EXERCISES

WALKING

2.1 Weighted Voting

1. **Nassau County Board of Supervisors (1990).** Table 2-13 shows the six districts in Nassau County and their votes in the County Board of Supervisors in 1990. Suppose the quota was set at 60% or more of the votes. Describe this weighted voting system using the standard notation $[q: w_1, w_2, \ldots, w_N]$.

District	Weight
Hempstead #1	30
Hempstead #2	28
Oyster Bay	22
North Hempstead	21
Long Beach	2
Glen Cove	2

■ TABLE 2-13

2. **The European Union Council (2010).** Table 2-14 shows the weights of the member nations in the European Union Council of Ministers in 2010. Suppose the quota is set at 74% or more of the votes.

   (a) Find the value of N (the number of players in this weighted voting system).

   (b) Find the value of V (the total number of votes in the system).

   (c) Describe this weighted voting system using the standard notation $[q: w_1, w_2, \ldots, w_N]$.

Member nation	Weight
France, Germany, Italy, United Kingdom	29
Spain, Poland	27
Romania	14
Netherlands	13
Belgium, Greece, Czech Republic, Hungary, Portugal	12
Austria, Bulgaria, Sweden	10
Denmark, Ireland, Lithuania, Slovakia, Finland	7
Cyprus, Estonia, Latvia, Luxembourg, Slovenia	4
Malta	3

■ TABLE 2-14

3. Consider the weighted voting system $[q: 6, 4, 3, 3, 2, 2]$.

   (a) What is the smallest value that the quota q can take?

   (b) What is the largest value that the quota q can take?

   (c) What is the value of the quota if *at least* three-fourths of the votes are required to pass a motion?

   (d) What is the value of the quota if *more* than three-fourths of the votes are required to pass a motion?

4. Consider the weighted voting system $[q: 10, 6, 5, 4, 2]$.

   (a) What is the smallest value that the quota q can take?

   (b) What is the largest value that the quota q can take?

   (c) What is the value of the quota if *at least* two-thirds of the votes are required to pass a motion?

   (d) What is the value of the quota if *more* than two-thirds of the votes are required to pass a motion?

5. In each of the following weighted voting systems, determine which players, if any, have veto power.

   (a) $[7: 4, 3, 3, 2]$

   (b) $[9: 4, 3, 3, 2]$

   (c) $[10: 4, 3, 3, 2]$

   (d) $[11: 4, 3, 3, 2]$

6. In each of the following weighted voting systems, determine which players, if any, have veto power.

   (a) $[9: 8, 4, 2, 1]$

   (b) $[12: 8, 4, 2, 1]$

   (c) $[14: 8, 4, 2, 1]$

   (d) $[15: 8, 4, 2, 1]$

7. Consider the weighted voting system $[q: 7, 5, 3]$. Find the *smallest* value of q for which

   (a) all three players have veto power.

   (b) P_2 has veto power but P_3 does not.

8. Consider the weighted voting system $[q: 10, 8, 6, 4, 2]$. Find the *smallest* value of q for which

   (a) all five players have veto power.

   (b) P_3 has veto power but P_4 does not.

9. A committee has four members (P_1, P_2, P_3, and P_4). In this committee P_1 has twice as many votes as P_2; P_2 has twice as many votes as P_3; P_3 and P_4 have the same number of votes. The quota is $q = 49$. For each of the given definitions of the *quota*, describe the committee using the notation $[q: w_1, w_2, w_3, w_4]$. (*Hint*: Write the weighted voting system as $[49: 4x, 2x, x, x]$, and then solve for x.)

(a) The quota is defined as a *simple majority* of the votes.

(b) The quota is defined as *more than two-thirds* of the votes.

(c) The quota is defined as *more than three-fourths* of the votes.

10. A committee has six members (P_1, P_2, P_3, P_4, P_5, and P_6). In this committee P_1 has twice as many votes as P_2; P_2 and P_3 each has twice as many votes as P_4; P_4 has twice as many votes as P_5; P_5 and P_6 have the same number of votes. The quota is $q = 121$. For each of the given definitions of the *quota*, describe the committee using the notation $[q: w_1, w_2, w_3, w_4, w_5, w_6]$. (*Hint*: Write the weighted voting system as $[121: 8x, 4x, 4x, 2x, x, x]$, and then solve for x.)

(a) The quota is defined as a *simple majority* of the votes.

(b) The quota is defined as *more than two-thirds* of the votes.

(c) The quota is defined as *more than three-fourths* of the votes.

2.2 Banzhaf Power

11. Consider the weighted voting system $[q: 7, 5, 3]$.

(a) What is the weight of the coalition formed by P_1 and P_3?

(b) For what values of the quota q is the coalition formed by P_1 and P_3 a winning coalition?

(c) For what values of the quota q is the coalition formed by P_1 and P_3 a losing coalition?

12. Consider the weighted voting system $[q: 10, 8, 6, 4, 2]$.

(a) What is the weight of the coalition formed by P_2, P_3, and P_4?

(b) For what values of the quota q is the coalition formed by P_2, P_3, and P_4 a winning coalition?

(c) For what values of the quota q is the coalition formed by P_2, P_3, and P_4 a losing coalition?

13. A weighted voting system with four players has the following winning coalitions (with critical players underlined):

$\{\underline{P_1}, \underline{P_2}, \underline{P_3}\}, \{\underline{P_1}, \underline{P_2}, \underline{P_4}\}, \{\underline{P_1}, \underline{P_2}, P_3, P_4\}$.

Find the Banzhaf power distribution of this weighted voting system.

14. A weighted voting system with five players has the following winning coalitions (with critical players underlined):

$\{\underline{P_1}, \underline{P_2}, \underline{P_3}\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_4}\}$, $\{\underline{P_1}, \underline{P_2}, P_3, P_4\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_3}, P_5\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_4}, P_5\}$, $\{\underline{P_1}, \underline{P_3}, \underline{P_4}, \underline{P_5}\}$,

$\{\underline{P_1}, P_2, P_3, P_4, P_5\}$.

Find the Banzhaf power distribution of this weighted voting system.

15. Consider the weighted voting system $[10: 6, 5, 4, 2]$.

(a) Which players are critical in the winning coalition $\{P_1, P_2, P_4\}$?

(b) Write down all winning coalitions.

(c) Find the Banzhaf power distribution of this weighted voting system.

16. Consider the weighted voting system $[5: 3, 2, 1, 1]$.

(a) Which players are critical in the winning coalition $\{P_1, P_3, P_4\}$?

(b) Write down all winning coalitions.

(c) Find the Banzhaf power distribution of this weighted voting system.

17. (a) Find the Banzhaf power distribution of the weighted voting system $[6: 5, 2, 1]$.

(b) Find the Banzhaf power distribution of the weighted voting system $[3: 2, 1, 1]$. Compare your answers in (a) and (b).

18. (a) Find the Banzhaf power distribution of the weighted voting system $[7: 5, 2, 1]$.

(b) Find the Banzhaf power distribution of the weighted voting system $[5: 3, 2, 1]$. Compare your answers in (a) and (b).

19. Consider the weighted voting system $[q: 5, 4, 3, 2, 1]$. Find the Banzhaf power distribution of this weighted voting system when

(a) $q = 10$

(b) $q = 11$

(c) $q = 12$

(d) $q = 15$

20. Consider the weighted voting system $[q: 8, 4, 2, 1]$. Find the Banzhaf power distribution of this weighted voting system when

(a) $q = 8$

(b) $q = 9$

(c) $q = 10$

(d) $q = 14$

21. In a weighted voting system with three players the winning coalitions are: $\{P_1, P_2\}$, $\{P_1, P_3\}$, and $\{P_1, P_2, P_3\}$.

(a) Find the critical players in each winning coalition.

(b) Find the Banzhaf power distribution of the weighted voting system.

22. In a weighted voting system with four players the winning coalitions are: $\{P_1, P_2\}$, $\{P_1, P_2, P_3\}$, $\{P_1, P_2, P_4\}$, and $\{P_1, P_2, P_3, P_4\}$.

(a) Find the critical players in each winning coalition.

(b) Find the Banzhaf power distribution of the weighted voting system.

23. **The Nassau County (N.Y.) Board of Supervisors (1960s version).** In the 1960s, the Nassau County Board of Supervisors operated as the weighted voting system $[58: 31, 31, 28, 21, 2, 2]$. Assume that the players are P_1 through P_6.

(a) List all the *two-* and *three-player* winning coalitions and find the critical players in each coalition.

(b) List all the winning coalitions that have P_4 as a member and find the critical players in each coalition.

(c) Use the results in (b) to find the Banzhaf power index of P_4.

(d) Use the results in (a) and (c) to find the Banzhaf power distribution of the weighted voting system.

24. **The Nassau County Board of Supervisors (1990s version).** In the 1990s, after a series of legal challenges, the Nassau County Board of Supervisors was redesigned to operate as the weighted voting system $[65: 30, 28, 22, 15, 7, 6]$.

(a) List all the *three-player* winning coalitions and find the critical players in each coalition.

(b) List all the *four-player* winning coalitions and find the critical players in each coalition. (*Hint*: There are 11 four-player winning coalitions.)

(c) List all the *five-player* winning coalitions and find the critical players in each coalition.

(d) Use the results in (a), (b), and (c) to find the Banzhaf power distribution of the weighted voting system.

25. A law firm is run by four partners $(A, B, C, \text{and } D)$. Each partner has one vote and decisions are made by majority rule, but in the case of a 2-2 tie, the coalition with D (the junior partner) loses. (For example, $\{A, B\}$ wins, but $\{A, D\}$ loses.)

(a) List all the winning coalitions in this voting system and find the critical players in each.

(b) Find the Banzhaf power distribution in this law firm.

26. A law firm is run by four partners $(A, B, C, \text{and } D)$. Each partner has one vote and decisions are made by majority rule, but in the case of a 2-2 tie, the coalition with A (the senior partner) wins.

(a) List all the winning coalitions in this voting system and the critical players in each.

(b) Find the Banzhaf power index of this law firm.

2.3 Shapley-Shubik Power

27. Table 2-15 shows the 24 sequential coalitions (with pivotal players underlined) in a weighted voting system with four players. Find the Shapley-Shubik power distribution of this weighted voting system.

$\langle P_1, P_2, \underline{P_3}, P_4 \rangle$	$\langle P_1, P_2, \underline{P_4}, P_3 \rangle$	$\langle P_1, P_3, \underline{P_2}, P_4 \rangle$	$\langle P_1, P_3, P_4, \underline{P_2} \rangle$
$\langle P_1, P_4, \underline{P_2}, P_3 \rangle$	$\langle P_1, P_4, P_3, \underline{P_2} \rangle$	$\langle P_2, P_1, \underline{P_3}, P_4 \rangle$	$\langle P_2, P_1, \underline{P_4}, P_3 \rangle$
$\langle P_2, P_3, \underline{P_1}, P_4 \rangle$	$\langle P_2, P_3, P_4, \underline{P_1} \rangle$	$\langle P_2, P_4, \underline{P_1}, P_3 \rangle$	$\langle P_2, P_4, P_3, \underline{P_1} \rangle$
$\langle P_3, P_1, \underline{P_2}, P_4 \rangle$	$\langle P_3, P_1, P_4, \underline{P_2} \rangle$	$\langle P_3, P_2, \underline{P_1}, P_4 \rangle$	$\langle P_3, P_2, P_4, \underline{P_1} \rangle$
$\langle P_3, P_4, P_1, \underline{P_2} \rangle$	$\langle P_3, P_4, P_2, \underline{P_1} \rangle$	$\langle P_4, P_1, \underline{P_2}, P_3 \rangle$	$\langle P_4, P_1, P_3, \underline{P_2} \rangle$
$\langle P_4, P_2, \underline{P_1}, P_3 \rangle$	$\langle P_4, P_2, P_3, \underline{P_1} \rangle$	$\langle P_4, P_3, P_1, \underline{P_2} \rangle$	$\langle P_4, P_3, P_2, \underline{P_1} \rangle$

TABLE 2-15

28. Table 2-16 shows the 24 sequential coalitions (with pivotal players underlined) in a weighted voting system with four players. Find the Shapley-Shubik power distribution of this weighted voting system.

$\langle P_1, \underline{P_2}, P_3, P_4 \rangle$	$\langle P_1, \underline{P_2}, P_4, P_3 \rangle$	$\langle P_1, \underline{P_3}, P_2, P_4 \rangle$	$\langle P_1, \underline{P_3}, P_4, P_2 \rangle$
$\langle P_1, P_4, \underline{P_2}, P_3 \rangle$	$\langle P_1, P_4, \underline{P_3}, P_2 \rangle$	$\langle P_2, \underline{P_1}, P_3, P_4 \rangle$	$\langle P_2, \underline{P_1}, P_4, P_3 \rangle$
$\langle P_2, P_3, \underline{P_1}, P_4 \rangle$	$\langle P_2, P_3, \underline{P_4}, P_1 \rangle$	$\langle P_2, P_4, \underline{P_1}, P_3 \rangle$	$\langle P_2, P_4, \underline{P_3}, P_1 \rangle$
$\langle P_3, \underline{P_1}, P_2, P_4 \rangle$	$\langle P_3, \underline{P_1}, P_4, P_2 \rangle$	$\langle P_3, P_2, \underline{P_1}, P_4 \rangle$	$\langle P_3, P_2, \underline{P_4}, P_1 \rangle$
$\langle P_3, P_4, \underline{P_1}, P_2 \rangle$	$\langle P_3, P_4, \underline{P_2}, P_1 \rangle$	$\langle P_4, P_1, \underline{P_2}, P_3 \rangle$	$\langle P_4, P_1, \underline{P_3}, P_2 \rangle$
$\langle P_4, P_2, \underline{P_1}, P_3 \rangle$	$\langle P_4, P_2, \underline{P_3}, P_1 \rangle$	$\langle P_4, P_3, \underline{P_1}, P_2 \rangle$	$\langle P_4, P_3, \underline{P_2}, P_1 \rangle$

TABLE 2-16

29. Consider the weighted voting system $[16: 9, 8, 7]$.

(a) Write down all the sequential coalitions, and in each sequential coalition identify the pivotal player.

(b) Find the Shapley-Shubik power distribution of this weighted voting system.

30. Consider the weighted voting system $[8: 7, 6, 2]$.

(a) Write down all the sequential coalitions, and in each sequential coalition identify the pivotal player.

(b) Find the Shapley-Shubik power distribution of this weighted voting system.

31. Find the Shapley-Shubik power distribution of each of the following weighted voting systems.

(a) $[15: 16, 8, 4, 1]$

(b) $[18: 16, 8, 4, 1]$

(c) $[24: 16, 8, 4, 1]$

(d) $[28: 16, 8, 4, 1]$

32. Find the Shapley-Shubik power distribution of each of the following weighted voting systems.

(a) $[8: 8, 4, 2, 1]$

(b) $[9: 8, 4, 2, 1]$

(c) $[12: 8, 4, 2, 1]$

(d) $[14: 8, 4, 2, 1]$

33. Find the Shapley-Shubik power distribution of each of the following weighted voting systems.

(a) $[51: 40, 30, 20, 10]$

(b) $[59: 40, 30, 20, 10]$ (*Hint*: Compare this situation with the one in (a).)

(c) $[60: 40, 30, 20, 10]$

34. Find the Shapley-Shubik power distribution of each of the following weighted voting systems.

(a) $[41: 40, 10, 10, 10]$

(b) $[49: 40, 10, 10, 10]$ (*Hint*: Compare this situation with the one in (a).)

(c) $[50: 40, 10, 10, 10]$

35. In a weighted voting system with three players the winning coalitions are: $\{P_1, P_2\}$, $\{P_1, P_3\}$, and $\{P_1, P_2, P_3\}$.

(a) List the sequential coalitions and identify the pivotal player in each one.

(b) Find the Shapley-Shubik power distribution of the weighted voting system.

36. In a weighted voting system with three players the winning coalitions are: $\{P_1, P_2\}$ and $\{P_1, P_2, P_3\}$.

(a) List the sequential coalitions and identify the pivotal player in each sequential coalition.

(b) Find the Shapley-Shubik power distribution of the weighted voting system.

37. Table 2-17 shows the 24 sequential coalitions in a weighted voting system with four players. In some cases the pivotal player is underlined, and in some cases it isn't. Find the Shapley-Shubik power distribution of this weighted voting system. (*Hint*: First find the pivotal player in the remaining sequential coalitions.)

$\langle P_1, \underline{P_2}, P_3, P_4\rangle$	$\langle P_2, P_1, P_3, P_4\rangle$	$\langle P_3, P_1, P_2, P_4\rangle$	$\langle P_4, P_1, P_2, P_3\rangle$
$\langle P_1, P_2, P_4, P_3\rangle$	$\langle P_2, P_1, P_4, P_3\rangle$	$\langle P_3, P_1, P_4, P_2\rangle$	$\langle P_4, P_1, P_3, P_2\rangle$
$\langle P_1, \underline{P_3}, P_2, P_4\rangle$	$\langle P_2, P_3, \underline{P_1}, P_4\rangle$	$\langle P_3, P_2, P_1, P_4\rangle$	$\langle P_4, P_2, P_1, P_3\rangle$
$\langle P_1, P_3, P_4, P_2\rangle$	$\langle P_2, P_3, \underline{P_4}, P_1\rangle$	$\langle P_3, P_2, P_4, P_1\rangle$	$\langle P_4, P_2, P_3, P_1\rangle$
$\langle P_1, P_4, \underline{P_2}, P_3\rangle$	$\langle P_2, P_4, \underline{P_1}, P_3\rangle$	$\langle P_3, P_4, \underline{P_1}, P_2\rangle$	$\langle P_4, P_3, P_1, P_2\rangle$
$\langle P_1, P_4, \underline{P_3}, P_2\rangle$	$\langle P_2, P_4, \underline{P_3}, P_1\rangle$	$\langle P_3, P_4, \underline{P_2}, P_1\rangle$	$\langle P_4, P_3, P_2, P_1\rangle$

TABLE 2-17

38. Table 2-18 shows the 24 sequential coalitions in a weighted voting system with four players. In some cases the pivotal player is underlined, and in some cases it isn't. Find the Shapley-Shubik power distribution of this weighted voting system. (*Hint*: First find the pivotal player in the remaining sequential coalitions.)

$\langle P_1, \underline{P_2}, P_3, P_4\rangle$	$\langle P_2, P_1, P_3, P_4\rangle$	$\langle P_3, P_1, P_2, P_4\rangle$	$\langle P_4, P_1, P_2, P_3\rangle$
$\langle P_1, P_2, P_4, P_3\rangle$	$\langle P_2, P_1, P_4, P_3\rangle$	$\langle P_3, P_1, P_4, P_2\rangle$	$\langle P_4, P_1, P_3, P_2\rangle$
$\langle P_1, P_3, \underline{P_2}, P_4\rangle$	$\langle P_2, P_3, \underline{P_1}, P_4\rangle$	$\langle P_3, P_2, P_1, P_4\rangle$	$\langle P_4, P_2, P_1, P_3\rangle$
$\langle P_1, P_3, \underline{P_4}, P_2\rangle$	$\langle P_2, P_3, P_4, \underline{P_1},\rangle$	$\langle P_3, P_2, P_4, P_1\rangle$	$\langle P_4, P_2, P_3, P_1\rangle$
$\langle P_1, P_4, \underline{P_2}, P_3\rangle$	$\langle P_2, P_4, \underline{P_1}, P_3\rangle$	$\langle P_3, P_4, \underline{P_1}, P_2\rangle$	$\langle P_4, P_3, P_1, P_2\rangle$
$\langle P_1, P_4, \underline{P_3}, P_2\rangle$	$\langle P_2, P_4, P_3, P_1\rangle$	$\langle P_3, P_4, P_2, P_1\rangle$	$\langle P_4, P_3, P_2, P_1\rangle$

TABLE 2-18

2.4 Subsets and Permutations

39. Let A be a set with 10 elements.

(a) Find the number of subsets of A.

(b) Find the number of subsets of A having one or more elements.

(c) Find the number of subsets of A having exactly one element.

(d) Find the number of subsets of A having two or more elements. [*Hint*: Use the answers to parts (b) and (c).]

40. Let A be a set with 12 elements.

(a) Find the number of subsets of A.

(b) Find the number of subsets of A having one or more elements.

(c) Find the number of subsets of A having exactly one element.

(d) Find the number of subsets of A having two or more elements. [*Hint*: Use the answers to parts (b) and (c).]

41. For a weighted voting system with 10 players,

(a) find the total number of coalitions.

(b) find the number of coalitions with two or more players.

42. Consider a weighted voting system with 12 players.

(a) Find the total number of coalitions in this weighted voting system.

(b) Find the number of coalitions with two or more players.

43. Consider a weighted voting system with six players (P_1 through P_6).

(a) Find the total number of coalitions in this weighted voting system.

(b) How many coalitions in this weighted voting system do not include P_1? (*Hint*: Think of all the possible coalitions of the remaining players.)

(c) How many coalitions in this weighted voting system do not include P_3? [*Hint*: Is this really different from (b)?]

(d) How many coalitions in this weighted voting system do not include both P_1 and P_3?

(e) How many coalitions in this weighted voting system include both P_1 and P_3? [*Hint*: Use your answers for (a) and (d).]

44. Consider a weighted voting system with five players (P_1 through P_5).

(a) Find the total number of coalitions in this weighted voting system.

(b) How many coalitions in this weighted voting system do not include P_1? (*Hint*: Think of all the possible coalitions of the remaining players.)

(c) How many coalitions in this weighted voting system do not include P_5? [*Hint*: Is this really different from (b)?]

(d) How many coalitions in this weighted voting system do not include P_1 or P_5?

(e) How many coalitions in this weighted voting system include both P_1 and P_5? [*Hint*: Use your answers for (a) and (d).]

For Exercises 45 through 48 you should use a calculator with a factorial key (typically, it's a key labeled either x! or n!). All scientific calculators and most business calculators have such a key.

45. Use a calculator to compute each of the following.

(a) 13!

(b) 18!

(c) 25!

(d) Suppose that you have a supercomputer that can list *one trillion* (10^{12}) sequential coalitions per second. Estimate (in years) how long it would take the computer to list all the sequential coalitions of 25 players.

46. Use a calculator to compute each of the following.

(a) 12!

(b) 15!

(c) 20!

(d) Suppose that you have a supercomputer that can list one billion (10^9) sequential coalitions per second. Estimate (in years) how long it would take the computer to list all the sequential coalitions of 20 players.

47. Use a calculator to compute each of the following.

(a) $\frac{13!}{3!}$ **(c)** $\frac{13!}{4!9!}$

(b) $\frac{13!}{3!10!}$ **(d)** $\frac{13!}{5!8!}$

48. Use a calculator to compute each of the following.

(a) $\frac{12!}{2!}$ **(c)** $\frac{12!}{3!9!}$

(b) $\frac{12!}{2!10!}$ **(d)** $\frac{12!}{4!8!}$

The purpose of Exercises 49 and 50 is for you to learn how to numerically manipulate factorials. If you use a calculator to answer these questions, you are defeating the purpose of the exercise. Please try Exercises 49 and 50 without using a calculator.

49. **(a)** Given that 10! = 3,628,800, find 9!

(b) Find $\frac{11!}{10!}$

(c) Find $\frac{11!}{9!}$

(d) Find $\frac{9!}{6!}$

(e) Find $\frac{101!}{99!}$

50. **(a)** Given that 20! = 2,432,902,008,176,640,000, find 19!

(b) Find $\frac{20!}{19!}$

(c) Find $\frac{201!}{199!}$

(d) Find $\frac{11!}{8!}$

51. Consider a weighted voting system with seven players (P_1 through P_7).

(a) Find the number of sequential coalitions in this weighted voting system.

(b) How many sequential coalitions in this weighted voting system have P_7 as the first player?

(c) How many sequential coalitions in this weighted voting system have P_7 as the last player?

(d) How many sequential coalitions in this weighted voting system do not have P_1 as the *first* player?

52. Consider a weighted voting system with six players (P_1 through P_6).

(a) Find the number of sequential coalitions in this weighted voting system.

(b) How many sequential coalitions in this weighted voting system have P_4 as the last player?

(c) How many sequential coalitions in this weighted voting system have P_4 as the *third* player?

(d) How many sequential coalitions in this weighted voting system do not have P_1 as the first player?

53. A law firm has seven partners: a senior partner (P_1) with 6 votes and six junior partners (P_2 through P_7) with 1 vote each. The quota is a *simple majority* of the votes. (This law firm operates as the weighted voting system $[7: 6, 1, 1, 1, 1, 1, 1]$.)

(a) In how many sequential coalitions is the senior partner P_1 the pivotal player? (*Hint*: First note that P_1 is the pivotal player in all sequential coalitions except those in which he is the first player.)

(b) Using your answer in (a), find the Shapley-Shubik power index of the senior partner P_1.

(c) Using your answer in (b), find the Shapley-Shubik power distribution in this law firm.

54. A law firm has six partners: a senior partner (P_1) with 5 votes and five junior partners (P_2 through P_6) with 1 vote each. The quota is a *simple majority* of the votes. (This law firm operates as the weighted voting system $[6: 5, 1, 1, 1, 1, 1]$.)

(a) In how many sequential coalitions is the senior partner P_1 the pivotal player? (*Hint*: First note that P_1 is the pivotal player in all sequential coalitions except those in which he is the first player.)

(b) Using your answer in (a), find the Shapley-Shubik power index of the senior partner P_1.

(c) Using your answer in (b), find the Shapley-Shubik power distribution in this law firm.

JOGGING

55. A partnership has four partners (P_1, P_2, P_3, and P_4). In this partnership P_1 has twice as many votes as P_2; P_2 has twice as many votes as P_3; P_3 has twice as many votes as P_4. The quota is a *simple majority* of the votes. Show that P_1 is always a *dictator*. (*Hint*: Write the weighted voting system in the form $[q: 8x, 4x, 2x, x]$, and express q in terms of x. Consider separately the case when x is even and the case when x is odd.)

56. In a weighted voting system with four players the winning coalitions are: $\{P_1, P_2, P_3\}$, $\{P_1, P_2, P_4\}$, $\{P_1, P_3, P_4\}$, and $\{P_1, P_2, P_3, P_4\}$.

(a) Find the Banzhaf power distribution of the weighted voting system.

(b) Find the Shapley-Shubik power distribution of the weighted voting system.

57. In a weighted voting system with three players, the six sequential coalitions (each with the pivotal player underlined) are: $\langle P_1, \underline{P_2}, P_3\rangle$, $\langle P_1, \underline{P_3}, P_2\rangle$, $\langle P_2, \underline{P_1}, P_3\rangle$, $\langle P_2, P_3, \underline{P_1}\rangle$, $\langle P_3, \underline{P_1}, P_2\rangle$, and $\langle P_3, P_2, \underline{P_1}\rangle$. Find the Banzhaf power distribution of the weighted voting system.

58. The Smith family has two parents (P_1 and P_2) and three children (c_1, c_2, and c_3). Family vacations are decided by a majority of the votes, but at least one parent must vote Yes (i.e., the three children don't have enough weight to carry the motion).

(a) If we use $[q: p, p, c, c, c]$ to describe this weighted voting system, find q, p, and c.

(b) Find the Banzhaf Power distribution of this weighted voting system.

59. A professional basketball team has four coaches, a head coach (H) and three assistant coaches (A_1, A_2, A_3). Player personnel decisions require at least three Yes votes, one of which must be H's.

(a) If we use $[q: h, a, a, a]$ to describe this weighted voting system, find q, h, and a.

(b) Find the Shapley-Shubik power distribution of the weighted voting system.

60. Veto power. A player P with weight w is said to have *veto power* if and only if $w < q$, and $V - w < q$ (where V denotes the total number of votes in the weighted voting system). Explain why each of the following is true:

(a) A player has veto power if and only if the player is a member of every winning coalition.

(b) A player has veto power if and only if the player is a critical player in the *grand* coalition.

61. Dummies. We defined a *dummy* as a player that is never critical. Explain why each of the following is true:

(a) If P is a dummy, then any winning coalition that contains P would also be a winning coalition without P.

(b) P is a dummy if and only if the Banzhaf power index of P is 0.

(c) P is a dummy if and only if the Shapley-Shubik power index of P is 0.

62. (a) Consider the weighted voting system $[22: 10, 10, 10, 10, 1]$. Are there any dummies? Explain your answer.

(b) Without doing any work [but using your answer for (a)], find the Banzhaf and Shapley-Shubik power distributions of this weighted voting system.

(c) Consider the weighted voting system $[q: 10, 10, 10, 10, 1]$. Find all the possible values of q for which P_5 is not a dummy.

(d) Consider the weighted voting system $[34: 10, 10, 10, 10, w]$. Find all positive integers w which make P_5 a dummy.

63. Consider the weighted voting system $[q: 8, 4, 1]$.

(a) What are the possible values of q?

(b) Which values of q result in a dictator? (Who? Why?)

(c) Which values of q result in exactly one player with veto power? (Who? Why?)

(d) Which values of q result in more than one player with veto power? (Who? Why?)

(e) Which values of q result in one or more dummies? (Who? Why?)

64. Consider the weighted voting system $[9: w, 5, 2, 1]$.

(a) What are the possible values of w?

(b) Which values of w result in a dictator? (Who? Why?)

(c) Which values of w result in a player with veto power? (Who? Why?)

(d) Which values of w result in one or more dummies? (Who? Why?)

65. (a) Verify that the weighted voting systems $[12: 7, 4, 3, 2]$ and $[24: 14, 8, 6, 4]$ result in exactly the same Banzhaf power distribution. (If you need to make calculations, do them for both systems side by side and look for patterns.)

(b) Based on your work in (a), explain why the two proportional weighted voting systems $[q: w_1, w_2, \ldots, w_N]$ and $[cq: cw_1, cw_2, \ldots, cw_N]$ always have the same Banzhaf power distribution.

66. **(a)** Verify that the weighted voting systems $[12: 7, 4, 3, 2]$ and $[24: 14, 8, 6, 4]$ result in exactly the same Shapley-Shubik power distribution. (If you need to make calculations, do them for both systems side by side and look for patterns.)

(b) Based on your work in (a), explain why the two proportional weighted voting systems $[q: w_1, w_2, \ldots, w_N]$ and $[cq: cw_1, cw_2, \ldots, cw_N]$ always have the same Shapley-Shubik power distribution.

67. A law firm has $N+1$ partners: the senior partner with N votes, and N junior partners with one vote each. The quota is a simple majority of the votes. Find the Shapley-Shubik power distribution in this weighted voting system. (*Hint*: Try Exercise 53 or 54 first.)

68. Consider the generic weighted voting system $[q: w_1, w_2, \ldots, w_N]$. (Assume $w_1 \geq w_2 \geq \cdots \geq w_N$.)

(a) Find all the possible values of q for which no player has veto power.

(b) Find all the possible values of q for which every player has veto power.

(c) Find all the possible values of q for which P_i has veto power but P_{i+1} does not. (*Hint*: See Exercise 60.)

69. The weighted voting system $[8: 6, 4, 2, 1]$ represents a partnership among four partners (P_1, P_2, P_3, and you!). You are the partner with just one vote, and in this situation you have no power (you dummy!). Not wanting to remain a dummy, you offer to buy one vote. Each of the other four partners is willing to sell you one of their votes, and they are all asking the same price. From which partner should you buy in order to get as much power for your buck as possible? Use the Banzhaf power index for your calculations. Explain your answer.

70. The weighted voting system $[27: 10, 8, 6, 4, 2]$ represents a partnership among five people (P_1, P_2, P_3, P_4, and P_5). You are P_5, the one with two votes. You want to increase your power in the partnership and are prepared to buy one share (one share equals one vote) from any of the other partners. Partners P_1, P_2, and P_3 are each willing to sell cheap ($1000 for one share), but P_4 is not being quite as cooperative—she wants $5000 for one of her shares. Given that you still want to buy one share, from whom should you buy it? Use the Banzhaf power index for your calculations. Explain your answer.

71. The weighted voting system $[18: 10, 8, 6, 4, 2]$ represents a partnership among five people (P_1, P_2, P_3, P_4, and P_5). You are P_5, the one with two votes. You want to increase your power in the partnership and are prepared to buy shares (one share equals one vote) from any of the other partners.

(a) Suppose that each partner is willing to sell one share and that they are all asking the same price. Assuming that you decide to buy only one share, from which partner should you buy? Use the Banzhaf power index for your calculations.

(b) Suppose that each partner is willing to sell two shares and that they are all asking the same price. Assuming that you decide to buy two shares from a single partner, from which partner should you buy? Use the Banzhaf power index for your calculations.

(c) If you have the money and the cost per share is fixed, should you buy one share or two shares (from a single person)? Explain.

72. **Mergers.** Sometimes in a weighted voting system two or more players decide to merge—that is to say, to combine their votes and always vote the same way. (Note that a merger is different from a coalition—coalitions are temporary, whereas mergers are permanent.) For example, if in the weighted voting system $[7: 5, 3, 1]$ P_2 and P_3 were to merge, the weighted voting system would then become $[7: 5, 4]$. In this exercise we explore the effects of mergers on a player's power.

(a) Consider the weighted voting system $[4: 3, 2, 1]$. In Example 2.9 we saw that P_2 and P_3 each have a Banzhaf power index of $1/5$. Suppose that P_2 and P_3 merge and become a single player P^*. What is the Banzhaf power index of P^*?

(b) Consider the weighted voting system $[5: 3, 2, 1]$. Find first the Banzhaf power indexes of players P_2 and P_3 and then the Banzhaf power index of P^* (the merger of P_2 and P_3). Compare.

(c) Rework the problem in (b) for the weighted voting system $[6: 3, 2, 1]$.

(d) What are your conclusions from (a), (b), and (c)?

73. **Decisive voting systems.** A weighted voting system is called *decisive* if for every losing coalition, the coalition consisting of the remaining players (called the *complement*) must be a winning coalition.

(a) Show that the weighted voting system $[5: 4, 3, 2]$ is decisive.

(b) Show that the weighted voting system $[3: 2, 1, 1, 1]$ is decisive.

(c) Explain why any weighted voting system with a dictator is decisive.

(d) Find the number of winning coalitions in a decisive voting system with N players.

74. **Equivalent voting systems.** Two weighted voting systems are *equivalent* if they have the same number of players and exactly the same winning coalitions.

(a) Show that the weighted voting systems $[8: 5, 3, 2]$ and $[2: 1, 1, 0]$ are equivalent.

(b) Show that the weighted voting systems $[7: 4, 3, 2, 1]$ and $[5: 3, 2, 1, 1]$ are equivalent.

(c) Show that the weighted voting system discussed in Example 2.12 is equivalent to $[3: 1, 1, 1, 1, 1]$.

(d) Explain why equivalent weighted voting systems must have the same Banzhaf power distribution.

(e) Explain why equivalent weighted voting systems must have the same Shapley-Shubik power distribution.

75. **Relative voting power.** The *relative voting weight* w_i of a player P_i is the fraction of votes controlled by that player.

A player's Banzhaf power index β_i can differ considerably from his relative voting weight w_i. One indicator of the relation between Banzhaf power and relative voting weight is the ratio between the two (called the *relative Banzhaf voting power*): $\pi_i = \frac{\beta_i}{w_i}$.

(a) Compute the relative Banzhaf voting power of California in the Electoral College (see Table 2-11).

(b) Compute the relative Banzhaf voting power of each player in Example 2.13.

RUNNING

76. The Cleansburg City Council. Find the Banzhaf power distribution in the Cleansburg City Council discussed in Example 2.17.

77. The Fresno City Council. In Fresno, California, the city council consists of seven members (the mayor and six other council members). A motion can be passed by the mayor and at least three other council members, or by at least five of the six ordinary council members.

(a) Describe the Fresno City Council as a weighted voting system.

(b) Find the Shapley-Shubik power distribution for the Fresno City Council. (*Hint*: See Example 2.17 for some useful ideas.)

78. Suppose that in a weighted voting system there is a player A who hates another player P so much that he will always vote the opposite way of P, regardless of the issue. We will call A the *antagonist* of P.

(a) Suppose that in the weighted voting system $[8: 5, 4, 3, 2]$, P is the player with two votes and his antagonist A is the player with five votes. The other two players we'll call P_2 and P_3. What are the possible coalitions under these circumstances? What is the Banzhaf power distribution under these circumstances?

(b) Suppose that in a generic weighted voting system with N players there is a player P who has an antagonist A. How many coalitions are there under these circumstances?

(c) Give examples of weighted voting systems where a player A can

(i) increase his Banzhaf power index by becoming an antagonist of another player.

(ii) decrease his Banzhaf power index by becoming an antagonist of another player.

(d) Suppose that the antagonist A has more votes than his enemy P. What is a strategy that P can use to gain power at the expense of A?

79. (a) Give an example of a weighted voting system with four players and such that the Shapley-Shubik power index of P_1 is $\frac{3}{4}$.

(b) Show that in any weighted voting system with four players a player cannot have a Shapley-Shubik power index of more than $\frac{3}{4}$ unless he or she is a dictator.

(c) Show that in any weighted voting system with N players a player cannot have a Shapley-Shubik power index of more than $\frac{(N-1)}{N}$ unless he or she is a dictator.

(d) Give an example of a weighted voting system with N players and such that P_1 has a Shapley-Shubik power index of $\frac{(N-1)}{N}$.

80. (a) Explain why in any weighted voting system with N players a player with veto power must have a Banzhaf power index bigger than or equal to $\frac{1}{N}$.

(b) Explain why in any weighted voting system with N players a player with veto power must have a Shapley-Shubik power index bigger than or equal to $\frac{1}{N}$.

PROJECTS AND PAPERS

1 The Johnston Power Index

The Banzhaf and Shapley-Shubik power indexes are not the only two mathematical methods for measuring power. The Johnston power index is a subtle but rarely used variation of the Banzhaf power index in which the power of a player is based not only on how often he or she is critical in a coalition, but also on the number of other players in the coalition. Specifically, being a critical player in a coalition of 2 players contributes $\frac{1}{2}$ toward your power score; being critical in a coalition of 3 players contributes $\frac{1}{3}$ toward your power score; and being critical in a coalition of 10 contributes only $\frac{1}{10}$ toward your power score.

A player's *Johnston power score* is obtained by adding all such fractions over all coalitions in which the player is critical. The player's *Johnston power index* is his or her Johnston power score divided by the sum of all players' power scores.

Prepare a presentation on the Johnston power index. Include a mathematical description of the procedure for computing Johnston power, give examples, and compare the results with the ones obtained using the Banzhaf method. Include your own personal analysis of the merits of the Johnston method compared with the Banzhaf method.

2 The Past, Present, and Future of the Electoral College

Starting with the Constitutional Convention of 1776 and ending with the Bush-Gore presidential election of 2000, give a historical and political analysis of the Electoral College. You should address some or all of the following issues: How did the Electoral College get started? Why did some of the Founding Fathers want it? How did it evolve? What has been its impact over the years in affecting presidential elections? (Pay particular attention to the 2000 presidential election.) What does the future hold