? If so, describe the share David should choose and give the value (as a percent) of that share to David. If the cut is not a 50-50 cut, give the values of the two shares to Paula.

3.3 The Lone-Divider Method

21. Three partners are dividing a plot of land among themselves using the lone-divider method. After the divider D divides the land into three shares s_1, s_2, and s_3, the choosers C_1 and C_2 submit their bids for these shares.

(a) Suppose that the choosers' bids are C_1: $\{s_2, s_3\}$; C_2: $\{s_2, s_3\}$. Describe *two* different fair divisions of the land.

(b) Suppose that the choosers' bids are C_1: $\{s_2, s_3\}$; C_2: $\{s_1, s_3\}$. Describe *three* different fair divisions of the land.

22. Three partners are dividing a plot of land among themselves using the lone-divider method. After the divider D divides the land into three shares s_1, s_2, and s_3, the choosers C_1 and C_2 submit their bids for these shares.

(a) Suppose that the choosers' bids are C_1: $\{s_2\}$; C_2: $\{s_1, s_3\}$. Describe *two* different fair divisions of the land.

(b) Suppose that the choosers' bids are C_1: $\{s_1, s_2\}$; C_2: $\{s_2, s_3\}$. Describe *three* different fair divisions of the land.

23. Four partners are dividing a plot of land among themselves using the lone-divider method. After the divider D divides the land into four shares s_1, s_2, s_3, and s_4, the choosers C_1, C_2, and C_3 submit their bids for these shares.

(a) Suppose that the choosers' bids are C_1: $\{s_2\}$; C_2: $\{s_1, s_3\}$; C_3: $\{s_2, s_3\}$. Find a fair division of the land. Explain why this is the only possible fair division.

(b) Suppose that the choosers' bids are C_1: $\{s_2, s_3\}$; C_2: $\{s_1, s_3\}$; C_3: $\{s_1, s_2\}$. Describe *two* different fair divisions of the land.

(c) Suppose that the choosers' bids are C_1: $\{s_2\}$; C_2: $\{s_1, s_3\}$; C_3: $\{s_1, s_4\}$. Describe *three* different fair divisions of the land.

24. Four partners are dividing a plot of land among themselves using the lone-divider method. After the divider D divides the land into four shares s_1, s_2, s_3, and s_4, the choosers C_1, C_2, and C_3 submit their bids for these shares.

(a) Suppose that the choosers' bids are C_1: $\{s_2\}$; C_2: $\{s_1, s_3\}$; C_3: $\{s_2, s_3\}$. Find a fair division of the land. Explain why this is the only possible fair division.

(b) Suppose that the choosers' bids are C_1: $\{s_2\}$; C_2: $\{s_1, s_3\}$; C_3: $\{s_1, s_4\}$. Describe *three* different fair divisions of the land.

(c) Suppose that the choosers' bids are C_1: $\{s_2\}$; C_2: $\{s_1, s_2, s_3\}$; C_3: $\{s_2, s_3, s_4\}$. Describe *three* different fair divisions of the land.

25. Mark, Tim, Maia, and Kelly are dividing a cake among themselves using the lone-divider method. The divider divides the cake into four slices (s_1, s_2, s_3, and s_4). Table 3-20 shows the values of the slices to each player expressed as a percent of the total value of the cake.

(a) Who was the divider?

(b) Find a fair division of the cake.

	s_1	s_2	s_3	s_4
Mark	20%	32%	28%	20%
Tim	25%	25%	25%	25%
Maia	15%	15%	30%	40%
Kelly	24%	24%	24%	28%

TABLE 3-20

26. Allen, Brady, Cody, and Diane are sharing a cake valued at \$20 using the lone-divider method. The divider divides the cake into four slices (s_1, s_2, s_3, and s_4). Table 3-21 shows the values of the slices in the eyes of each player.

(a) Who was the divider?

(b) Find a fair division of the cake.

	s_1	s_2	s_3	s_4
Allen	\$4.00	\$5.00	\$4.00	\$7.00
Brady	\$6.00	\$6.50	\$4.00	\$3.50
Cody	\$5.00	\$5.00	\$5.00	\$5.00
Diane	\$7.00	\$4.50	\$4.00	\$4.50

TABLE 3-21

27. Four partners are dividing a plot of land among themselves using the lone-divider method. After the divider D divides the land into four shares s_1, s_2, s_3, and s_4, the choosers C_1, C_2, and C_3 submit the following bids: C_1: $\{s_2\}$; C_2: $\{s_1, s_2\}$; C_3: $\{s_1, s_2\}$. For each of the following possible divisions, determine if it is a fair division or not. If not, explain why not.

(a) D gets s_3; s_1, s_2, and s_4 are recombined into a single piece that is then divided fairly among C_1, C_2, and C_3 using the lone-divider method for three players.

(b) D gets s_1; s_2, s_3, and s_4 are recombined into a single piece that is then divided fairly among C_1, C_2, and C_3 using the lone-divider method for three players.

(c) D gets s_4; s_1, s_2, and s_3 are recombined into a single piece that is then divided fairly among C_1, C_2, and C_3 using the lone-divider method for three players.

(d) D gets s_3; C_1 gets s_2; and s_1, s_4 are recombined into a single piece that is then divided fairly between C_2 and C_3 using the divider-chooser method.

28. Four partners are dividing a plot of land among themselves using the lone-divider method. After the divider D divides the land into four shares s_1, s_2, s_3, and s_4, the choosers C_1, C_2, and C_3 submit the following bids: C_1: $\{s_3, s_4\}$; C_2: $\{s_4\}$; C_3: $\{s_3\}$. For each of the following possible divisions, determine if it is a fair division or not. If not, explain why not.

(a) D gets s_1; s_2, s_3, and s_4 are recombined into a single piece that is then divided fairly among C_1, C_2, and C_3 using the lone-divider method for three players.

(b) D gets s_3; s_1, s_2, and s_4 are recombined into a single piece that is then divided fairly among C_1, C_2, and C_3 using the lone-divider method for three players.

(c) D gets s_2; s_1, s_3, and s_4 are recombined into a single piece that is then divided fairly among C_1, C_2, and C_3 using the lone-divider method for three players.

(d) C_2 gets s_4; C_3 gets s_3; s_1, s_2 are recombined into a single piece that is then divided fairly between C_1 and D using the divider-chooser method.

29. Five players are dividing a cake among themselves using the lone-divider method. After the divider D cuts the cake into five slices (s_1, s_2, s_3, s_4, s_5), the choosers C_1, C_2, C_3, and C_4 submit their bids for these shares.

(a) Suppose that the choosers' bids are C_1: $\{s_2, s_4\}$; C_2: $\{s_2, s_4\}$; C_3: $\{s_2, s_3, s_5\}$; C_4: $\{s_2, s_3, s_4\}$. Describe *two* different fair divisions of the cake. Explain why that's it—why there are no others.

(b) Suppose that the choosers' bids are C_1: $\{s_2\}$; C_2: $\{s_2, s_4\}$; C_3: $\{s_2, s_3, s_5\}$; C_4: $\{s_2, s_3, s_4\}$. Find a fair division of the cake. Explain why that's it—there are no others.

30. Five players are dividing a cake among themselves using the lone-divider method. After the divider D cuts the cake into five slices (s_1, s_2, s_3, s_4, s_5), the choosers C_1, C_2, C_3, and C_4 submit their bids for these shares.

(a) Suppose that the choosers' bids are C_1: $\{s_2, s_3\}$; C_2: $\{s_2, s_4\}$; C_3: $\{s_1, s_2\}$; C_4: $\{s_1, s_3, s_4\}$. Describe *three* different fair divisions of the land. Explain why that's it—why there are no others.

(b) Suppose that the choosers' bids are C_1: $\{s_1, s_4\}$; C_2: $\{s_2, s_4\}$; C_3: $\{s_2, s_4, s_5\}$; C_4: $\{s_2\}$. Find a fair division of the land. Explain why that's it—why there are no others.

31. Four partners (Egan, Fine, Gong, and Hart) jointly own a piece of land with a market value of $480,000. The partnership is breaking up, and the partners decide to divide the land among themselves using the lone-divider method. Using a map, the divider divides the property into four parcels s_1, s_2, s_3, and s_4. Table 3-22 shows the value of some of the parcels in the eyes of each partner.

	s_1	s_2	s_3	s_4
Egan	$80,000	$85,000		$195,000
Fine		$100,000	$135,000	$120,000
Gong			$120,000	
Hart	$95,000	$100,000		$110,000

TABLE 3-22

(a) Who was the divider? Explain.

(b) Determine each chooser's bid.

(c) Find a fair division of the property.

32. Four players (Abe, Betty, Cory, and Dana) are dividing a pizza worth $18.00 among themselves using the lone-divider method. The divider divides the pizza into four shares s_1, s_2, s_3, and s_4. Table 3-23 shows the value of some of the slices in the eyes of each player.

	s_1	s_2	s_3	s_4
Abe	$5.00	$5.00	$3.50	
Betty		$4.50		
Cory	$4.80	$4.20	$4.00	
Dana	$4.00	$3.75	$4.25	

TABLE 3-23

(a) Who was the divider? Explain.

(b) Determine each chooser's bid.

(c) Find a fair division of the pizza.

Exercises 33 and 34 refer to the following situation: Jared, Karla, and Lori are planning to divide the half vegetarian–half meatball foot-long sub sandwich shown in Fig. 3-23 among themselves using the lone-divider method. Jared likes the meatball and vegetarian parts equally well; Karla is a strict vegetarian and does not eat meat at all; Lori likes the meatball part twice as much as the vegetarian part. (Assume that when the sandwich is cut, the cuts are always made perpendicular to the length of the sandwich. You can describe different shares of the sandwich using the ruler and interval notation—for example, [0, 6] *describes the vegetarian half,* [6, 8] *describes one-third of the meatball half, etc.)*

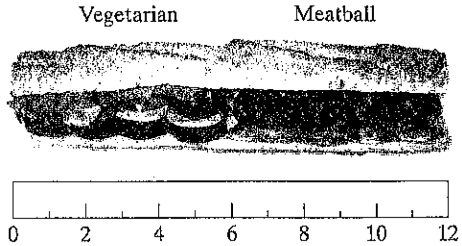


FIGURE 3-23

33. Suppose that Jared is the divider.

(a) Describe how Jared should cut the sandwich into three shares. Label the three shares s_1 for the leftmost piece, s_2 for the middle piece, and s_3 for the rightmost piece. Use the ruler and interval notation to describe the three shares. (Assume that Jared knows nothing about Karla and Lori's likes and dislikes.)

(b) Which of the three shares are fair shares to Karla?

(c) Which of the three shares are fair shares to Lori?

(d) Find three different fair divisions of the sandwich.

34. Suppose that Lori ends up being the divider.

(a) Describe how Lori should cut the sandwich into three shares. Label the three shares s_1 for the leftmost piece, s_2 for the middle piece, and s_3 for the rightmost piece. Use the ruler and interval notation to describe the three shares. (Assume that Lori knows nothing about Karla and Jared's likes and dislikes.)

(b) Which of the three shares are fair shares to Jared?

(c) Which of the three shares are fair shares to Karla?

(d) Suppose that Lori gets s_3. Describe how to proceed to find a fair division of the sandwich.

3.4 The Lone-Chooser Method

Exercises 35 through 38 refer to the following situation: Angela, Boris, and Carlos are dividing the vanilla-strawberry cake shown in Fig. 3-24(a) using the lone-chooser method. Figure 3-24(b) shows how each player values each half of the cake. In your answers assume that all cuts are normal "cake cuts" from the center to the edge of the cake. You can describe each piece of cake by giving the angles of the vanilla and strawberry parts, as in "15° strawberry–40° vanilla" or "60° vanilla only."

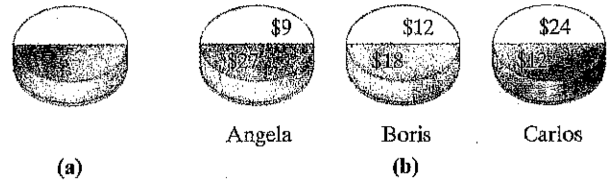


FIGURE 3-24

35. Suppose that Angela and Boris are the dividers and Carlos is the chooser. In the first division, Boris cuts the cake vertically through the center as shown in Fig. 3-25, with Angela choosing s_1 (the left half) and Boris s_2 (the right half). In the second division, Angela subdivides s_1 into three pieces and Boris subdivides s_2 into three pieces.

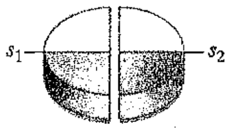


FIGURE 3-25

(a) Describe how Angela would subdivide s_1 into three pieces.

(b) Describe how Boris would subdivide s_2 into three pieces.

(c) Based on the subdivisions in (a) and (b), describe a possible final fair division of the cake.

(d) For the final fair division you described in (c), find the value (in dollars and cents) of each share in the eyes of the player receiving it.

36. Suppose that Carlos and Angela are the dividers and Boris is the chooser. In the first division, Carlos cuts the cake vertically through the center as shown in Fig. 3-25, with Angela choosing s_1 (the left half) and Carlos s_2 (the right half). In the second division, Angela subdivides s_1 into three pieces and Carlos subdivides s_2 into three pieces.

(a) Describe how Carlos would subdivide s_2 into three pieces.

(b) Describe how Angela would subdivide s_1 into three pieces.

(c) Based on the subdivisions in (a) and (b), describe a possible final fair division of the cake.

(d) For the final fair division you described in (c), find the value (in dollars and cents) of each share in the eyes of the player receiving it.

37. Suppose that Angela and Boris are the dividers and Carlos is the chooser. In the first division, Angela cuts the cake into two shares: s_1 (a 120° strawberry-only piece) and s_2 (a 60° strawberry–180° vanilla piece) as shown in Fig. 3-26. Boris picks the share he likes best, and Angela gets the other share. In the second division, Angela subdivides her share of the cake into three pieces and Boris subdivides his share of the cake into three pieces.

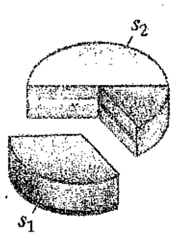


FIGURE 3-26

(a) Describe which share (s_1 or s_2) Boris picks and how he might subdivide it.

(b) Describe how Angela would subdivide her share of the cake.

(c) Based on the subdivisions in (a) and (b), describe a possible final fair division of the cake.

(d) For the final fair division you described in (c), find the value (in dollars and cents) of each share in the eyes of the player receiving it.

38. Suppose that Carlos and Angela are the dividers and Boris is the chooser. In the first division, Carlos cuts the cake into two shares: s_1 (a 135° vanilla-only piece) and s_2 (a 45° vanilla–180° strawberry piece) as shown in Fig. 3-27. Angela picks the share she likes better and Carlos gets the other share. In the second division, Angela subdivides her share of the cake into three pieces and Carlos subdivides his share of the cake into three pieces.

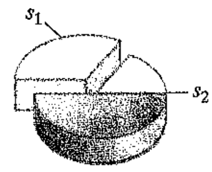


FIGURE 3-27

(a) Describe which share (s_1 or s_2) Angela picks and how she might subdivide it.

(b) Describe how Carlos might subdivide his share of the cake.

(c) Based on the subdivisions in (a) and (b), describe a possible final fair division of the cake.

(d) For the final fair division you described in (c), find the value (in dollars and cents) of each share in the eyes of the player receiving it.

Exercises 39 and 40 refer to the following: Arthur, Brian, and Carl are dividing the cake shown in Fig. 3-28 using the lone-chooser method. Arthur loves chocolate cake and orange cake equally but hates strawberry cake and vanilla cake. Brian loves chocolate cake and strawberry cake equally but hates orange cake and vanilla cake. Carl loves chocolate cake and vanilla cake equally but hates orange cake and strawberry cake. In your answers, assume all cuts are normal "cake cuts" from the center to the edge of the cake. You can describe each piece of cake by giving the angles of its parts, as in "15° strawberry–40° chocolate" or "60° orange only."

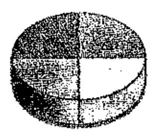


FIGURE 3-28

39. Suppose that Arthur and Brian are the dividers and Carl is the chooser. In the first division, Arthur cuts the cake vertically through the center as shown in Fig. 3-29 and Brian picks the share he likes better. In the second division, Brian subdivides the share he chose into three pieces and Arthur subdivides the other share into three pieces.

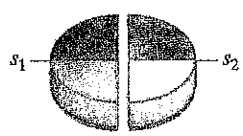


FIGURE 3-29

(a) Describe which share (s_1 or s_2) Brian picks and how he might subdivide it.

(b) Describe how Arthur might subdivide the other share.

(c) Based on the subdivisions in (a) and (b), describe a possible final fair division of the cake.

(d) For the final fair division you described in (c), find the value of each share (as a percentage of the total value of the cake) in the eyes of the player receiving it.

40. Suppose that Carl and Arthur are the dividers and Brian is the chooser. In the first division, Carl makes the cut shown in Fig. 3-30 and Arthur picks the share he likes better. In the second division, Arthur subdivides the share he chose into three pieces and Carl subdivides the other share into three pieces.

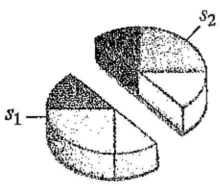


FIGURE 3-30

(a) Describe which share (s_1 or s_2) Arthur picks and how he might subdivide it.

(b) Describe how Carl might subdivide the other share.

(c) Based on the subdivisions in (a) and (b), describe a possible final fair division of the cake.

(d) For the final fair division you described in (c), find the value of each share (as a percentage of the total value of the cake) in the eyes of the player receiving it.

41. Jared, Karla, and Lori are dividing the foot-long half meatball–half vegetarian sub shown in Fig. 3-31 using the lone-chooser method. Jared likes the vegetarian and meatball parts equally well, Karla is a strict vegetarian and does not eat meat at all, and Lori likes the meatball part twice as much as she likes the vegetarian part. Suppose that Karla and Jared are the dividers and Lori is the chooser. In the first division, Karla divides the sub into two shares (a left share s_1 and a right share s_2) and Jared picks the share he likes better. In the second division, Jared subdivides the share he picks into three pieces (a "left" piece J_1, a "middle" piece J_2, and a "right" piece J_3) and Karla subdivides the other share into three pieces (a "left" piece K_1, a "middle" piece K_2, and a "right" piece K_3). Assume that all cuts are perpendicular to the length of the sub. (You can describe the pieces of sub using the ruler and interval notation, as in [3, 7] for the piece that starts at inch 3 and ends at inch 7.)

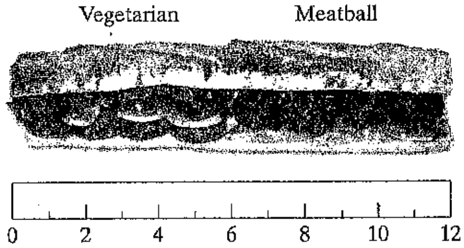


FIGURE 3-31

(a) Describe Karla's first division into s_1 and s_2.

(b) Describe which share (s_1 or s_2) Jared picks and how he would then subdivide it into the three pieces J_1, J_2, and J_3.

(c) Describe how Karla would subdivide her share into three pieces K_1, K_2, and K_3.

(d) Based on the subdivisions in (a), (b), and (c), describe the final fair division of the sub and give the value of each player's share (as a percentage of the total value of the sub) in the eyes of the player receiving it.

42. Jared, Karla, and Lori are dividing the foot-long half meatball–half vegetarian sub shown in Fig. 3-31 using the lone-chooser method. Jared likes the vegetarian and meatball parts equally well, Karla is a strict vegetarian and does not eat meat at all, and Lori likes the meatball part twice as much as she likes the vegetarian part. Suppose that Karla and Lori are the dividers and Jared is the chooser. In the first division, Lori divides the sub into two shares (a left share s_1 and a right share s_2) and Karla picks the share she likes better. In the second division, Karla subdivides the share she picks into three pieces (a "left" piece K_1, a "middle" piece K_2, and a "right" piece K_3) and Lori subdivides the other share into three pieces (a "left" piece L_1, a "middle" piece L_2, and a "right" piece L_3). Assume that all cuts are perpendicular to the length of the sub. (You can describe the pieces of sub using the ruler and interval notation, as in [3, 7] for the piece that starts at inch 3 and ends at inch 7.)

(a) Describe Lori's first division into s_1 and s_2.

(b) Describe which share (s_1 or s_2) Karla picks and how she would then subdivide it into the three pieces K_1, K_2, and K_3.

(c) Describe how Lori would subdivide her share into three pieces L_1, L_2, and L_3.

(d) Based on the subdivisions in (a), (b), and (c), describe the final fair division of the sub and give the value of each player's share (as a percentage of the total value of the sub) in the eyes of the player receiving it.

3.5 The Method of Sealed Bids

43. Ana, Belle, and Chloe are dividing four pieces of furniture using the method of sealed bids. Table 3-24 shows the players' bids on each of the items.

	Ana	Belle	Chloe
Dresser	$150	$300	$275
Desk	$180	$150	$165
Vanity	$170	$200	$260
Tapestry	$400	$250	$500

TABLE 3-24

(a) Find the value of each player's fair share.

(b) Describe the first settlement (who gets which item and how much do they pay or get in cash).

(c) Find the surplus after the first settlement is over.

(d) Describe the final settlement (who gets which item and how much do they pay or get in cash).

44. Andre, Bea, and Chad are dividing an estate consisting of a house, a small farm, and a painting using the method of sealed bids. Table 3-25 shows the players' bids on each of the items.

	Andre	Bea	Chad
House	\$150,000	\$146,000	\$175,000
Farm	\$430,000	\$425,000	\$428,000
Painting	\$50,000	\$59,000	\$57,000

■ TABLE 3-25

(a) Describe the first settlement of this fair division and compute the surplus.

(b) Describe the final settlement of this fair-division problem.

45. Five heirs (A, B, C, D, and E) are dividing an estate consisting of six items using the method of sealed bids. The heirs' bids on each of the items are given in Table 3-26.

	A	B	C	D	E
Item 1	\$352	\$295	\$395	\$368	\$324
Item 2	\$98	\$102	\$98	\$95	\$105
Item 3	\$460	\$449	\$510	\$501	\$476
Item 4	\$852	\$825	\$832	\$817	\$843
Item 5	\$513	\$501	\$505	\$505	\$491
Item 6	\$725	\$738	\$750	\$744	\$761

■ TABLE 3-26

(a) Find the value of each player's fair share.

(b) Describe the first settlement (who gets which item and how much do they pay or get in cash).

(c) Find the surplus after the first settlement is over.

(d) Describe the final settlement (who gets which item and how much do they pay or get in cash).

46. Oscar, Bert, and Ernie are using the method of sealed bids to divide among themselves four items they commonly own. Table 3-27 shows the bids that each player makes for each item.

(a) Find the value of each player's fair share.

(b) Describe the first settlement (who gets which item and how much do they pay or get in cash).

(c) Find the surplus after the first settlement is over.

(d) Describe the final settlement (who gets which item and how much do they pay or get in cash).

	Oscar	Bert	Ernie
Item 1	\$8600	\$5500	\$3700
Item 2	\$3500	\$4200	\$5000
Item 3	\$2300	\$4400	\$3400
Item 4	\$4800	\$2700	\$2300

■ TABLE 3-27

47. Anne, Bette, and Chia jointly own a flower shop. They can't get along anymore and decide to break up the partnership using the method of sealed bids, with the understanding that one of them will get the flower shop and the other two will get cash. Anne bids \$210,000, Bette bids \$240,000, and Chia bids \$225,000. How much money do Anne and Chia each get from Bette for their third share of the flower shop?

48. Al, Ben, and Cal jointly own a fruit stand. They can't get along anymore and decide to break up the partnership using the method of sealed bids, with the understanding that one of them will get the fruit stand and the other two will get cash. Al bids \$156,000, Ben bids \$150,000, and Cal bids \$171,000. How much money do Al and Ben each get from Cal for their one-third share of the fruit stand?

49. Ali, Briana, and Caren are roommates planning to move out of their apartment. They identify four major chores that need to be done before moving out and decide to use the method of sealed bids to reverse auction the chores. Table 3-28 shows the bids that each roommate made for each chore. Describe the final outcome of the division (which chores are done by each roommate and how much each roommate pays or gets paid.)

	Ali	Briana	Caren
Chore 1	\$65	\$70	\$55
Chore 2	\$100	\$85	\$95
Chore 3	\$60	\$50	\$45
Chore 4	\$75	\$80	\$90

■ TABLE 3-28

50. Anne, Bess, and Cindy are roommates planning to move out of their apartment. They identify five major chores that need to be done before moving out and decide to use the method of sealed bids to reverse auction the chores. Table 3-29 shows the bids that each roommate made for each chore. Describe the final outcome of the division (which chores are done by each roommate and how much each roommate pays or gets paid.)

	Anne	Bess	Cindy
Chore 1	\$20	\$30	\$40
Chore 2	\$50	\$10	\$22
Chore 3	\$30	\$20	\$15
Chore 4	\$30	\$20	\$10
Chore 5	\$20	\$40	\$15

■ TABLE 3-29

3.6 The Method of Markers

51. Three players (A, B, and C) are dividing the array of 13 items shown in Fig. 3-32 using the method of markers. The players' bids are as indicated in the figure.

(a) Which items go to A?

(b) Which items go to B?

(c) Which items go to C?

(d) Which items are left over?

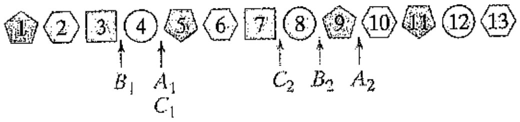


FIGURE 3-32

52. Three players (A, B, and C) are dividing the array of 13 items shown in Fig. 3-33 using the method of markers. The players' bids are as indicated in the figure.

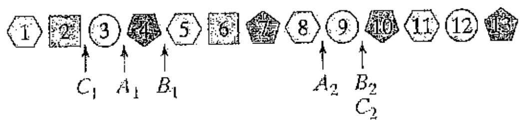


FIGURE 3-33

(a) Which items go to A?

(b) Which items go to B?

(c) Which items go to C?

(d) Which items are left over?

53. Three players (A, B, and C) are dividing the array of 12 items shown in Fig. 3-34 using the method of markers. The players' bids are as indicated in the figure.

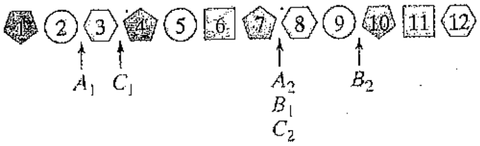


FIGURE 3-34

(a) Which items go to A?

(b) Which items go to B?

(c) Which items go to C?

(d) Which items are left over?

54. Three players (A, B, and C) are dividing the array of 12 items shown in Fig. 3-35 using the method of markers. The players' bids are indicated in the figure.

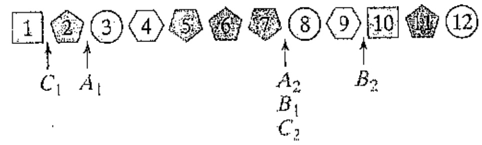


FIGURE 3-35

(a) Which items go to A?

(b) Which items go to B?

(c) Which items go to C?

(d) Which items are left over?

55. Five players (A, B, C, D, and E) are dividing the array of 20 items shown in Fig. 3-36 using the method of markers. The players' bids are as indicated in the figure.

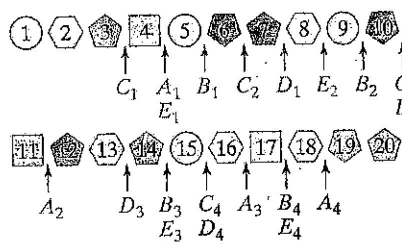


FIGURE 3-36

(a) Describe the allocation of items to each player.

(b) Which items are left over?

56. Four players (A, B, C, and D) are dividing the array of 15 items shown in Fig. 3-37 using the method of markers. The players' bids are as indicated in the figure.

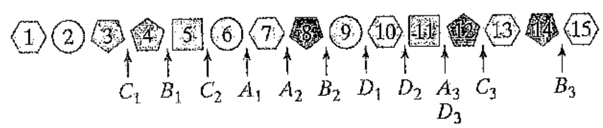


FIGURE 3-37

(a) Describe the allocation of items to each player.

(b) Which items are left over?

57. Quintin, Ramon, Stephone, and Tim are dividing a collection of 18 classic superhero comic books using the method of markers. The comic books are randomly lined up in the array shown below. (The W's are Wonder Woman comic books, the S's are Spider-Man comic books, the G's are Green Lantern comic books, and the B's are Batman comic books.)

W S S G S W W B G G G S G S G S B B

The value of the comic books in the eyes of each player is shown in Table 3-30.

	Quintin	Ramon	Stephone	Tim
Each W is worth	$12	$9	$8	$5
Each S is worth	$7	$5	$7	$4
Each G is worth	$4	$5	$6	$4
Each B is worth	$6	$11	$14	$7

■ **TABLE 3-30**

(a) Describe the placement of each player's markers. (Use Q_1, Q_2, and Q_3 for Quintin's markers, R_1, R_2, and R_3 for Ramon's markers, and so on.)

(b) Describe the allocation of comic books to each player and describe what comic books are left over.

58. Queenie, Roxy, and Sophie are dividing a set of 15 CDs—3 Beach Boys CDs, 6 Grateful Dead CDs, and 6 opera CDs using the method of markers. Queenie loves the Beach Boys but hates the Grateful Dead and opera. Roxy loves the Grateful Dead and the Beach Boys equally well but hates opera. Sophie loves the Grateful Dead and opera equally well but hates the Beach Boys. The CDs are lined up in an array as follows:

O O O GD GD GD BB BB BB GD GD GD O O O

(O represents the opera CDs, GD the Grateful Dead CDs, and BB the Beach Boys CDs.)

(a) Describe the placement of each player's markers. (Use Q_1 and Q_2 for Queenie's markers, R_1 and R_2 for Roxy's markers, etc.)

(b) Describe the allocation of CDs to each player and describe what CDs are left over.

(c) Suppose that the players agree that each one gets to pick an extra CD from the leftover CDs. Suppose that Queenie picks first, Sophie picks second, and Roxy picks third. Describe which leftover CDs each one would pick.

59. Ana, Belle, and Chloe are dividing 3 Snickers bars, 3 Milky Way bars, and 3 candy necklaces among themselves using the method of markers. The players' value systems are as follows: (1) Ana likes all candy bars the same; (2) Belle loves Milky Way bars but hates Snickers bars and candy necklaces; (3) Chloe likes candy necklaces twice as much as she likes Snickers or Milky Way bars. Suppose that the candy is lined up exactly as shown in Fig. 3-38.

FIGURE 3-38

(a) Describe the placement of each player's markers. (Use A_1 and A_2 for Ana's markers, B_1 and B_2 for Belle's markers, and C_1 and C_2 for Chloe's markers.) (*Hint*: For each player, compute the value of each piece as a fraction of the value of the booty first. This will help you figure out where the players would place their markers.)

(b) Describe the allocation of candy to each player and which pieces of candy are left over.

(c) Suppose that the players decide to divide the leftover pieces by a random lottery in which each player gets to choose one piece. Suppose that Belle gets to choose first, Chloe second, and Ana last. Describe the division of the leftover pieces.

60. Arne, Bruno, Chloe, and Daphne are dividing 3 Snickers bars, 3 Milky Way bars, 3 candy necklaces, and 6 3Musketeers among themselves using the method of markers. Arne hates Snickers Bars but likes Milky Way bars, candy necklaces, and 3Musketeers equally well. Bruno hates Milky Way bars but likes Snickers bars, candy necklaces, and 3Musketeers equally well. Chloe hates candy necklaces and 3Musketeers and likes Snickers three times as much as Milky Way bars. Daphne hates Snickers and Milky Way bars and values a candy necklaces as equal to two-thirds the value of a 3Musketeers (i.e., 2 3Musketeers bars equal 3 candy necklaces). Suppose the candy is lined up exactly as shown in Fig. 3-39.

FIGURE 3-39

(a) Describe the placement of each player's markers. (*Hint*: For each player, compute the value of each piece as a fraction of the value of the booty first. This will help you figure out where the players would place their markers.)

(b) Describe the allocation of candy to each player and which pieces of candy are left over.

(c) After the allocation, each player is allowed to pick one piece of candy. Will there be any arguments?

JOGGING

61. Consider the following method for dividing a continuous asset S among three players (two dividers and one chooser):

Step 1. Divider 1 (D_1) cuts S into two pieces s_1 and s_2 that he considers to be worth, $\frac{1}{3}$ and $\frac{2}{3}$ of the value of S, respectively.

Step 2. Divider 2 (D_2) cuts the second piece s_2 into two halves s_{21} and s_{22} that she considers to be of equal value.

Step 3. The chooser C chooses one of the three pieces (s_1, s_{21}, or s_{22}), D_1 chooses next, and D_2 gets the last piece.

(a) Explain why under this method C is guaranteed a fair share.

(b) Explain why under this method D_1 is guaranteed a fair share.

(c) Illustrate with an example why under this method D_2 is not guaranteed a fair share.

62. Consider the following method for dividing a continuous asset S among three players:

Step 1. Divider 1 (D_1) cuts S into two pieces s_1 and s_2 that he considers to be worth, $\frac{1}{3}$ and $\frac{2}{3}$ of the value of S, respectively.

Step 2. Divider 2 (D_2) gets a shot at s_1. If he thinks that s_1 is worth $\frac{1}{3}$ of S or less, he can pass (case 1); if he thinks that s_1 is worth more than $\frac{1}{3}$, he can trim the piece to a smaller piece s_{11} that he considers to be worth exactly $\frac{1}{3}$ of S (case 2).

Step 3. The chooser C gets a shot at either s_1 (in case 1) or at s_{11} (in case 2). If she thinks the piece is a fair share, she gets to keep it (case 3). Otherwise, the piece goes to the divider that considers it to be worth $\frac{1}{3}$ (D_1 in case 1, D_2 in case 2).

Step 4. The two remaining players (D_2 and C in case 1, D_1 and C in case 2, D_1 and D_2 in case 3) get to divide the "remainder" (whatever is left of S) between themselves using the divider-chooser method.

Explain why the above is a fair-division method that guarantees that if played properly, each player will get a fair share.

63. Two partners (A and B) jointly own a business but wish to dissolve the partnership using the method of sealed bids. One of the partners will keep the business; the other will get cash for his half of the business. Suppose that A bids \$$x$ and B bids \$$y$. Assume that B is the high bidder. Describe the final settlement of this fair division in terms of x and y.

64. Three partners (A, B, and C) jointly own a business but wish to dissolve the partnership using the method of sealed bids. One of the partners will keep the business; the other two will each get cash for their one-third share of the business. Suppose that A bids \$$x$, B bids \$$y$, and C bids \$$z$. Assume that C is the high bidder. Describe the final settlement of this fair division in terms of x, y, and z. (*Hint:* Try Exercises 47 and 48 first.)

65. Three players (A, B, and C) are sharing the chocolate-strawberry-vanilla cake shown in Fig. 3-40(a). Figure 3-40(b) shows the relative value that each player gives to each of the three parts of the cake. There is a way to divide this cake into three pieces (using just three cuts) so that each player ends up with a piece that he or she will value at exactly 50% of the value of the cake. Find such a fair division.

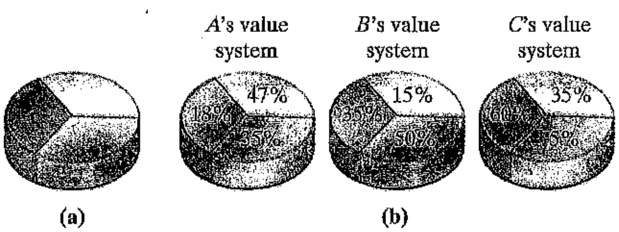


FIGURE 3-40

66. Angelina and Brad are planning to divide the chocolate-strawberry cake shown in Fig. 3-41(a) using the divider-chooser method, with Angelina being the divider. Suppose that Angelina values chocolate cake *three* times as much as she values strawberry cake. Figure 3-41(b) shows a generic cut made by Angelina dividing the cake into two shares s_1 and s_2 of equal value to her. Think of s_1 as an "$x°$ chocolate–$y°$ strawberry" share. For each given measure of the angle x, find the corresponding measure of the angle y.

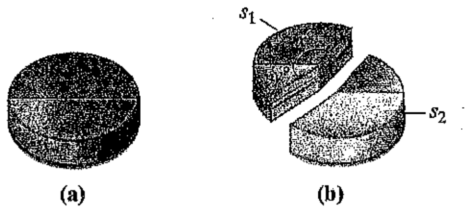


FIGURE 3-41

(a) $x = 60°$.

(b) $x = 72°$.

(c) $x = 108°$.

(d) $x = 120°$.

67. Standoffs in the lone-divider method. In the lone-divider method, a *standoff* occurs when a set of k choosers are bidding for less than k items. The types of *standoffs* possible depend on the number of players. With $N = 3$ players, there is only *one* type of standoff—the two choosers are bidding for the same item. With $N = 4$ players, there are *three* possible types of standoffs: two choosers are bidding on the same item, or three choosers are bidding on the same item, or three choosers are bidding on just two items. With $N = 5$ players, there are *six* possible types of standoffs, and the number of possible types of standoffs increases rapidly as the number of players increases.

(a) List the six possible types of *standoffs* with $N = 5$ players.

(b) What is the number of possible types of *standoffs* with $N = 6$ players?

(c) What is the number of possible types of *standoffs* with N players? Give your answer in terms of N. (*Hint:* You will need to use the formula given in Chapter 1 for the number of pairwise comparisons among a set of objects.)

68. Efficient and envy-free fair divisions. A fair division is called **efficient** if there is no other fair division that gives *every* player a share that is as good or better (i.e., any other fair division that gives some players a better share must give some other players a worse share). A fair division is called **envy-free** if every player ends up with a share that he or she feels is *as good as or better than* that of any other player.

Suppose that three partners (A, B, and C) jointly own a piece of land that has been subdivided into six parcels ($s_1, s_2, \ldots, s_6$). The partnership is splitting up, and the partners are going to divide fairly the six parcels among themselves. Table 3-31 shows the value of each parcel in the eyes of each partner.

	A	B	C
s_1	\$20,000	\$16,000	\$19,000
s_2	\$19,000	\$18,000	\$18,000
s_3	\$18,000	\$19,000	\$15,000
s_4	\$16,000	\$20,000	\$12,000
s_5	\$15,000	\$15,000	\$20,000
s_6	\$12,000	\$12,000	\$16,000

■ TABLE 3-31

(a) Find a fair division of the six parcels among the three partners that is *efficient.*

(b) Find a fair division of the six parcels among the three partners that is *envy-free.*

(c) Find a fair division of the six parcels among the three partners that is *efficient but not envy-free.*

(d) Find a fair division of the six parcels among the three partners that is *envy-free but not efficient.*

RUNNING

69. Suppose that N players bid on M items using the method of sealed bids. Let T denote the table with M rows (one for each item) and N columns (one for each player) containing all the players' bids (i.e., the entry in column j, row k represents player j's bid for item k). Let $c_1, c_2, \ldots, c_N$ denote, respectively, the sum of the entries in column 1, column 2, ..., column N of T, and let $r_1, r_2, \ldots, r_M$ denote, respectively, the sum of the entries in row 1, row 2, ..., row M of T. Let $w_1, w_2, \ldots, w_M$ denote the winning bids for items 1, 2, ..., M, respectively (i.e., w_1 is the largest entry in row 1 of T, w_2 is the largest entry in row 2, etc.). Let S denote the surplus money left after the first settlement.

(a) Show that

$$S = (w_1 + w_2 + \cdots + w_M) - \frac{(c_1 + c_2 + \cdots + c_N)}{N}.$$

(b) Using (a), show that

$$S = \left(w_1 - \frac{r_1}{N}\right) + \left(w_2 - \frac{r_2}{N}\right) + \cdots + \left(w_M - \frac{r_M}{N}\right).$$

(c) Using (b), show that $S \geq 0$.

(d) Describe the conditions under which $S = 0$.

70. Asymmetric method of sealed bids. Suppose that an estate consisting of M indivisible items is to be divided among N heirs $(P_1, P_2, \ldots, P_N)$ using the method of sealed bids. Suppose that Grandma's will stipulates that P_1 is entitled to $x_1\%$ of the estate, P_2 is entitled to $x_2\%$ of the estate, ..., P_N is entitled to $x_N\%$ of the estate. The percentages add up to 100%, but they are not all equal (Grandma loved some grandchildren more than others). Describe a variation of the method of sealed bids that ensures that each player receives a "fair share" (i.e., P_1 receives a share that she considers to be worth at least $x_1\%$ of the estate, P_2 receives a share that he considers to be worth at least $x_2\%$ of the estate, etc.).

71. Lone-chooser is a fair-division method. Suppose that N players divide a cake using the *lone-chooser method.* The chooser is C and the dividers are $D_1, D_2, \ldots, D_{N-1}$. Explain why, when properly played, the method guarantees to each player a fair share. (You will need one argument for the dividers and a different argument for the chooser.)

PROJECTS AND PAPERS

1 Envy-Free Fair Division

An *envy-free fair division* is a fair division in which each player ends up with a share that he or she feels is as good as or better than that of any other player. Thus, in an envy-free fair division a player would never envy or covet another player's share. Over the last 20 years several envy-free fair-division methods have been developed.

Write a paper discussing the topic of envy-free fair division.

Notes: Some ideas of topics for your paper: (1) Discuss how envy-free fair division differs from the (proportional) type of fair division discussed in this chapter. (2) Describe a continuous envy-free fair-division method for $N = 3$ players. (3) Give an outline of the Brams-Taylor method for continuous envy-free fair division for any number of players.

2 Fair Divisions with Unequal Shares

All the fair-division problems we discussed in this chapter were based on the assumption of *symmetry* (i.e., all players have equal rights in the division). Sometimes, players are not all equal and are entitled to larger or smaller shares than other players. This type of fair-division problem is called an *asymmetric* fair division (*asymmetric* means that the players are not all equal in their rights). For example, Grandma's will may stipulate that her estate is to be divided as follows: Art is entitled to 25%, Betty is entitled to 35%, Carla is entitled to 30%, and Dave is entitled to 10%. (After all, it is her will, and if she wants to be difficult, she can!)

Write a paper discussing how some of the fair-division methods discussed in this chapter can be adapted for the case of asymmetric fair division. Discuss at least one discrete and one continuous asymmetric fair-division method.