Conformal Mapping in Linear Time

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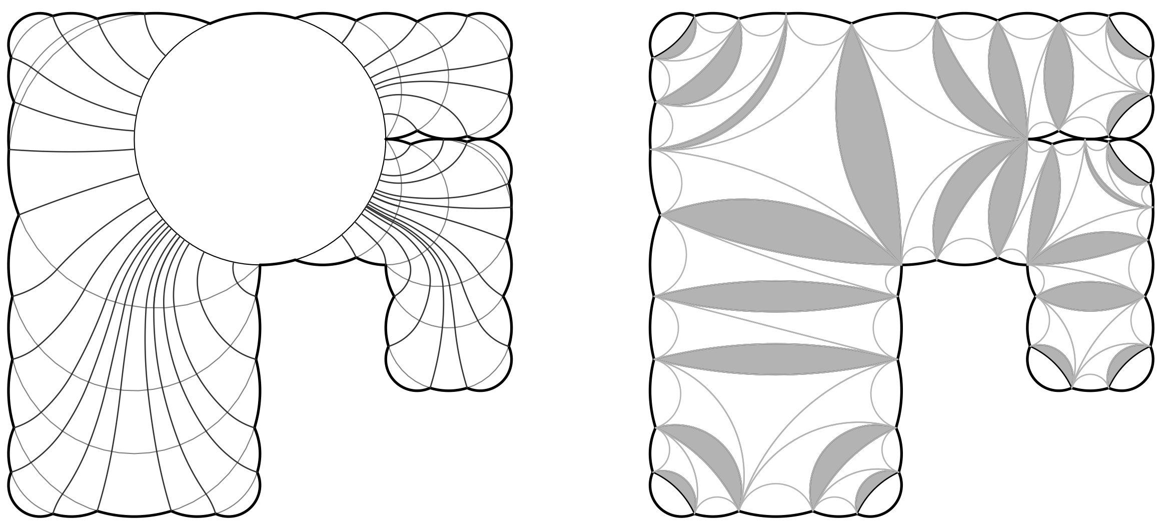
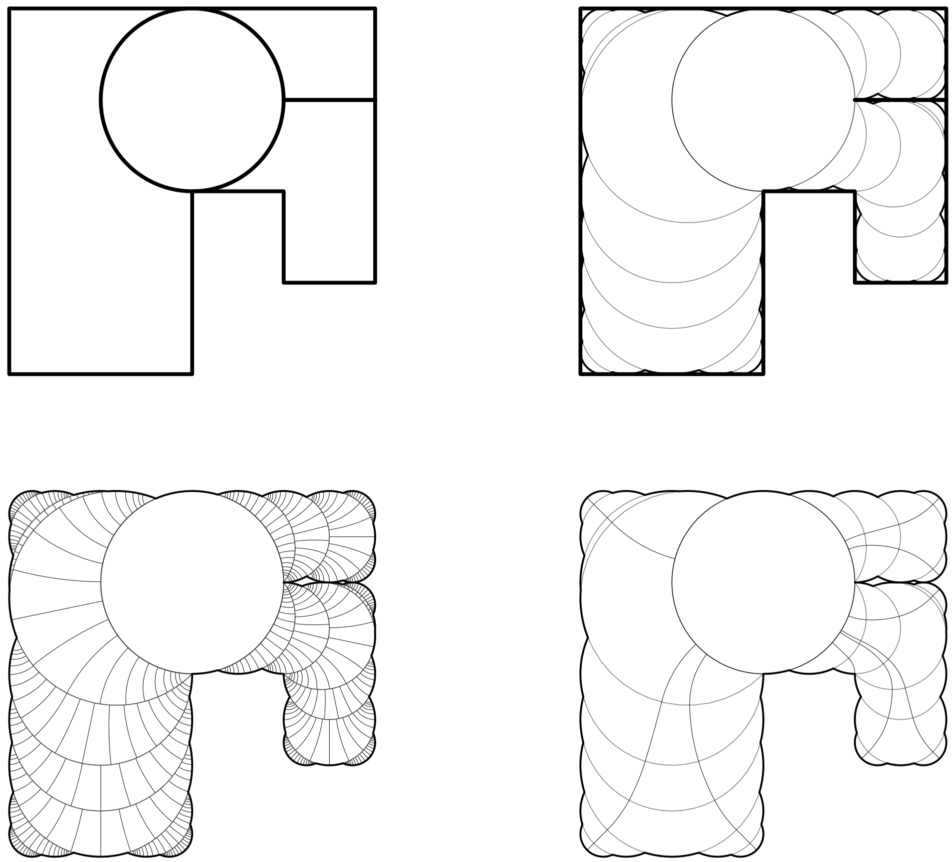
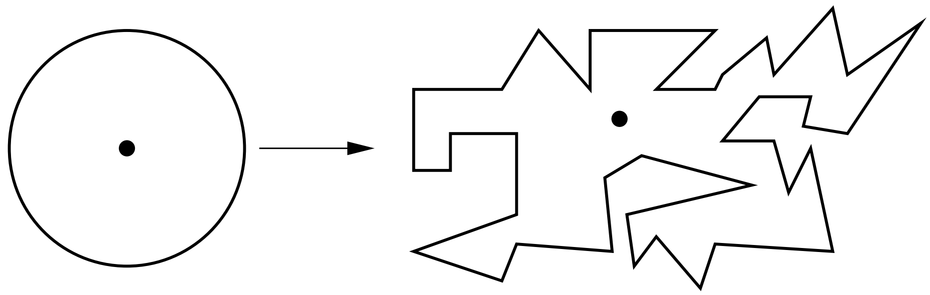


copies of lecture slides available at
`www.math.sunysb.edu/~bishop/lectures`



- Fast to compute using medial axis.
- Close to Riemann map.
- Motivated by hyperbolic 3-manifolds.

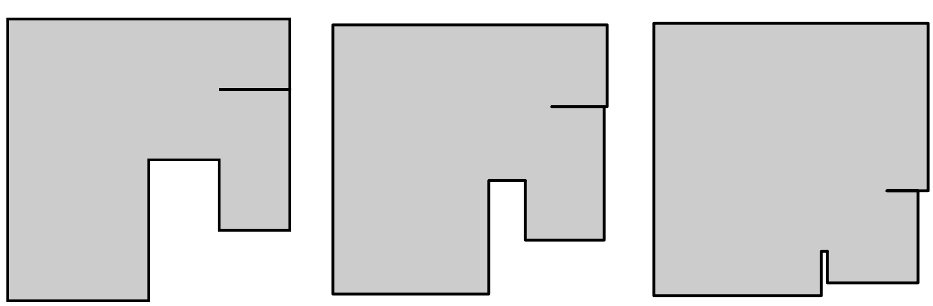
Riemann Mapping: If Ω is simply connected, then there is a conformal $f : \mathbb{D} \to \Omega$.



Schwarz-Christoffel formula for polygons

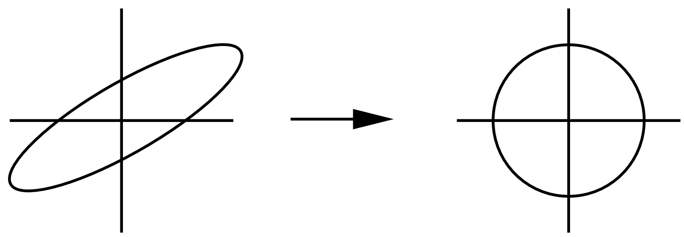
$$f(z) = A + C \int^{z} \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{\alpha_k - 1} dw.$$

α's are interior angles, z's are pre-vertices.



A mapping f is K-quasiconformal if either:

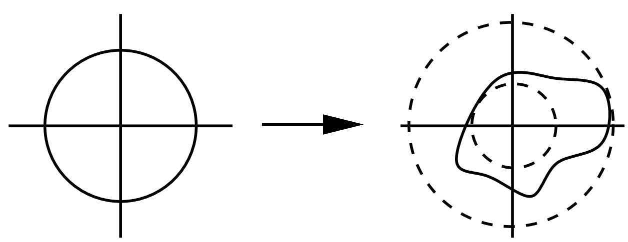
Analytic definition: $|f_{\bar{z}}| \leq \frac{K-1}{K+1}|f_z|$



$$f_z = \frac{1}{2}(f_x - if_y),\ f_{\bar{z}} = \frac{1}{2}(f_x + if_y).$$

Metric definition: For every $x \in \Omega$, $\epsilon > 0$ and small enough $r > 0$, there is $s > 0$ so that

$$D(f(x), s) \subset f(D(x, r)) \subset D(f(x), s(K+\epsilon)).$$



Notation for today: ϵ-conformal $= e^{\epsilon}$-quasiconformal.

- f determined $\mu_f = f_{\bar{z}}/f_z$. Conformal iff $\mu \equiv 0$
- If ϵ-QC and fixes $1, -1, i$ then $|f(z) - z| = O(\epsilon)$.

Theorem: If $\partial\Omega$ is an n-gon we can compute a ϵ-QC map between Ω and $\mathbb{D}$ in time $O(n \log^2 \frac{1}{\epsilon} \log\log \frac{1}{\epsilon})$.

Theorem: Suppose $\partial\Omega$ is an n-gon. We can construct points $\mathbf{w} = \{w_1, \ldots, w_n\} \subset \mathbb{T}$ so that:

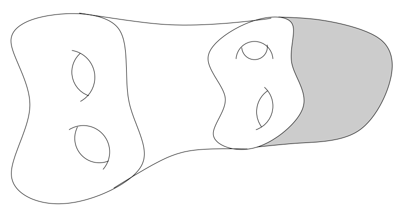
1. requires at most $O(n \log^2 \frac{1}{\epsilon} \log\log \frac{1}{\epsilon})$ steps.
2. $d_{QC}(\mathbf{w}, \mathbf{z}) < \epsilon$.

$\mathbf{z}$ = conformal pre-vertices.

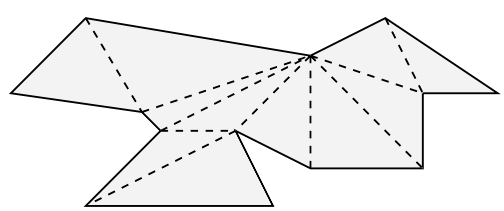
$d_{QC}(\mathbf{w}, \mathbf{z}) = \inf\{\log K : \exists h \in \mathrm{QC_K}, \mathrm{h}(\mathbf{w}) = \mathbf{z}\}$.
$\mathrm{QC_K}$ = K-quasiconformal maps.

Proof is amusing because it involves (at least) two results which don't seem to involve conformal mappings:

Theorem (Sullivan, Epstein-Marden): If M is a hyperbolic 3-manifold and $C(M)$ is the convex core of M, then there is a biLipschitz map between $\partial_\infty M$ and $\partial C(M)$.



Theorem (Chazelle): A simple n-gon can be triangulated in time $O(n)$.



Proof of theorem is in two steps:

Step 1: Given $\epsilon < \epsilon_0$ and ϵ-QC $f_n : \Omega \to \mathbb{D}$ construct $C\epsilon^2$-QC map $f_{n+1} : \Omega \to \mathbb{D}$. Construction takes time $C(\epsilon) = C + C \log^2 \frac{1}{\epsilon} \log\log \frac{1}{\epsilon}$.

Step 2: Build domains, maps and finite sets

$$(\Omega_0, V_0) \xrightarrow{g_0}, \dots, \xrightarrow{g_{N-1}} (\Omega_N, V_N)$$

so that

- $\Omega_0 = \mathbb{D}$,
- $\Omega_N = \Omega$, $V_N = V$,
- δ-QC maps $g_k : \Omega_k \to \Omega_{k+1}$, $V_k \to V_{k+1}$.

If $\delta < \epsilon_0/2$ then find conformal maps by induction (use previously found map $f_k : \mathbb{D} \to \Omega_k$ composed with g_k as starting point of iteration in Step 1 to find next map $f_{k+1} : \mathbb{D} \to \Omega_{k+1}$ with accuracy $\epsilon/2$).

Amazing Fact 1: ϵ_0 is independent of Ω, n.

Amazing Fact 2: N is independent of Ω, n.

Consequence: Can build chain of domains and maps and get ϵ_0 approximation in time $O(n)$ (independent of Ω). Then just repeat Step 1 until get desired accuracy :

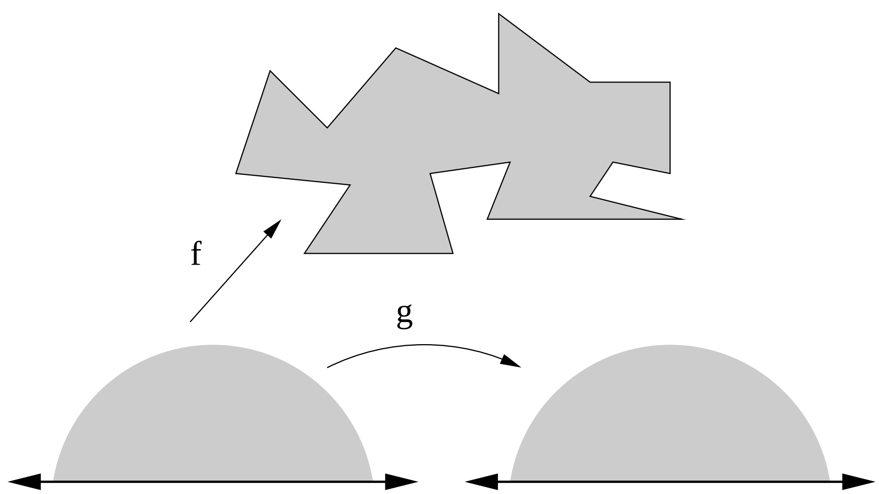
$$\epsilon_0, C\epsilon_0^2, \ldots C^k \epsilon_0^{2^k}.$$

About $\log\log\epsilon$ iterations suffice and time for kth iteration is $O(k2^{2k})$, so work dominated by final step.

Idea for Step 1: Suppose

$$f : \mathbb{H} \to \Omega, \qquad g : \mathbb{H} \to \mathbb{H}, \qquad \mu_f = \mu_g.$$

Then $f \circ g^{-1} : \mathbb{H} \to \Omega$ is conformal.

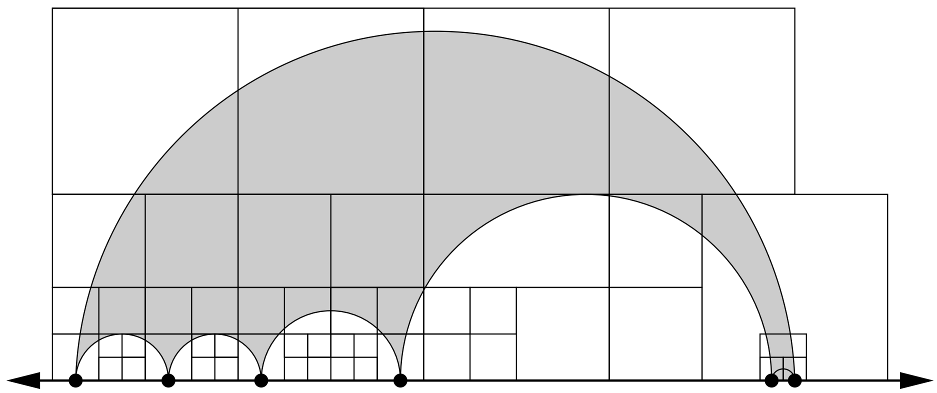


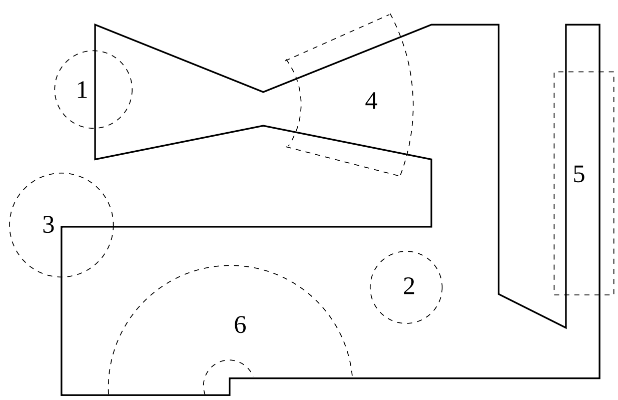
Can't solve Beltrami equation $g_{\bar{z}} = \mu g_z$ exactly in finite time, but can quickly solve

$$g_{\bar{z}} = (\mu + O(\|\mu\|^2))g_z,$$

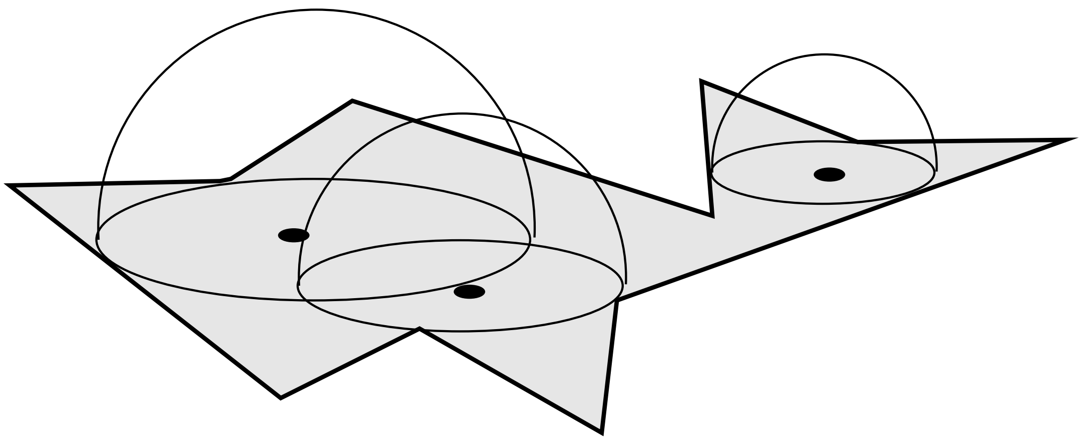
using fast multipole method of Greengard and Rokhlin. Then $f \circ g^{-1}$ is $(1 + C\|\mu\|^2)$-QC.

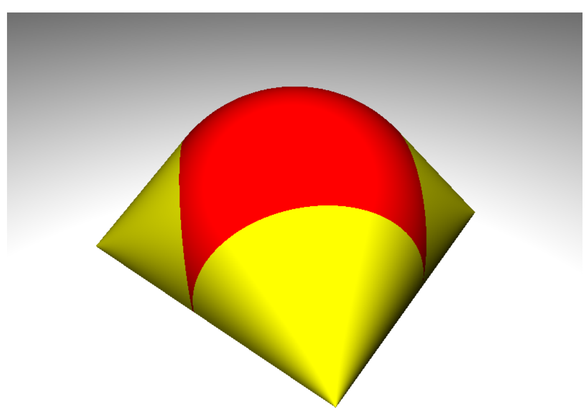
Cut $\mathbb{H}$ into $O(n)$ pieces on which f, f^α or $\log f$ has nice series representation. Need $p = O(|\log \epsilon|)$ terms on each piece to get ϵ accuracy.



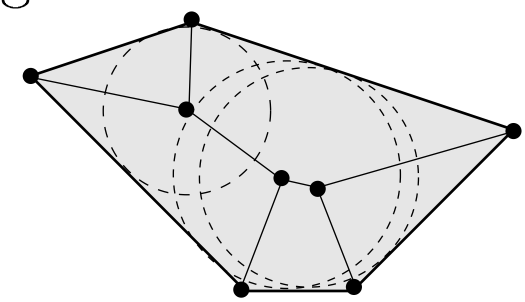


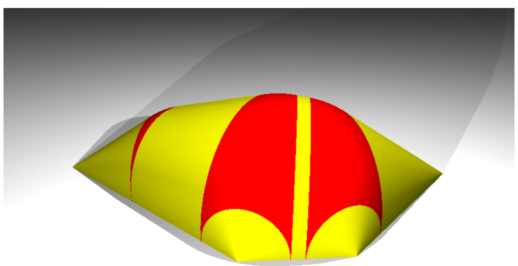
The **dome** of Ω is boundary of union of all hemispheres with bases contained in Ω.

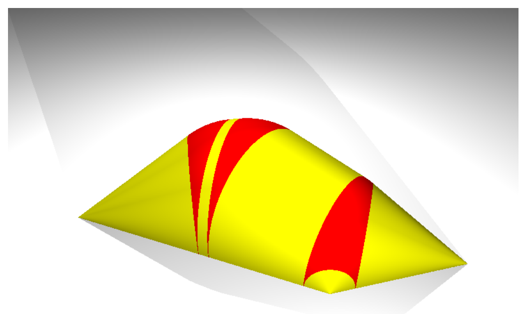




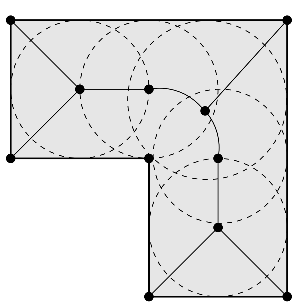
A convex polygon:

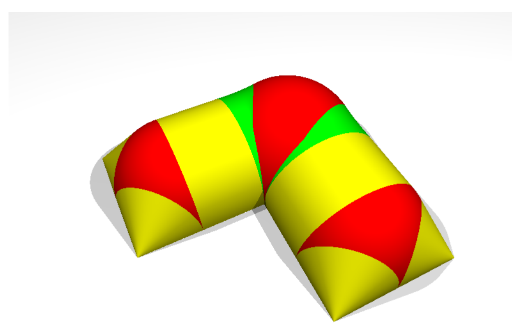






A non-convex polygon:

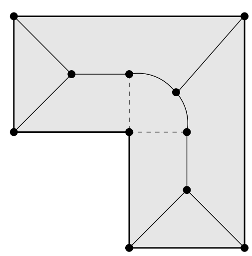




Each point on Dome(Ω) is on dome of a maximal disk D in Ω. Must have $|\partial D \cap \partial\Omega| \geq 2$. The centers of these disks form the **medial axis**.

For polygons is a finite tree with 3 types of edges:
- point-point bisectors (straight)
- edge-edge bisectors (straight)
- point-edge bisector (parabolic arc)

MA is boundary of Voronoi cells in polygon.

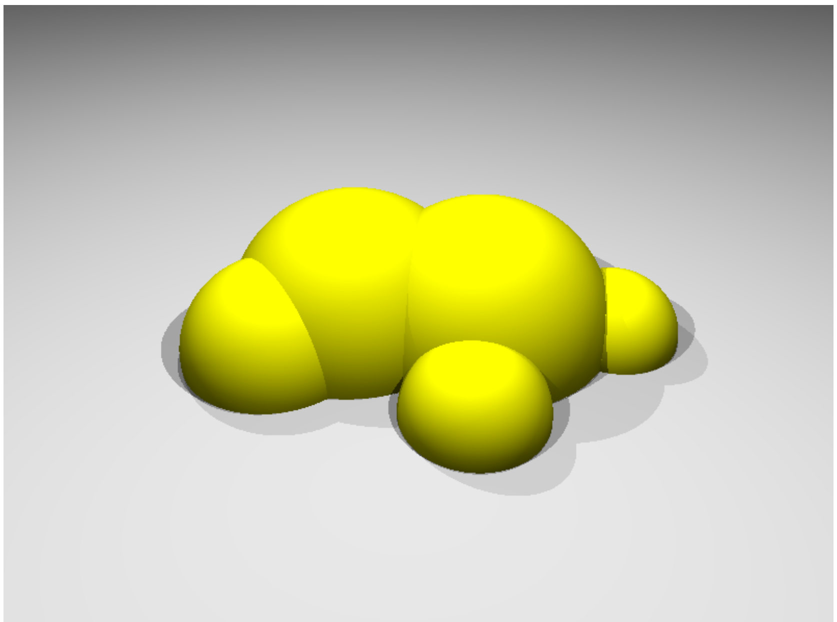


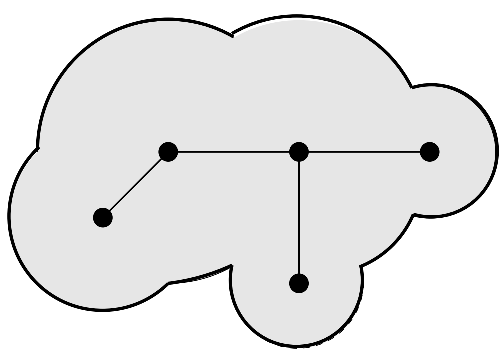
For applications see:

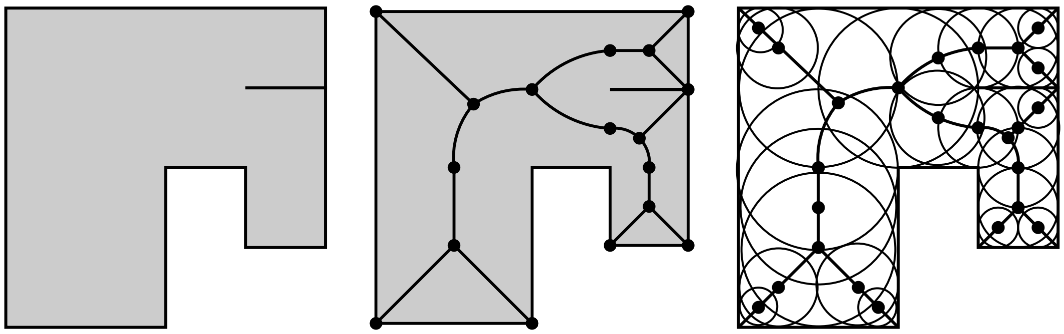
www.ics.uci.edu/ eppstein/gina/medial.html+

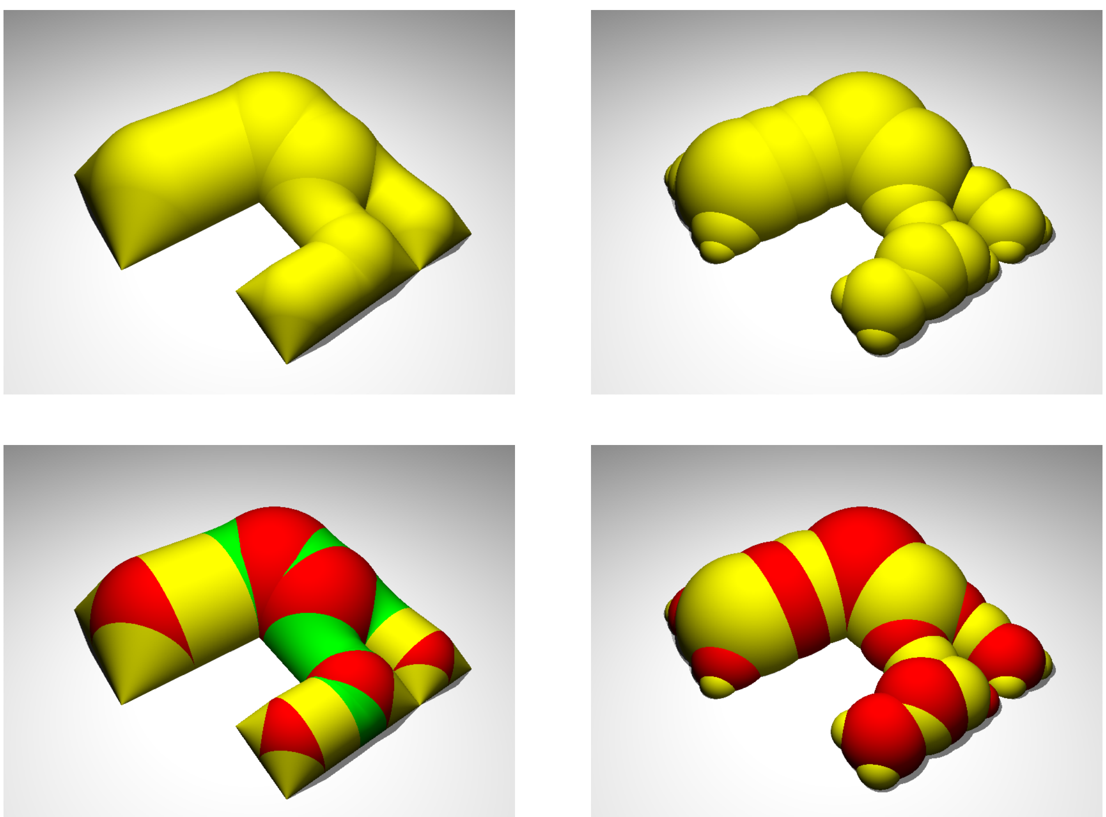
MA can be computed in linear time (Chin, Snoeyink, Wang, 1999): cut polygon into histograms, triangulate using Chazelle's method, compute Voronoi diagrams for each and merge results.

Finitely bent domain (= finite union of disks).





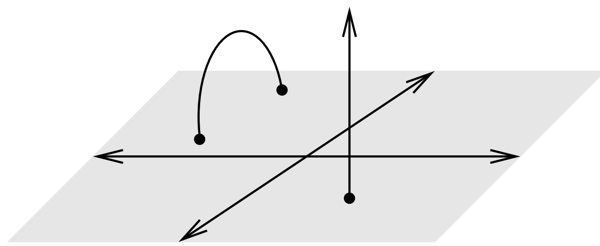




Hyperbolic space: Metric on $\mathbb{R}^n_+$,

$$d\rho = |dz|/\text{dist}(z, \mathbb{R}^{n-1}).$$

Geodesics are circles or lines orthogonal to $\mathbb{R}^{n-1}$.



Dome of Ω bounds hyperbolic convex hull of Ω^c. The hyperbolic metric on a simply connected plane domain Ω is defined by transferring the metric on half-plane by conformal map.

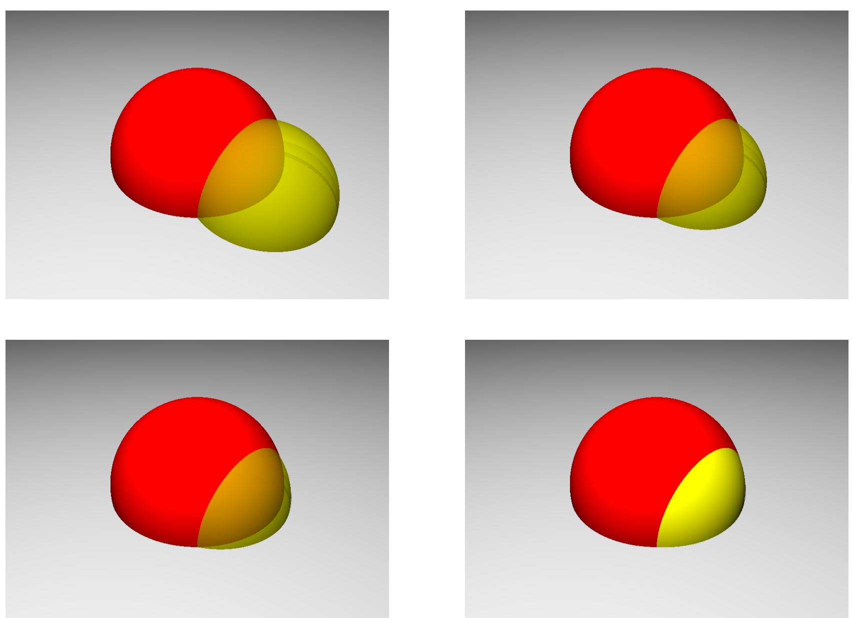
Fact: $\rho \simeq \tilde{\rho}$ (pseudo-hyperbolic metric)

$$d\tilde{\rho} = \frac{|dz|}{\text{dist}(z, \partial\Omega)},$$

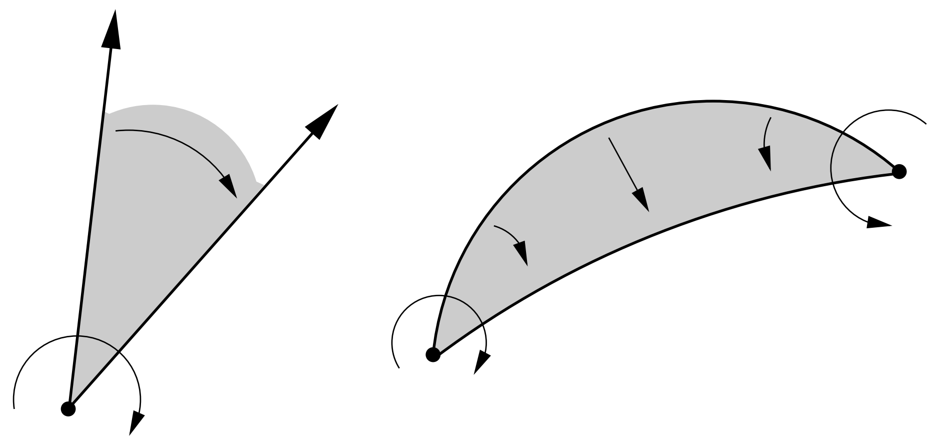
Let ρ_S be the hyperbolic path metric on S.

Theorem (Thurston): There is an isometry ι from (S, ρ_S) to the hyperbolic disk.

For finitely bent domains rotate around each bending geodesic by an isometry to remove the bending (more obvious if vertices are 0 and ∞).

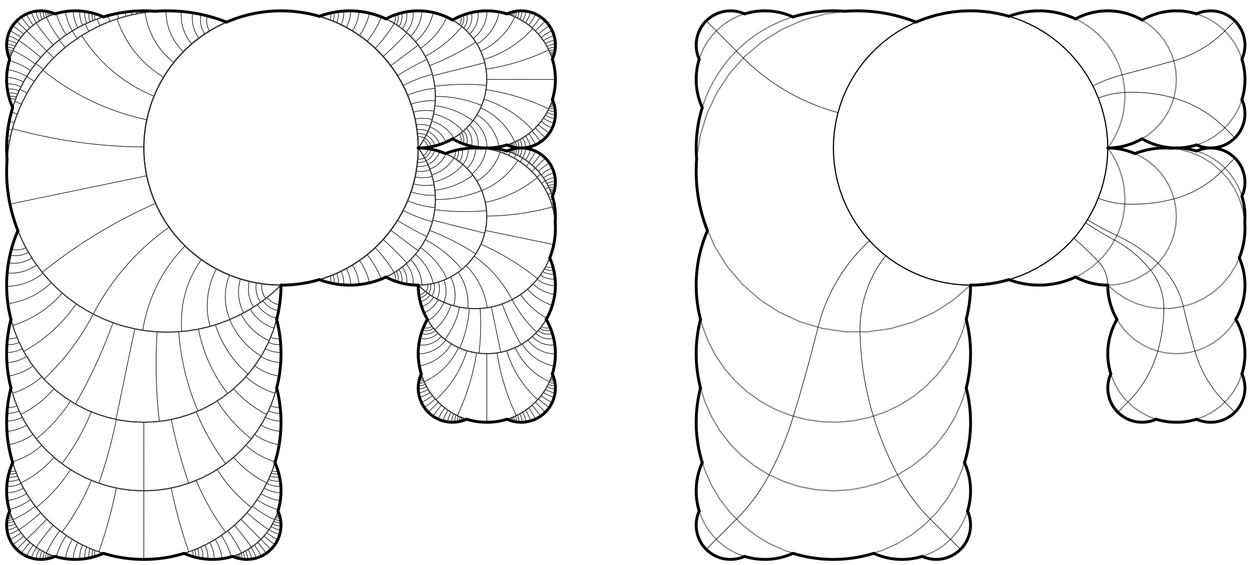


Elliptic Möbius transformation is conjugate to a rotation.



Elliptic transformation determined by fixed points and angle of rotation θ. It identifies sides of a crescent of angle θ: think of flow along circles orthogonal to boundary arcs.

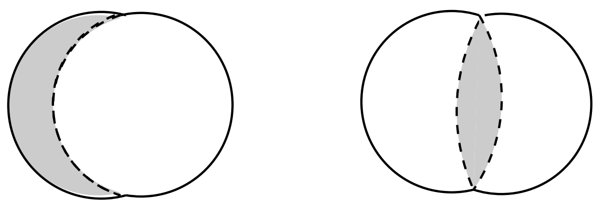
Visualize ι as a flow: Write finitely bent Ω as a disk D and a union of crescents. Foliate crescents by orthogonal circles. Following leaves of foliation in $\Omega \setminus D$ gives $\iota : \partial\Omega \to \partial D$.



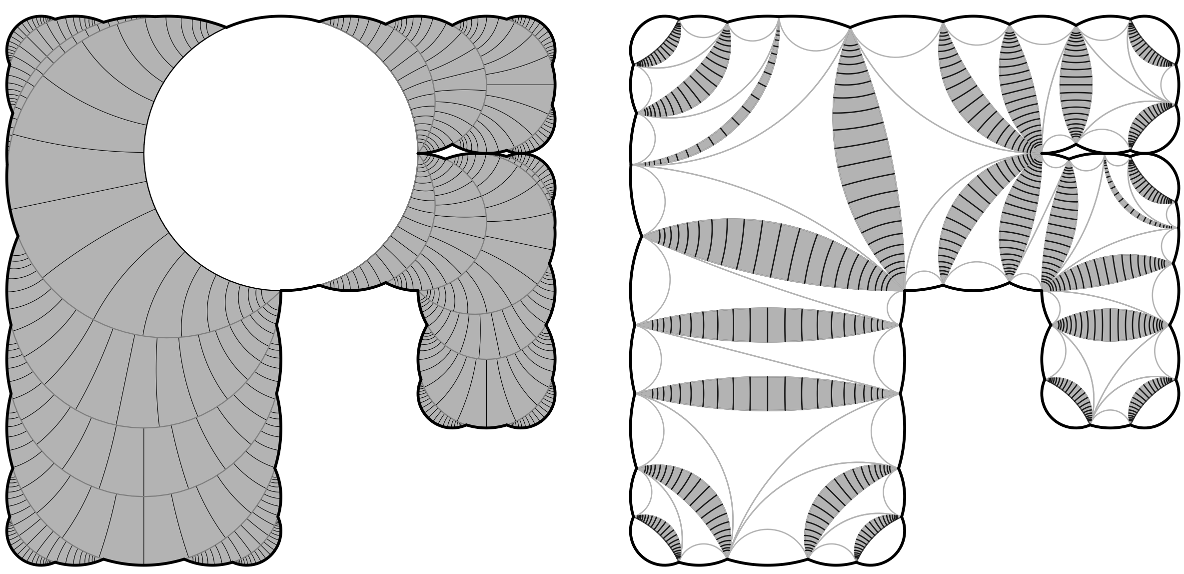
Has continuous extension to interior: identity on disk and collapses orthogonal arcs to points.

- ι is "Riemann map" from dome to disk.
- ι has K-QC extension to interior of base.
- ι can be evaluated at n points in time $O(n)$.

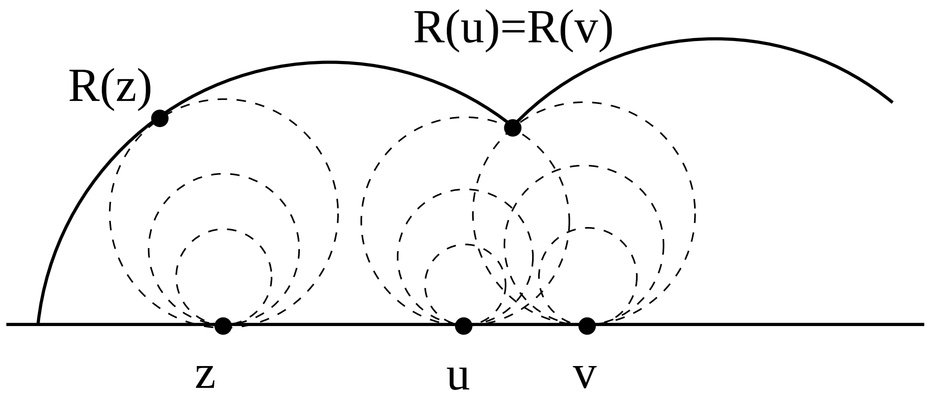
There are at least two ways to decompose a finite union of disks using crescents (with same angles and vertices in both cases).



We call these **tangential** and **normal** crescents. A finitely bent domain can be decomposed with either kind of crescent.



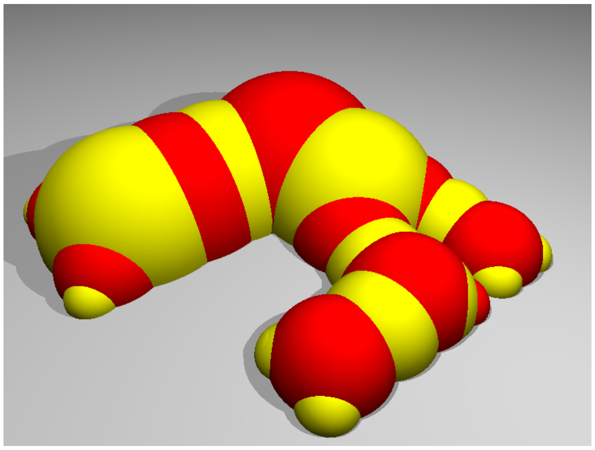
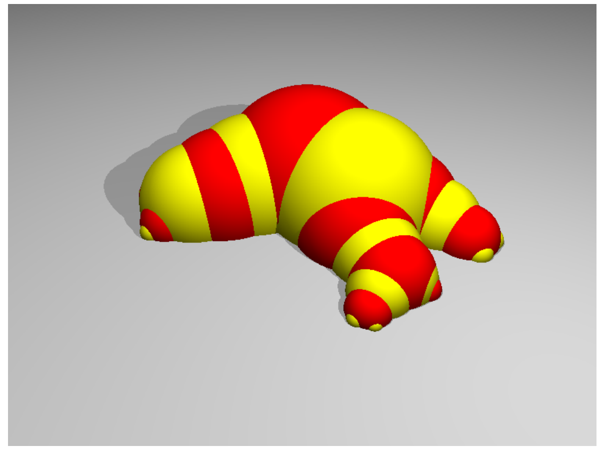
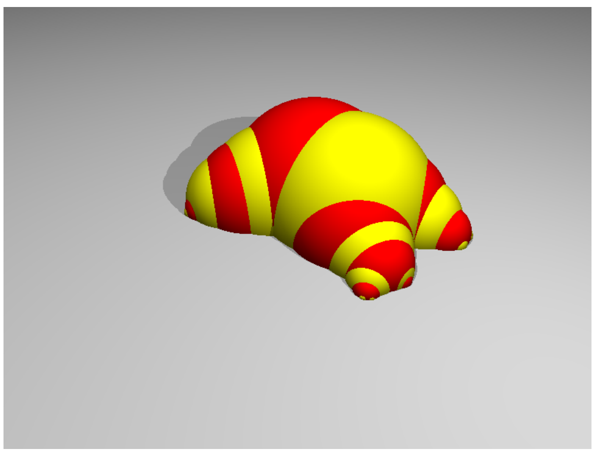
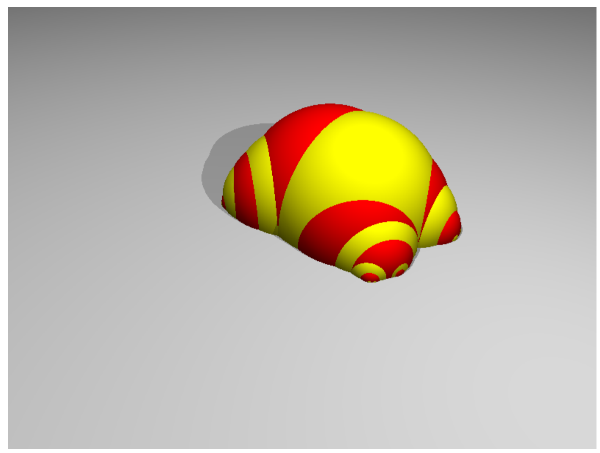
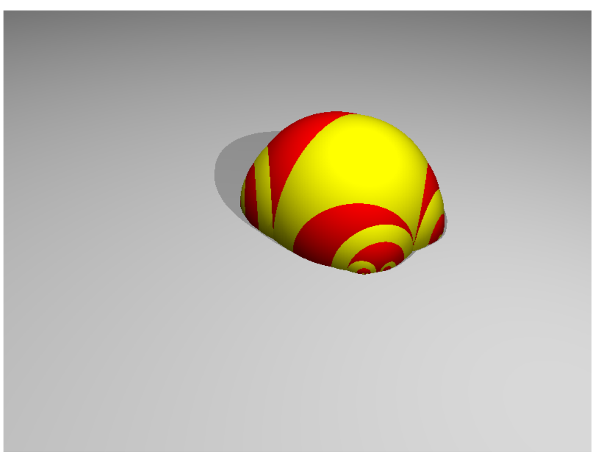
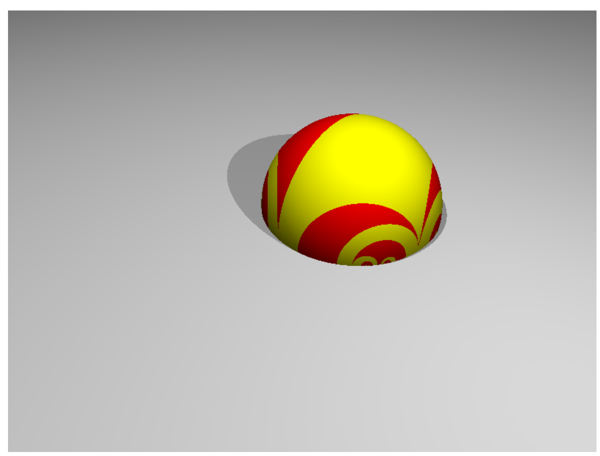
Nearest point retraction $R : \Omega \to \mathrm{Dome}(\Omega)$**:**
Expand ball tangent at $z \in \Omega$ until it hits a point $R(z)$ of the dome.

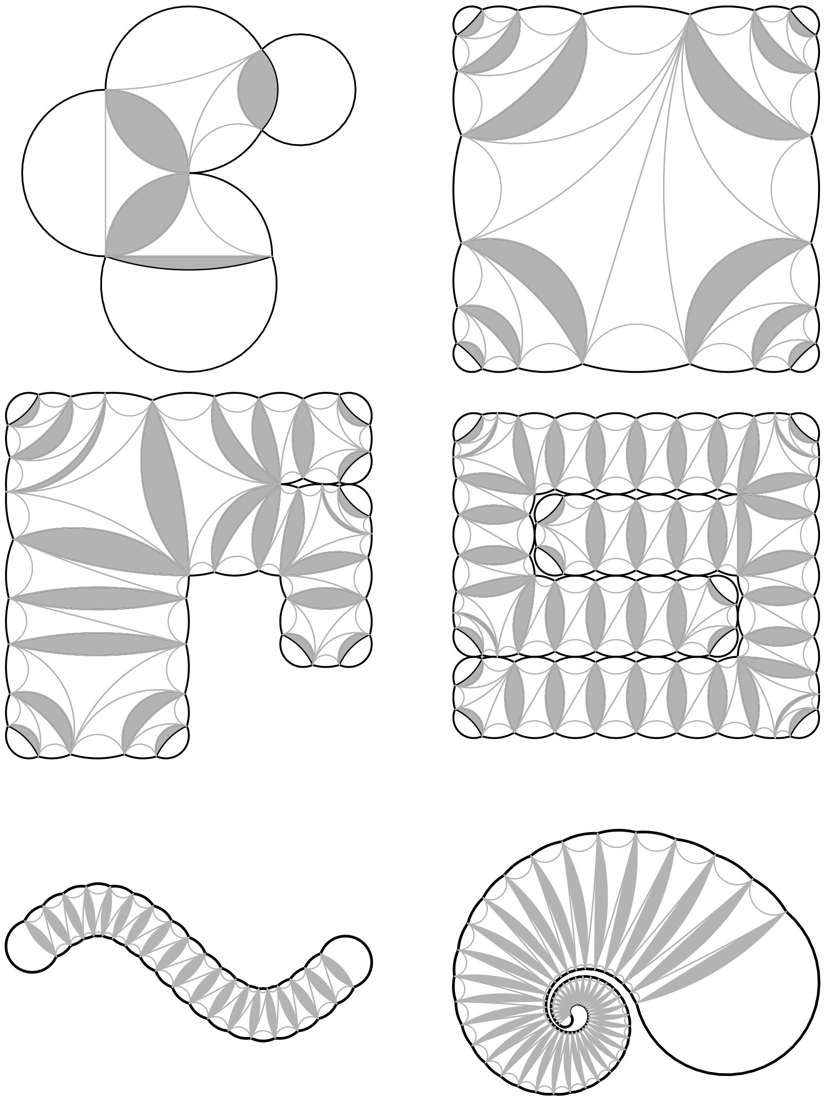


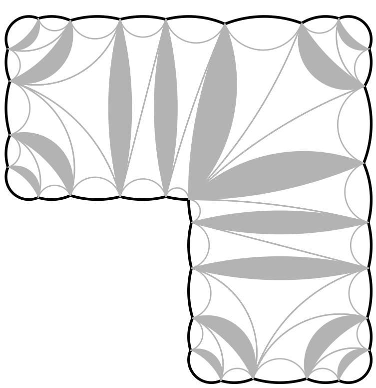
$$\text{normal crescents} = R^{-1}(\text{bending lines})$$
$$\text{gaps} = R^{-1}(\text{faces})$$

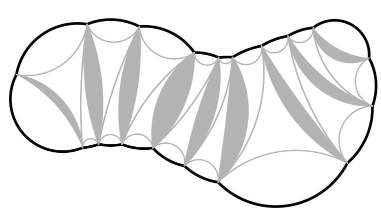
collapsing crescents = nearest point retraction

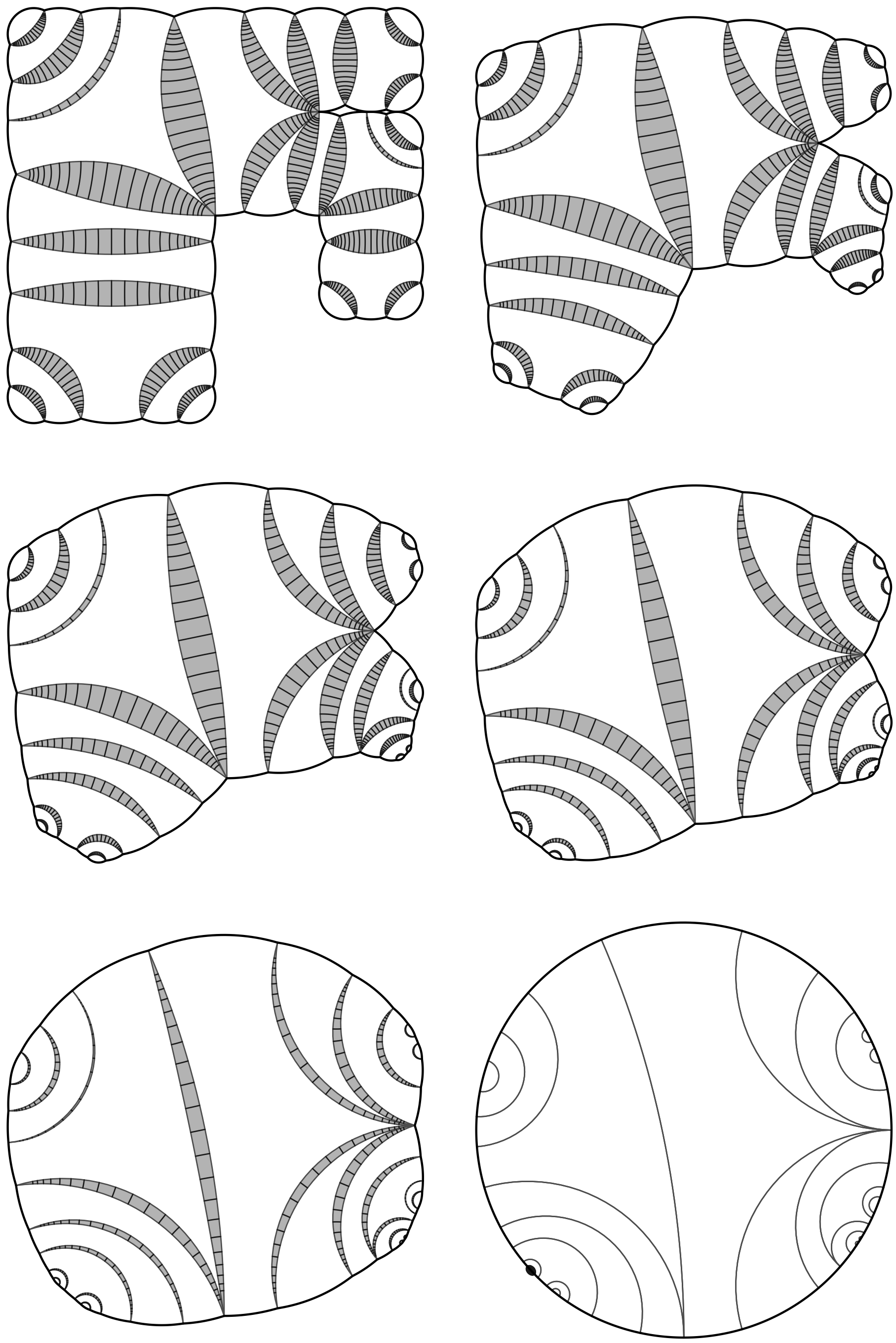
• Get map from Ω to $\mathbb{D}$ by mapping Ω to dome, followed by ι map to disk.









- f is a **bi-Lipschitz** if

$$\frac{1}{A}\rho(x,y) \le \rho(f(x), f(y)) \le A\rho(x,y).$$

- f is a **quasi-isometry** if

$$\frac{1}{A}\rho(x,y) - B \le \rho(f(x), f(y)) \le A\rho(x,y) + B.$$

- QI=BL at "large scales".

- On hyperbolic disk, BL $\Rightarrow$ QC $\Rightarrow$ QI.

Theorem: $f : \mathbb{T} \to \mathbb{T}$ has a QC-extension to interior iff it has QI-extension (hyperbolic metric) iff it has a BL-extension.

Theorem: Nearest point retraction is a quasi-isometry with constants independent of Ω.

Corollary (Sullivan, Epstein-Marden): There is a K-QC map $\sigma : \Omega \to$ Dome so that $\sigma = \text{Id}$ on $\partial\Omega$.

Corollary: $\iota : \partial\Omega \to \partial\mathbb{D}$ extends K-QC to Ω.

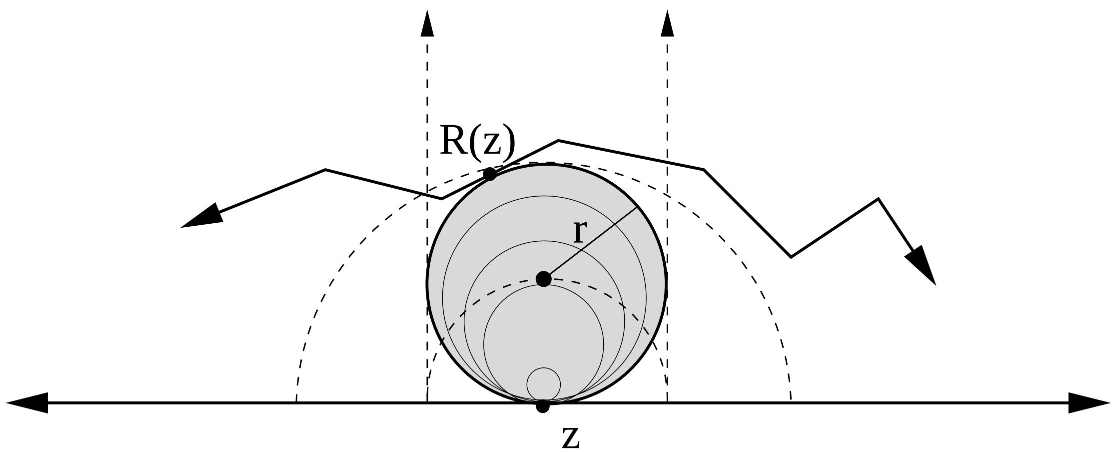
S-E-M theorem comes from hyperbolic 3-manifolds. If Ω is invariant under Möbius group G, $M = \mathbb{R}^3_+/G$ is hyperbolic manifold,

$$\partial_\infty M = \Omega/G, \quad \partial C(M) = \text{Dome}(\Omega)/G.$$

Thurston conjectured $K = 2$ is possible, but shown false by Epstein and Markovic. Best known upper bound is $K < 7.82$.

Fact 1: If $z \in \Omega$, $\infty \notin \Omega$,

$$r \simeq \mathrm{dist}(z, \partial\Omega) \simeq \mathrm{dist}(R(z), \mathbb{R}^2) \simeq |z - R(z)|.$$



Fact 2: R is Lipschitz.

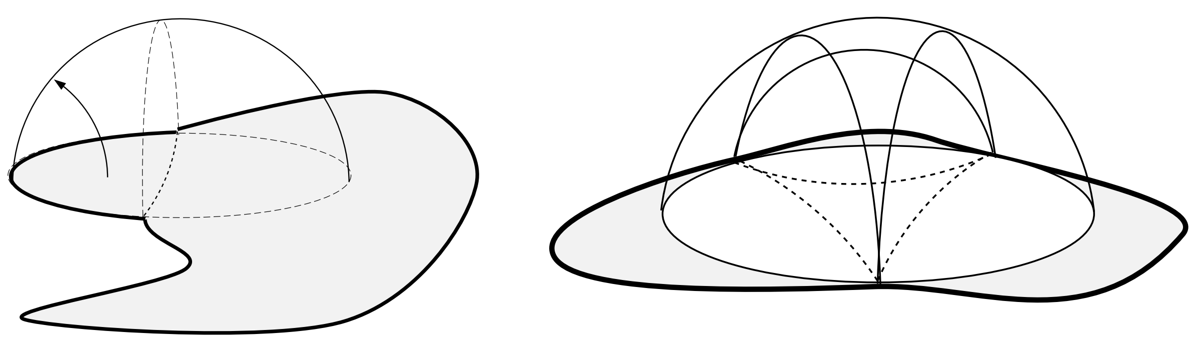
Ω simply connected $\Rightarrow$

$$d\rho \simeq \frac{|dz|}{\mathrm{dist}(z, \partial\Omega)}.$$

$z \in D \subset \Omega$ and $R(z) \in \mathrm{Dome(D)} \Rightarrow$

$$\mathrm{dist}(z, \partial\Omega)/\sqrt{2} \leq \mathrm{dist}(z, \partial D) \leq \mathrm{dist}(z, \partial\Omega)$$

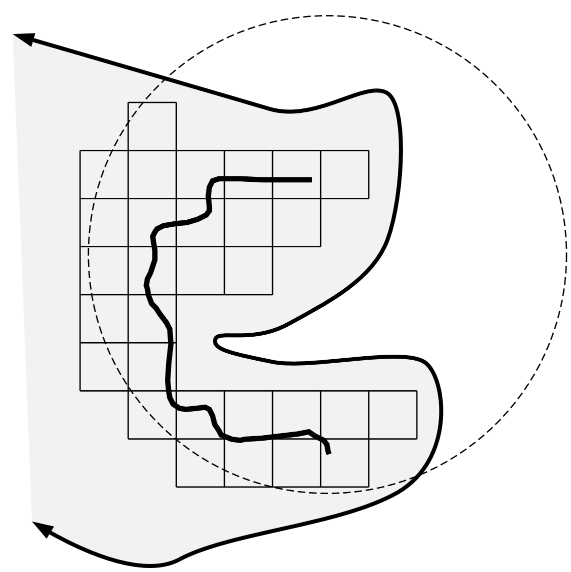
$$\Rightarrow \quad d\rho_\Omega(z) \simeq d\rho_D(z) = d\rho_{\mathrm{Dome}}(R(z)).$$



Fact 3: $\rho_S(R(z), R(w)) \leq 1 \Rightarrow \rho_\Omega(z, w) \leq C$.

Suppose $\mathrm{dist}(R(z), \mathbb{R}^2) = r$ and γ is geodesic from z to w.

$$\begin{aligned} \Rightarrow \qquad & \mathrm{dist}(\gamma, \mathbb{R}^2) \simeq r \\ \Rightarrow \qquad & \mathrm{dist}(R^{-1}(\gamma), \partial\Omega) \simeq r, \\ & \qquad R^{-1}(\gamma) \subset D(z, Cr) \\ \Rightarrow \qquad & \rho_\Omega(z, w) \leq C \end{aligned}$$



Moreover, $g = \iota \circ \sigma : \Omega \to \mathbb{D}$ is locally Lipschitz. Standard estimates show

$$|g'(z)| \simeq \frac{\text{dist}(g(z), \partial\mathbb{D})}{\text{dist}(z, \partial\Omega)}.$$

Use Fact 1

$$\begin{aligned}
\text{dist}(z, \partial\Omega) &\simeq \text{dist}(\sigma(z), \mathbb{R}^2) \\
&\simeq \exp(-\rho_{\mathbb{R}^3_+}(\sigma(z), z_0)) \\
&\gtrsim \exp(-\rho_S(\sigma(z), z_0)) \\
&= \exp(-\rho_D(g(z), 0)) \\
&\simeq \text{dist}(g(z), \partial D)
\end{aligned}$$