Boundary Behavior of Uniformly Convergent Conformal Maps

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Abstract of the Dissertation

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In the first section of this thesis we prove that for any sequence $\{\phi_n\}$ of conformal maps of the unit disk with limit map ϕ, uniformly convergent on compacta, and any positive, decreasing, continuous kernel function $K(|t|)$ which grows faster at the origin than $\log(\frac{1}{|t|})$, there is a subsequence $\{\phi_{n_k}\}$ and a Borel set $E \subset \partial\mathbb{D}$ of zero K-capacity so that off of E each element in the subsequence has well-defined radial extension to the boundary $\phi_n(x)$, and furthermore that $\phi_n(x) \rightarrow \phi(x)$. We provide an example to show that the theorem is sharp–one cannot, in general, take the set E to have zero logarithmic capacity.

In the second section of this thesis we present a new proof of the fact that to any orientation-reversing, quasisymmetric involution h of the unit circle, fixing ± 1, there is associated a quasiarc γ in the complex plane so that the conformal map $\phi(z)$ of the exterior of the unit disk to the complement of the quasiarc identifies x with $h(x)$. We present an explicit construction of approximating maps converging to $\phi(z)$ and provide computer-generated images of the associated quasiarcs for several maps h.

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Um beijão for Candida.

For my dad

Introduction

Recall that a homeomorphism h of the unit circle is a *welding function* if there are conformal maps f and f^* on $\mathbb{D}$ and $\mathbb{D}^*$, respectively, so that $f(\mathbb{D})$ and $f^*(\mathbb{D}^*)$ are the two complementary components of a Jordan curve $\Gamma \subset \mathbb{C}$, and so that $h(x) = f^{-1} \circ f^*(x)$ for all $x \in \partial\mathbb{D}$. The maps f and f^* are the *welding functions* associated h. The well-known fact that there are always such maps when h is quasisymmetric is sometimes referred to as the Fundamental Theorem of Conformal Welding (FTCW). Many authors have proven that conformal welding is possible for homeomorphisms which are nice in senses like that of quasisymmetry. Bishop showed [Bis03] at the other extreme that log-singular functions are always welding maps as well. Recall that a homeomorphism h of the unit circle is log-singular if there is a decomposition $\partial\mathbb{D} = F_1 \cup F_2$ so that F_1 and $h(F_2)$ both have logarithmic capacity zero.

In the first chapter we prove a theorem motivated by a result of Hamilton in his paper Generalized Conformal Welding [Ham91]. A homeomorphism h of the circle is said to be a generalized conformal welding on a set $E \subset \partial\mathbb{D}$ if there are maps f and f^* as above so that f^* has radial limits on E, f has radial limits on $h(E)$ and so that $h = f^{-1} \circ f^*(x)$ for all $x \in E$. One then asks the question, for an arbitrary homeomorphism of the circle, how large is the set on which it is a generalized conformal welding? This has been answered for several classes of homeomorphisms. In Theorem 1 of his paper [Ham91] Hamilton shows that any regular homeomorphism h of the unit circle is a generalized conformal welding on a set E where $\partial\mathbb{D} \setminus E$ has Lebesgue measure zero. Recall that a *regular* homeomorphism is one for which the forward and backward images of any zero-Hausdorff dimension set have Lebesgue measure zero. He shows first that any regular homeomorphism of the unit circle can be approximated by a sequence of bilipschitz homeomorphisms $h_n(x)$ which converge uniformly to h on $\partial\mathbb{D}$. Of course, the bilipschitz constants k_n of the maps h_n do not remain bounded—if they did, then h itself would be bilipschitz. For each approximating map h_n he applies the FTCW and associates to it a quasicircle Γ_n and conformal maps f_n and f_n^* onto the complementary components of Γ_n with $f_n^{-1} \circ f_n^*(x) = h_n(x)$ for all $x \in \partial\mathbb{D}$. With suitable normalization, the sequences $\{f_n\}$ and $\{f_n^*\}$ converge respectively to maps f and f^*. Note that the limit domains $\Omega = f(\mathbb{D})$ and $\Omega^* = f^*(\mathbb{D}^*)$ need not be the complementary components of a Jordan curve, but we will have the relation $f^{-1} \circ f^*(x) = h(x)$ for any x in $\partial\mathbb{D}$ satisfying the four conditions below.

- $\lim_{r\to 1} f_n^*(r \cdot x)$ exists for an n
- $\lim_{r\to 1} f_n^{-1}(r \cdot h(x))$ exists for all n
- $\lim_{n\to\infty} f_n^*(x) = f^*(x)$
- $\lim_{n\to\infty} f_n^{-1}(x) = f^{-1}(h(x))$

By a well-known result of Beurling [Pom92], the first condition will hold off of a set of logarithmic capacity zero. The first two conditions together may require that we throw out a set E_1 of logarithmic capacity zero as well as its image under h, and a set E_2 of logarithmic capacity zero as well as its preimage under h. Together, $E_1 \cup E_2 \cup h(E_1) \cup h^{-1}(E_2)$ form a set of Lebesgue measure zero. This is what the regularity condition on h implied. The second two conditions are the subject of his Theorem 2:

Theorem (Hamilton) *Let $\{h_n(z)\}$ be analytic in $\mathbb{D}$, with $h_n(\mathbb{D})$ having area no greater than 1. Then there exists a subsequence $\{h_{n_k}\}$ and a limit $h(z)$ so that for any $\alpha, \epsilon > 0$ there is $E \subset \partial\mathbb{D}$ with α-dimensional Hausdorff measure $\mathcal{H}^\alpha(E) \leq \epsilon$ and $h_{n_k}(z) \to h(z)$ on $\partial\mathbb{D} \setminus E$. One cannot take the set E to have logarithmic capacity zero.*

In other words, the last two conditions imply that we must throw out a set of Hausdorff dimension zero and its h-image for each of the maps f^* and f^{-1}. Again by the regularity assumption on h, this is a set of Lebesgue measure zero, so his sequence of approximations leads to generalized conformal welding off of a set of Lebesgue measure zero.

Recall [Car67] that a set has positive Hausdorff measure $\mathcal{H}^\alpha$, $\alpha > 0$ if and only if it has positive capacity for the kernel $K_\alpha = t^{-\alpha}$, so Hamilton's theorem states that for a sequence of analytic maps satisfying the given criteria and any $\alpha > 0$, there is a subsequence $\{h_{n_k}\}$ such that $h_{n_k}(z) \to h(z)$ off of a set of $|t|^{-\alpha}$-capacity zero, but not necessarily for the logarithmic kernel function $K(|t|) = \log(\frac{1}{|t|})$. This raises the question: Does the result hold for kernels which lie between the $|t|^{-\alpha}$'s and $\log(\frac{1}{|t|})$? By 'between' we mean that a kernel K satisfies

$$\lim_{|t|\to 0} \frac{K(|t|}{|t|^{-\alpha}} = 0 \text{ for all } \alpha, \text{ but } \lim_{|t|\to 0} \frac{K(|t|}{\log(\frac{1}{|t|})} = \infty.$$

We answer this question in Chapter 1, for the case of conformal maps, with the following theorem. Let $\tilde{\phi}$ denote the radial extension to the boundary for a conformal map ϕ, where it exists.

Theorem *Let $\{\phi_n\}$ be a sequence of conformal maps of the unit disk converging uniformly on compacta to the conformal map ϕ. Then for any function K satisfying*

$$\lim_{t \to 0} \frac{K(t)}{\log \frac{1}{t}} = \infty$$

there is a subsequence $\{\phi_{n_k}\}$ and a set E_K with $\mathrm{cap}_K E_K = 0$ so that $\tilde{\phi}_{n_k}(x) \to \tilde{\phi}(x)$ for $x \in \partial\mathbb{D} \setminus E_K$.

We provide an example showing that the theorem is sharp.

Following Hamilton's proof of his Theorem 1 and considering the theorem above, it might seem that his result could not be improved to the class of log-regular homeomorphisms, but as Bishop shows in [Bis03] this is not the case. Bishop used a very different approach to address the question of the size of the set where a circle homeomorphism is a generalized conformal welding. He shows that if a homeomorphism h is log-regular, then it is a generalized conformal welding on a set of full Lebesgue measure. Bishop shows that a sequence of conformal maps of $\mathbb{D}$ arising as approximations to a welding map f cannot have the property that their boundary values fail to converge on a set of positive logarithmic capacity. Bishop constructs his approximating maps using Koebe's circle domain theorem. Given a homeomorphism h of $\partial\mathbb{D}$ he identifies each of n evenly-spaced points $x_1, \dots, x_n$ with a dilated copy of its image under h by the Koebe map on the left side of Figure 1.

The arcs connecting each x_j to $2h(x_j)$ can be chosen arbitrarily so long as they

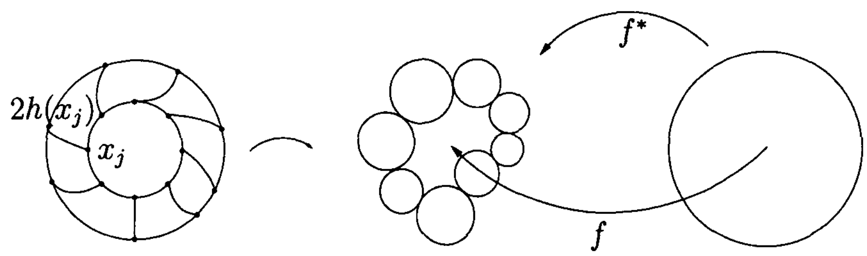

Figure 1: Bishop's approximate welding maps

do not intersect, and the choice will not affect the resulting domain.

In the second chapter we use an idea similar to Bishop's circle-chain construction to give a new proof of the fact that to any orientation-reversing, quasisymmetric involution h of the unit circle, fixing ± 1, there is associated a quasiarc γ in the complex plane so that the conformal map $\phi(z)$ of the exterior of the unit disk to the complement of the quasiarc identifies x with $h(x)$. We similarly identify n evenly-spaced points on the upper semi-circle with their h-images on the lower-semicircle. One could apply Koebe's theorem as in [Bis03] to create chains of circles converging to a quasiarc, but we instead apply a composition of n explicit 'pinching' maps which identify x_j with $h(x_j)$. We will not have a chain of *circles* as a result, making it more difficult to show that the chains are converging to a quasiarc, but because of the explicit description of the maps we can create computer-generated images of the quasiarcs for a given map h. Several examples are presented..

In the third chapter we present a (possibly) new proof of Koebe's well-known circle domain theorem for finitely connected domains. Koebe's Theorem has been related to the field of conformal welding by Bishop [Bis03] as mentioned above. Koebe's theorem for simply connected domains is just the Riemann mapping theorem. The Riemann mapping theorem is usually proven with the Schwarz lemma. In Chapter 3 we use an analogous lemma, the Schwarz-Pick lemma for multiply connected domains [HS93], to prove Koebe's theorem for finitely connected domains. We cannot be certain that the proof presented is new—such an old theorem has many difficult-to-find proofs—but a search of readily available literature did not locate such a proof.

Chapter 1

Boundary Behavior of Uniformly Convergent Conformal Maps

1.1 Introduction

It is well known [Pom92] that any conformal map of the unit disk has well-defined radial extension to all x in $\partial\mathbb{D}$ with the exception of a set E of zero logarithmic capacity. For a given conformal map of the disk ϕ we denote by $\tilde{\phi}$ the radial extension of ϕ to $\partial\mathbb{D}$, where it exists. Suppose now that we are given a *sequence* of conformal maps of the disk, $\{\phi_n\}$, converging uniformly on compact subsets to a map ϕ. For each n there is a set of zero logarithmic capacity off of which $\tilde{\phi}_n$ is well defined, but is the set E where $\lim_{n\to\infty} \tilde{\phi}_n(x) \neq \tilde{\phi}(x)$ also so small? In general it is not. In fact, a uniformly convergent sequence of conformal maps of the unit disk may have the property that every one of its subsequences has boundary-value functions which fail to converge to $\tilde{\phi}$ on a set of positive logarithmic capacity. We provide such an example in Section 1.3. We show also, in Section 1.4, that the set E cannot be any larger than positive logarithmic capacity in the following sense:

Theorem 1. *Let $\{\phi_n\}$ be a sequence of conformal maps of the unit disk converging uniformly on compacta to the conformal map ϕ. Then for any function K satisfying*

$$\lim_{t\to 0}\frac{K(t)}{\log\frac{1}{t}}=\infty$$

there is a subsequence $\{\phi_{n_k}\}$ and a set E_K with $\mathrm{cap}_K E_K=0$ so that $\tilde{\phi}_{n_k}(x)\to\tilde{\phi}(x)$ for $x\in\partial\mathbb{D}\setminus E_K$.

In the context of conformal welding, Hamilton [Ham91] showed that for a uniformly convergent sequence $\{h_n\}$ of *analytic* maps of the unit disk there is a subsequence $\{h_{n_k}\}$, a limit map h, and a set $E\subset\partial\mathbb{D}$ such that E has Hausdorff dimension zero and $\tilde{h}_{n_k}(z)\to\tilde{h}(z)$ for all $z\in\partial\mathbb{D}\setminus E$. Hamilton also states that one cannot take the set E to have logarithmic capacity zero. Our example in Section 1.3 shows that the same is true for the class of conformal maps.

Hamilton's result also motivates Theorem 1 as we now describe. Recall [Car67] that for any α the Hausdorff α-measure $H_\alpha(E)$ of a set E is infinite if and only if the associated α-capacity $C_\alpha(E)>0$. In terms of α-capacities, Hamilton's result states that for any kernel function $K=t^{-\alpha}$, the set E has zero K-capacity. We show that for *conformal* maps of the unit disk we can choose E to have zero K-capacity for any kernel that grows faster at the origin than the logarithmic kernel, and the example in Section 1.3 makes our theorem sharp.

1.2 Background and Definitions

Recall that a set E is said to have positive logarithmic capacity if it supports a probability distribution μ so that the energy integral

$$I(\mu) = \int\limits_E \int\limits_E \log \frac{1}{|x-y|} d\mu(x) d\mu(y)$$

is finite. If such a μ exists we define the Robin's constant of E to be $\gamma(E) = \inf_\mu I(\mu)$, and the logarithmic capacity of E to be $\mathrm{cap}(E) = e^{-\gamma(E)}$. The distribution achieving the minimal energy integral is called the *equilibrium distribution* for E. It is usually denoted μ_E. If no μ yielding a finite energy integral exists, we say that the set has zero logarithmic capacity.

The concept of capacity can be generalized to other kernel functions K, where we say that a set E has positive K-capacity if there is a probability distribution μ supported on E so that

$$I_K(\mu) = \int\limits_E \int\limits_E K(|x-y|) d\mu(x) d\mu(y)$$

is finite. We then define $\gamma_K(E)$ and $\mathrm{cap}_K(E)$ analogously. In this paper we follow Carleson [Car67] and consider only kernels which are continuous, decreasing, and non-negative. We include for reference several properties of capacities and some tools commonly employed to estimate them.

Different authors use different definitions of capacity. The definition we chose to use here is that in [Pom92]. Carleson [Car67], for example, defines it as

$$\mathrm{cap}_K(E) = \frac{1}{\gamma_K(E)}$$

where $\gamma_K(E)$ is as defined above. The two definitions of capacity yield the same sets of zero capacity, but capacity by Carleson's definition has the convenience of countable subadditivity.

For the definition we have chosen to use, we don't have countable subadditivity, but we do have that for a countable collection of sets E_j having Robin constants $\gamma_K(E_j)$, their union E satisfies

$$\mathrm{cap}_K(E) \leq \exp\left[\frac{-1}{\sum \frac{1}{\gamma_K(E_j)}}\right]$$

From [Car67] we have that

$$\frac{1}{\gamma_k(E)} \leq \sum \frac{1}{\gamma_K(E_j)},$$

(this is the countable subadditivity for cap $= \frac{1}{\gamma}$). Equation 1.2 follows immediately. For clarity later we summarize this property as Lemma 1.

Lemma 1. *For any $\epsilon > 0$ and any kernel function K there is an increasing sequence of positive real numbers $\{g_j(\epsilon)\}$ so that if E_j is a set in $\partial\mathbb{D}$ with $\gamma_K(E_j) \geq g_j$, then $E = \cup E_j$ has*

$$\mathrm{cap}(E) \leq \epsilon.$$

We will need to use estimates of harmonic measure in simply connected domains. The definition of harmonic measure most often used is the following:

Definition 1. *Let Ω be a simply connected domain in the complex plane and let $z \in \Omega$. Let ϕ be the conformal map from $\mathbb{D}$ onto Ω with $\phi(0) = z$. Then for a Borel set $E \subset \partial\Omega$ we define the harmonic measure of E in Ω from z as*

$$\omega(z, E, \Omega) = \frac{|\phi^{-1}(E)|}{2\pi}$$

It is clear from the definition that harmonic measure is a conformal invariant. We will use the alternate (but equivalent) definition below, which can be applied to non-simply connected domains.

Definition 2. *Let Ω be a domain in the complex plane, and let E be a subset of $\partial\Omega$. Let $U_E = \{u : u \text{ is harmonic in } \Omega, u \leq \chi_E \text{ on } \partial\Omega\}$, where χ_E is the characteristic function of E. For $z \in \Omega$ we define the harmonic measure of E in Ω from z as*

$$\omega(z, E, \Omega) = \sup_{u \in U_E} u(z).$$

Also applicable to non-simply connected domains is a third formulation of the concept of harmonic measure. It was established by Kakutani in [Kak44] that harmonic measure in planar domains is closely related to Brownian motion. He showed that for a domain $\Omega \subset \mathbb{C}$, a set Borel set $E \subset \partial\Omega$, and a point $z \in \Omega$, the harmonic measure $\omega(z, E, \Omega)$ is equal to the probability that a Brownian particle starting at z will pass through E when it first exits Ω.

Returning to the second definition, for fixed E and Ω we define $u(z) = \omega(z, E, \Omega)$. Then $u(z)$ is harmonic in Ω, and we refer to $u(z)$ as the *harmonic* *measure function* for E.

We recall a useful property of harmonic functions [Rud66]:

Theorem 2 (Harnack). *Let $u(z)$ be a positive harmonic function in $D(a,R)$. Then for $r < R$*

$$\frac{R-r}{R+r}u(a) \leq u(a+re^{i\theta}) \leq \frac{R+r}{R-r}u(a)$$

Many estimates of harmonic measure come from bounds on another conformal invariant: moduli of families of curves. A curve family, usually denoted Γ is defined in [Pom92] as the collection of open, half open, or closed arcs in a Borel set $B \subset \mathbb{C}$ satisfying a set of prescribed conditions. Commonly used examples are the family of curves joining (or separating) the boundary components of an annulus $\mathbb{A} = \{r < |z| < R\}$, Figure 1.1a(b), or the families of curves joining (or separating) vertical sides of a rectangle $\{|\mathrm{Re}(z)| < \frac{a}{2}, |\mathrm{Im}(z)| < \frac{b}{2}\}$, Figure 1.1 c(d).

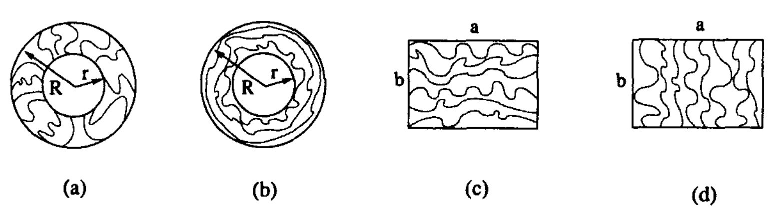

Figure 1.1: Typical curve families

Definition 3. *A metric ρ is* admissible *for a curve family Γ if $\int_C \rho(z)|dz| \geq 1$ for all curves $C \in \Gamma$.*

The modulus of a curve family Γ in a domain B can then be defined.

Definition 4.

$$\operatorname{mod}(\Gamma) = \inf\left\{ \iint_B \rho^2(z)dxdy|\rho \text{ admissible} \right\}$$

In other words, the modulus of the curve family Γ in smallest area given to the domain B by a metric which gives length at least one to every member of Γ. The moduli of the curve families in Figure 1.1 are well known. They are:

$$\text{(a)}\frac{2\pi}{\log\frac{R}{r}}, \qquad \text{(b)}\frac{\log\frac{R}{r}}{2\pi}, \qquad \text{(c)}\ \frac{b}{a}, \qquad \text{(d)}\frac{a}{b}$$

One relation between harmonic measure and the modulus of a curve family is the following, also found in [Pom92]. It also provides a bound for logarithmic capacity. It states that a subset E of the boundary a domain Ω has small harmonic measure if it is hard to reach in the sense that a Brownian particle is unlikely to make its first exit through E. See Figure 1.2.
Let $\phi : \mathbb{D} \to \mathbb{C}$ be conformal, and define

$$d_\phi(z) = \operatorname{dist}(\phi(z), \partial(\phi(\mathbb{D}))), \text{ for } z \in \mathbb{D}. \tag{1.1}$$

Theorem 3. *Let $\phi : \mathbb{D} \to \Omega$ be a conformal map fixing the origin. Let $E \subset \partial\mathbb{D}$ be such that any curve C joining 0 to E has image $\phi(C)$ which must travel a distance d through a region H with $\operatorname{dist}(0, H) \geq d_\phi(0)$. Then*

$$\omega(0, E, \mathbb{D}) \leq \operatorname{cap}(E) \leq \frac{15}{\pi} e^{-\frac{\pi d^2}{\operatorname{area}(H)}}$$

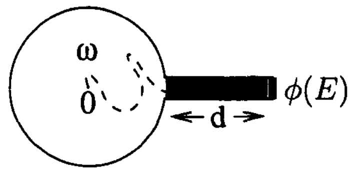

Figure 1.2: The shaded region is H

Pfluger's Theorem, found for instance in [Pom92], relates the modulus of a curve family in a domain Ω to the capacity of a set E in $\partial\Omega$.

Theorem 4 (Pfluger). *Let E be a Borel set on $\partial\mathbb{D}$ and let $\Gamma_E(r)$ be the family of curves in the annulus $\{r < |z| < 1\}$ connecting $\{|z| = r\}$ to E. Then for $0 < r \leq \frac{1}{3}$*

$$\frac{\sqrt{r}}{1+r}\mathrm{cap}(E) \leq \exp\left(-\frac{\pi}{\mathrm{mod}(\Gamma_E(r))}\right) \leq \frac{\sqrt{r}}{1-r}\mathrm{cap}(E)$$

In particular,

$$\mathrm{cap}(E) = \lim_{r\to 0}\frac{1}{\sqrt{r}}\exp\left(-\frac{\pi}{\mathrm{mod}(\Gamma_E(r))}\right).$$

In section 1.4.1 we use the reformulation of Pfluger's Theorem given below.

Corollary 1. *Let E be a Borel set on $\partial\mathbb{D}$ and let $\Gamma_E(R)$ be the family of curves in the annulus $\{R < |z| < 1\}$ connecting $\{|z| = R\}$ to E. Then for $R > \frac{1}{3}$*

$$\gamma(E) \geq \frac{\pi}{\mathrm{mod}(\Gamma_E(R))} - \log\frac{1}{\sqrt{R}}.$$

Proof:

Let $0 < r < R < 1$. Then it is easy to show that

$$\frac{1}{\mathrm{mod}(\Gamma_E(r))} \geq \frac{\log \frac{R}{r}}{2\pi} + \frac{1}{\mathrm{mod}(\Gamma_E(R))}$$

where the first term on the right hand side is the modulus of the family of curves connecting $\{|z| = r\}$ to $\{|z| = R\}$.

The left hand inequality in Pfluger's Theorem is then

$$\begin{aligned}
\mathrm{cap}(E) &\leq \frac{1+r}{\sqrt{r}} e^{-\frac{\pi}{\mathrm{mod}(\Gamma_E(r))}} \\
&\leq \frac{1+r}{\sqrt{r}} e^{-\pi\left(\frac{\log \frac{R}{r}}{2\pi} + \frac{1}{\mathrm{mod}(\Gamma_E(R))}\right)} \\
&\leq \frac{1+r}{\sqrt{R}} e^{-\frac{\pi}{\mathrm{mod}(\Gamma_E(R))}}
\end{aligned}$$

From the relation $\gamma(E) = -\log \mathrm{cap}(E)$ and letting $r \longrightarrow 0$ we get the desired result. □

Lastly, we include a commonly used elementary result for conformal maps of the unit disk. It is a corollary of the Koebe Distortion Theorem [Pom92].

Theorem 5. *Let $d_f(z)$ be defined as in Equation 1.1. Then*

$$\frac{1}{4}(1 - |z|^2)|f'(z)| \leq d_f(z) \leq (1 - |z|^2)|f'(z)|.$$

1.3 Theorem 1 is Sharp

Let Ω_n be the unit disk with n radial slits $\{s_j\}_1^n$ of the form $s_j = re^{2\pi \frac{j-1}{n}}, r \in [\frac{1}{2}, 1]$, removed. Let $\phi_n : \mathbb{D} \to \Omega_n$ be the conformal map fixing the origin and satisfying $\phi_n(1) = \frac{1}{2}$, and define $E_n = \phi_n^{-1}(\partial\Omega_n \cap \partial\mathbb{D})$.

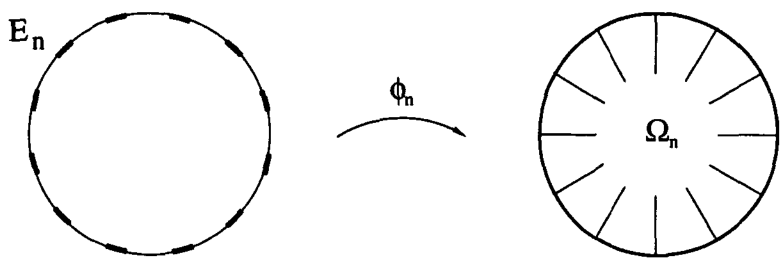

Figure 1.3: Definition of the set E_n

The maps ϕ_n converge uniformly to the map $\phi(z) = \frac{z}{2}$ on $\mathbb{D}$, so $|\phi(z)| = \frac{1}{2}$ for all $z \in \partial\mathbb{D}$. The sets $\{E_n\}$, being the points at distance $\frac{1}{2}$ from their corresponding ϕ-values are contained, respectively, in the set of points where the boundary values of ϕ_n disagree with those of ϕ. If a point $z \in \partial\mathbb{D}$ is in infinitely many E_{n_k}'s, then the limit $\lim_{n\to\infty} \phi_n(z) \neq \phi(z)$. The set of points in infinitely many E_n's is precisely the set $\cap_m \cup_{n>m} E_n$. We show that for *every* subsequence $\{\phi_{n_k}\}$ of $\{\phi_n\}$ there is a set $E \subset \cup_m \cap_{k>m} E_{n_k}$ having logarithmic capacity at least c_0, or equivalently, that there is a set of logarithmic capacity at least c_0 where the boundary values of $\{\phi_{n_k}\}$ fail to converge to those of ϕ.

1.3.1 Construction of the set E

To prove that the set E we construct has positive logarithmic capacity we use the fact that a set has *zero* logarithmic capacity if and only if it has harmonic measure zero in any simply connected domain Ω which contains it in its interior. Moreover, for a fixed simply connected domain Ω, if E is compactly contained in its interior then the harmonic measure $\omega(0, E, \Omega \setminus E)$ is comparable to the reciprocal of the Robin's constant of E. This follows by conformal invariance of harmonic measure from the following theorem [GM05]. We assume without loss of generality that $0 \in \Omega \setminus E$.

Theorem 6. *Let E be contained in the annulus $\{0 < \delta \leq |z| \leq r < 2\}$. Then there are constants $c_1(\delta)$ and $c_2(\delta, r)$ so that*

$$\frac{c_1}{\gamma(E)} \leq \omega(0, E, D(0,2) \setminus E) \leq \frac{c_2}{\gamma(E)}.$$

We show that for any $\epsilon > 0$, we can choose our set E such that

$$\omega(0, E, D(0,2) \setminus E) \geq (1 - \epsilon).$$

Let $c_0 = \exp[-c_2(1,1)]$, where $c_2(1,1)$ is the constant on the right hand side of Theorem 6. Then the relation $\text{cap}(E) = \exp[-\gamma(E)]$ implies that $\text{cap}(E) \geq c_0$.

Our plan is to build a set of the form

$$E = \{E_{1_1} \cup E_{1_2} \cup \ldots \cup E_{1_{N_1}}\} \cap \{E_{2_1} \cup E_{2_2} \cup \ldots \cup E_{2_{N_2}}\} \cap \ldots$$

with $m > k \implies m_1 > k_{N_k}$ to ensure that $E \subset \cap_m \cup_{k>m} E_{n_k}$.

Our proof is based on the following two lemmas.

Lemma 2. *For any $\epsilon > 0$ and each $k \in \mathbb{N}$ we can choose a finite subcollection of E_{n_j}'s so that $F_k = \{E_{k_1} \cup \ldots \cup E_{k_K}\}$ satisfies the following two criteria:*

- $\omega(0, F_k, D(0,2) \setminus F_k) \geq (1 - \epsilon 2^{-k})$

- $m > k \to m_1 > k_K$

Lemma 3. *Let A be a finite collection of intervals in $\{|z| = 1\}$. Then for any $\eta > 0$ there is k sufficiently large so that,*

$$\omega(0, A \cap F_k, D(0,2) \setminus A \cap F_k) \geq (1 - \eta)\omega(0, A, D(0,2) \setminus A)$$

Lemma 2 states that we can take a finite collection of E_{n_k}'s with arbitrarily large index so that their union has almost the same harmonic measure as $\{|z| = 1\}$ in $D(0,2)$, namely 1. Lemma 3 states that a similar result holds if we restrict our sets F_k to a finite collection of intervals of $\{|z| = 1\}$. Together these lemmas allow us to inductively select a sequence of sets F_{k_m} so that $E = \cap_m F_{k_m}$ satisfies

$$\omega(0, E, D(0,2) \setminus E) \geq 1 - \epsilon.$$

First, we choose F_{k_1} so that

$$\omega(0, F_{k_1}, D(0,2) \setminus F_{k_1}) \geq (1 - \frac{\epsilon}{2}).$$

Since F_{k_1} is a finite collection of intervals, we then choose F_{k_2} so that

$$\omega(0, F_{k_1} \cap F_{k_2}, D(0,2) \setminus F_{k_1} \cap F_{k_2}) \geq (1 - \epsilon 2^{-2})\omega(0, F_{k_1}, D(0,2) \setminus F_{k_1}).$$

We continue inductively, choosing F_{k_m} so that

$$\omega(0, \cap_{j=1}^{m} F_{k_j}, D(0,2) \setminus \cap_{j=1}^{m} F_{k_j}) \geq (1 - \epsilon 2^{-m}) \omega(0, \cap_{j=1}^{m-1} F_{k_j}, D(0,2) \setminus \cap_{j=1}^{m-1} F_{k_j}).$$

Then $E = \cap_m F_{k_m}$ by construction satisfies

$$\omega(0, E, D(0,2) \setminus E) \geq \prod_{j=1}^{\infty} (1 - \epsilon 2^{-j}) \geq 1 - \epsilon,$$

where the last inequality comes from the fact that

$$\prod_{j=1}^{N} (1 - \epsilon 2^{-j}) \geq 1 - \left(\sum_{j=1}^{N} 2^{-j} \right) \epsilon.$$

1.3.2 Proofs of Lemmas 2 and 3

Proof of Lemma 2

Proof(of Lemma 2): Choose $N \in \mathbb{N}$, and assume that the index on any set E_n is at least N. Let $u_n(z) = \omega(z, E_n, D(0,2) \setminus E_n)$ be the harmonic measure function in $D(0,2) \setminus E_n$. Then we can find a lower bound on the harmonic measure of a union of E_n's, such as a $F_k = \cup_{k_1}^{k_K} E_{k_j}$, by summing the harmonic measure functions of the component sets and normalizing the boundary values. That is,

$$\omega(0, F_k, D(0,2) \setminus F_k) \geq \frac{\sum_j u_{k_j}(0)}{\sup_{z \in F_k} \{\sum_j u_{k_j}(z)\}} \tag{1.2}$$

To have any hope of making Equation 1.2 close to 1 we must show that the values of $u_n(0)$ are bounded below. This follows from Theorem 6 and the lemma below.

Lemma 4. *There is a universal bound on $\gamma(E_n)$, $0 \leq \gamma(E_n) \leq \gamma_0 = \log(2\sqrt{2})$.*

Proof: We use Pfluger's theorem to put a lower bound on the logarithmic capacity of E_n. Fix n. For a curve family Γ in $\partial\mathbb{D}$, let $\tilde{\Gamma}$ denote the image under ϕ_n of Γ. First observe from Theorem 5 that $\frac{1}{2} \leq |\phi_n'(0)| \leq 2$. This implies that for any $r > 0$ the image of the curve $|z| = r$ lies in the annulus $\{\frac{r}{8} \leq |z| \leq 2r\}$, so that $\mathrm{mod}(\Gamma_r(E_n)) \geq \frac{2\pi}{\log\frac{8}{r}}$. From Theorem 4 we see that $\mathrm{cap}(E_n) \geq \frac{1}{2\sqrt{2}}$. From the relation $\gamma(E) = -\log(\mathrm{cap}(E))$ we have that $\gamma(E_n) \leq \log(2\sqrt{2})$ for all n, and thereby $u_n(0) \geq \omega_0 = \frac{c_1(1)}{\log 2\sqrt{2}}$

□

We show next that if $z \in F_k$ is a point of E_{k_j} then the value of $u_{k_m}(z)$, $m \neq j$ is not too much larger than $u_{k_j}(0)$. We would like to say that for any $R > 1$ if we choose our indices $\{k_1, k_2, \dots, k_k\}$ carefully we can ensure that

$$\sup_{z \in E_{k_m}} u_{k_j}(z) \leq R u_{k_j}(0), j \neq m.$$

This is not quite possible, but as we show in Lemma 5 below, the linear measure of the subset of $\{|z| = 1\}$ where a given u_n assumes values greater than $Ru_n(0)$ is approaching zero as $n \to \infty$.

Lemma 5. *Let the sets $\{E_n\}$ and the functions $u_n(z)$ be as defined above. Let $R > 1$ and define the set $E_n^R = \{z : |z| = 1, u_n(z) > Ru_n(0)\}$. Then for any $\eta > 0$ there is N sufficiently large so that if $n > N$ then the linear measure of*

E_n^R *is smaller than* η.

It should be believable that Lemma 5 holds, considering that the mean value property of harmonic functions requires that $u_n(0) = \int u_n(e^{2\pi i\theta})d\theta$. There are a couple of details to work out, so we save the proof of Lemma 5 for last. It is in Section 1.3.3.

Suppose for now that Lemma 5 is true. Set $R = 1 + 2^{-k-1}$ and choose E_{k_1} so that $|E_{k_1}^R| \le 2^{-k-2}$. For each $n > k_1$ define $\tilde{E}_n = E_n \setminus E_{k_1}^R$ and let $\tilde{u}_n(z)$ be the harmonic measure function for the set $\tilde{E}_n$. By adding a set of the form $\tilde{E}_n$ next, as opposed to the entirety of one of the E_n's we ensure that the contribution of $\tilde{u}_n(z)$ to $\sup\{\sum_j u_{k_j}(z), z \in F_k\}$ is smaller than $Ru_n(0)$. However, by throwing out part of E_n, we decrease the value of the harmonic measure function at the origin. It is not difficult to see, though, that for n large $\tilde{u}_n(0) \ge (1 - \frac{|E_{n_1}^R|}{2\pi})u_n(0)$. More generally, we can say:

Lemma 6. *Let* $A \subset \{|z| = 1\}$. *Then for* E_n *as defined above and* n *sufficiently large,* $\omega(0, E_n \setminus A, D(0,2) \setminus E_n) > (1 - \omega(0, A, \mathbb{D}))\omega(0, E_n, D(0,2) \setminus E_n)$.

Proof(of Lemma 6):

Clearly

$$\omega(0, E_n, D(0,2) \setminus E_n) = \omega(0, E_n \setminus A, D(0,2) \setminus E_n) + \omega(0, E_n \cap A, D(0,2) \setminus E_n)$$

and so we need only to show that

$$\omega(0, E_n \cap A, D(0,2) \setminus E_n) < \omega(0, A, \mathbb{D})\omega(0, E_n, D(0,2) \setminus E_n).$$

First note that $\omega(0, E_n \cap A, D(0,2) \setminus E_n) < \omega(0, E_n \cap A, \mathbb{C} \setminus E_n)$, and recall that

$\omega(0, E_n \cap A, \mathbb{C} \setminus E_n)$ is the mass given to $E_n \cap A$ by the equilibrium distribution for E_n. Since E_n is comprised of n evenly distributed intervals it will give equal mass $\frac{1}{n}$ to each of them. Therefore for large n, $E_n \cap A$ will contain $n \cdot \omega(0, A, \mathbb{D})$ intervals of E_n. In other words, for large n, $\omega(0, E_n \cap A, \mathbb{C} \setminus E_n) = \omega(0, A, \mathbb{D})$.

□

We choose k_2 sufficiently large so that $\omega(0, E_{k_2} \cap E_{k_1}^R, D(0,2) \setminus E_{k_2}) = \omega(0, E_{k_1}^R, \mathbb{D}) \leq 2^{-k-2}$, and so that the linear measure of $E_{k_2}^R$ is smaller than 2^{-k-3}. We continue this process inductively, choosing k_j large enough that $\tilde{E}_{k_j} = E_{k_j} \setminus \{\cup_{l=1}^{j-1} E_{k_l}^R\}$ has $\tilde{u}_{k_j}(0) \geq (1 - \sum_{m=1}^{j} 2^{-k-m-1}) u_{k_j}(0)$, and so that $|E_{k_j}^R| \leq 2^{-k-j-1}$.

For each j, then, we have the following estimates:

$$\begin{aligned} \tilde{u}_{k_j}(0) &\geq (1 - 2^{-k-1}) u_{k_j}(0), \\ \tilde{u}_{k_j}(z) &\leq u_{k_j}(z) \leq R u_{k_j}, \text{ for } z \in F_k \setminus \tilde{E}_{k_j} \end{aligned}$$

With these estimates we can write Equation 1.2 as

$$\begin{aligned} \omega(z, F_k, D(0,2) \setminus F_k) &\geq \frac{(1 - 2^{-k-1}) \sum_{j=1}^{N_k} u_{k_j}(0)}{1 + R \sum_{j=1}^{N_k} u_{k_j}(0)} \\ &\geq \frac{(1 - 2^{-k-1}) \sum_{j=1}^{N_k} u_{k_j}(0)}{1 + (1 + 2^{-k-1}) \sum_{j=1}^{N_k} u_{k_j}(0)} \end{aligned}$$

Since each $u_n(0)$ is bounded below by ω_0, the sum $\sum_{j=1}^{N_k} u_{k_j}(0) \geq N_k \omega_0$. By choosing N_k sufficiently large, the sum in Equation 1.2 can be made arbitrarily close to $(1 - 2^{-k})$, proving Lemma 2.

$\square$

Proof of Lemma 3

Proof:

The probability that a Brownian particle starting at the origin will hit $A \cap F_k$ before hitting $\partial D(0,2)$ is bounded below by the product of the chance it will hit A before hitting $\partial D(0,2)$, and the chance that *from* A it will hit $A \cap F_k$ before hitting $\partial D(0,2)$ as in Figure 1.4. That is,

$$\omega(0, A \cap F_k, D(0,2) \setminus A \cap F_k) \geq \omega(0, A, D(0,2) \setminus A) \cdot \inf_{a \in A} \omega(a, A \cap F_k, D(0,2) \setminus A \cap F_k)$$

Ideally, it would be true that,

$$\inf_{a \in A} \omega(a, A \cap F_k, D(0,2) \setminus A \cap F_k) \geq (1 - \eta),$$

but this cannot be assumed. We show instead that there is a subset $\hat{A} \subset A$ of very small harmonic measure off of which $\inf_{a \in A} \omega(a, A \cap F_k, D(0,2) \setminus A \cap F_k)$ is as large as we wish. This is the content of the next lemma.

Lemma 7. *For any interval A on $\{|z| = 1\}$ and any $\eta > 0$ there is a subset $\hat{A} \subset A$ with*

$$\omega(0, \hat{A}, D(0,2) \setminus A) \geq (1 - \frac{\eta}{2}) \omega(0, A, D(0,2) \setminus A)$$

such that for sufficiently large k and any $a \in \hat{A}$

$$\omega(a, A \cap F_k, D(0,2) \setminus A \cap F_k) \geq (1 - \frac{\eta}{2})$$

It will follow from Lemma 7 that

$$\begin{aligned} \omega(0, A \cap F_k, D(0,2) \setminus A \cap F_k) &\geq \omega(0, \hat{A}, D(0,2) \setminus A) \cdot (1 - \frac{\eta}{2}) \\ &\geq (1 - \frac{\eta}{2})\omega(0, A, D(0,2) \setminus A)(1 - \frac{\eta}{2}) \\ &\geq (1 - \eta)\omega(0, A, D(0,2) \setminus A) \end{aligned}$$

Proof: The set $\hat{A}$ is obtained from A by removing two sets. First, for each interval A_j of A, let $\hat{A}_j$ be the closed subinterval of A_j obtained by removing an open neighborhood of each endpoint so small that $\omega(0, \hat{A}_j, D(0,2) \setminus A_j) \geq (1 - \frac{\eta}{4})\omega(0, A_j, D(0,2) \setminus A_j)$, so that the set $\cup \hat{A}_j$ satisfies

$$\omega(0, \cup \hat{A}_j, D(0,2) \setminus A) \geq (1 - \frac{\eta}{4})\omega(0, A, D(0,2) \setminus A)$$

Second, define the set

$$X_{k,\eta,\delta} = \{x : |x| = 1, \omega(x, \partial D(x,\delta), D(x,\delta) \setminus F_k) > \frac{\eta}{2}\}.$$

If we could ensure that for large enough k the set $X_{k,\eta,\delta}$ satisfied

$$\omega(0, A \setminus X_{k,\eta,\delta}, D(0,2) \setminus A) \geq (1 - \frac{\eta}{4})\omega(0, A, D(0,2) \setminus A), \qquad (1.3)$$

Then $\hat{A} = \cup \hat{A}_j \setminus X_{k,\eta,\delta}$ has

$$\omega(0, \hat{A}, D(0,2) \setminus A) \geq (1 - \frac{\eta}{2})\omega(0, A, D(0,2) \setminus A)$$

but by definition of $X_{k,\eta,\delta}$, this will show that

$$\omega(0, A \cap F_k, D(0,2) \setminus A \cap F_k) \geq \omega(0, \hat{A}, D(0,2) \setminus A) \cdot (1 - \frac{\eta}{2}),$$

since each $a \in \hat{A}$ is in the complement of $X_{k,\eta,\delta}$. Combining the two preceding inequalities produces

$$\omega(0, A \cap F_k, D(0,2) \setminus A \cap F_k) \geq (1 - \eta)\omega(0, A, D(0,2) \setminus A),$$

proving the lemma.

It remains to be shown that the sets X can be chosen as prescribed. This is proven in Lemma 8 below.

Lemma 8. *Define $X_{k,\eta,\delta}$ as above. Then for any $\eta, \delta, \epsilon > 0$ and there is k sufficiently large so that $|X_{k,\eta,\delta}| < \epsilon$.*

We want to show that for our set A, we can choose $X_{k,\eta,\delta}$ so that Equation 1.3 is satisfied. Before presenting the proof of Lemma 8 we point out why it is sufficient to show that the linear measure of the set $X_{k,\eta,\delta}$ can be made arbitrarily small.

The harmonic measure of $A' \subset A$ can be expressed as the integral with respect to arclength over A' of the continuous function $f_A = \frac{d\omega}{d\theta}$:

$$\omega(0, A', D(0,2) \setminus A) = \int_{A'} \frac{d\omega}{d\theta} d\theta$$

On the compact subset of $\cup \hat{A}_j$ of A, the function f_A is absolutely continuous, and so there is an $l_\epsilon > 0$ such that if $|X_{k,\eta,\delta}| \leq l_\epsilon$ then

$$\begin{aligned} \omega(0, X_{k,\eta,\delta} \cap A, D(0,2) \setminus A) &= \int_{X_{k,\eta,\delta}} \frac{d\omega}{d\theta} d\theta \\ &\leq \frac{\eta}{2} \omega(0, A, D(0,2) \setminus A) \end{aligned}$$

□

Proof(of Lemma 8):

We know that $\omega(0, \partial D(0,2), D(0,2) \setminus F_k) \leq 2^{-k}$. Alternately, we can say that the harmonic measure is bounded below by the product of the probabilities that a Brownian particle starting at the origin will first hit $X_{k,m,\delta}$, then move a distance δ away without hitting F_n, and finally hit $\partial D(0,2)$ upon first exiting the annulus $\mathbb{A}(1,2) = \{1 < |z| < 2\}$.

That is,

$$\begin{aligned} 2^{-k} &\geq \omega(0, \partial D(0,2), D(0,2) \setminus F_n) \\ &\geq \omega(0, X_{k,\eta,\delta}, \mathbb{D}) \cdot \frac{\eta}{8} \cdot \left(\frac{3/4}{\log 2} \delta \right), \end{aligned} \tag{1.4}$$

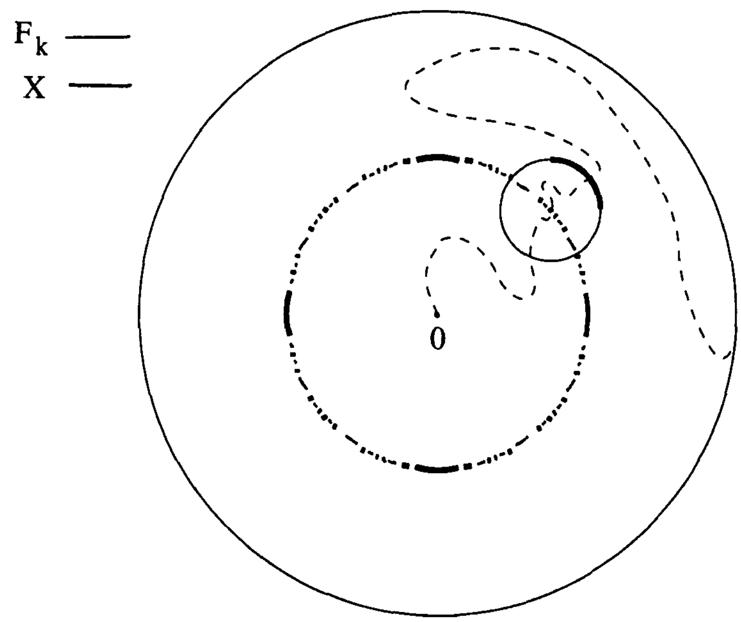

Figure 1.4: The set $X_{k,\eta,\delta}$

where the last element in the product is computed from the formula

$$\omega(z, |z| = 2, \mathbb{A}(1,2)) = \frac{\log|z|}{\log 2}.$$

If $|z|$ is very close to 1, then $\log|z|$ is approximately $|z| - 1$. In our case, if z is on the quarter of $\partial D(x,\delta)$ farthest from the origin, then $1 + \frac{3}{4}\delta < |z| < 1 + \delta$. So Equation 1.4 implies that

$$\omega(0, X_{k,\eta,\delta}, \mathbb{D}) \leq \frac{32\log 2}{3\eta\delta} 2^{-k}$$

We can ensure that this is smaller than ϵ by choosing k large.

□

1.3.3 Proof of Lemma 5

We now prove Lemma 5:

Proof(of Lemma 5):

We use the mean value property of harmonic functions to write $u_n(0)$ as

$$u_n(0) = \int u_n(e^{i\theta})d\theta. \tag{1.5}$$

From the symmetry of the sets E_n we see that the minimum value of $u_n(e^{i\theta})$ occurs at the midpoints of the n intervals comprising the complement of E_n. Let $\tilde{z}$ denote one such midpoint.

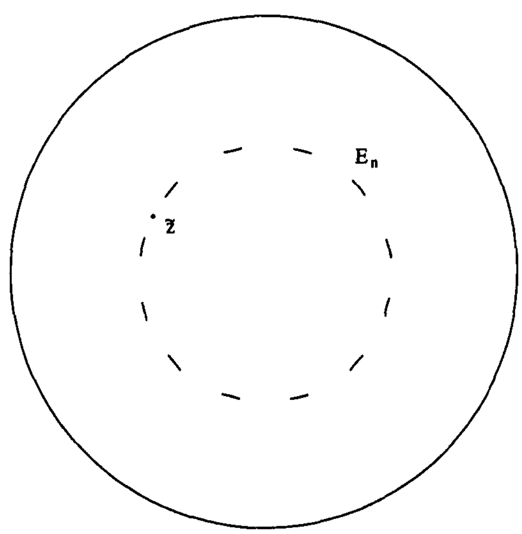

Figure 1.5: Minimum of $u_n|_{\partial\mathbb{D}}$ occurs at points like $\tilde{z}$

We show that as n becomes large, $u_n(\tilde{z}) \to u_n(0)$.

Lemma 9. *Let $u_n(z)$ and $\tilde{z}$ be defined as above. Then for any $\epsilon > 0$ there is N sufficiently large so that $u_n(\tilde{z}) \geq (1-\epsilon)u_n(0)$ for $n > N$.*

Demonstrating that the values of $u_n(e^{i\theta})$ cannot stay bounded away from

$u_n(0)$ as $n \to \infty$ proves Lemma 5, since Equation 1.5 can then be expressed as

$$\begin{aligned} u_n(0) &= \int_{E_n^R} u_n(e^{i\theta})d\theta + \int_{\partial\mathbb{D}\setminus E_n^R} u_n(e^{i\theta})d\theta \\ &\geq Ru_n(0)\cdot\frac{|E_n^R|}{2\pi} + (1-\epsilon)u_n(0)\cdot\left(1-\frac{|E_n^R|}{2\pi}\right) \end{aligned} \tag{1.6}$$

which shows that

$$|E_n^R| \leq \frac{2\pi\epsilon}{R-1+\epsilon}$$

so fixing $R > 1$ and choosing ϵ to be small forces $|E_n^R|$ to be small. We now prove Lemma 9.

Proof(of Lemma 9):

Suppose that for $c < 1$ there is n arbitrarily large so that $u_n(\tilde{z}) \leq c \cdot u_n(0)$. Then Harnack's inequality [Rud66] provides an estimate of the size of the neighborhood of $\tilde{z}$ where the harmonic function $u_n(z)$ stays bounded below $\frac{1+c}{2}u_n(0)$. Under our hypotheses it would follow that $u_n|_{\{|z|=1\}} \leq \frac{1+c}{2}u_n(0)$ for $\{|z-\tilde{z}| \leq \frac{\pi}{n}\frac{1-c}{1+3c}\}$. Let V_n be the collection of intervals on $|z| = 1$ where this holds. The linear measure of the sets V_n is always greater than $\frac{1-c}{1+3c}$. The linear measure of the sets E_n decays exponentially:

Lemma 10. *Let $\Omega_n = \phi_n(\mathbb{D})$ be the disk with n radial slits of length r_0 removed. Then the harmonic measure of one of the intervals I comprising $E = \phi_n^{-1}(\partial\Omega_n \cap \partial\mathbb{D})$ satisfies*

$$\omega(0, I, \mathbb{D}) \leq \frac{15}{\pi}e^{-\frac{r_0^2}{(1-r_0^2)}n}$$

Proof: This is a direct application of Theorem 3 with $\delta = r_0$ and area$(H) = \frac{\pi}{n}(1 - r_0^2)$

$\square$

It seems reasonable to believe then that $\omega(0, E_n, D(0,2) \setminus \{E_n \cup V_n\}) \to 0$ with increasing n. This is indeed the case, and it is sufficient to prove Lemma 9. This is because if

$$\omega(0, E_n, D(0,2) \setminus E_n \cup V_n) \leq \epsilon\omega(0, E_n \cup V_n, D(0,2) \setminus E_n \cup V_n),$$

then

$$\begin{aligned} u_n(0) &\leq \omega(0, V_n, D(0,2) \setminus E_n \cup V_n)\frac{1+c}{2c}u_n(0) + \\ &\quad \epsilon \cdot \omega(0, E_n, D(0,2) \setminus E_n \cup V_n) \cdot 1 \\ &\leq 1 \cdot \frac{1+c}{2c}u_n(0) + \epsilon \cdot 1 \end{aligned}$$

so that $\frac{1+c}{2c} \geq 1 - \epsilon u_n(0)$. Since $u_n(0)$ is uniformly bounded below, we see that $c \to 1$ as $n \to \infty$.

We show first that the capacity of the sets V_n is approaching 1. Consider a construction of the sets V_n analogous to that of the sets E_n as the preimage of $\partial\mathbb{D}$ under the Riemann map from $\mathbb{D}$ onto a radially slit disk $\mathbb{D} \setminus \cup_{j=1}^{n} s_j$, where $s_j = \{re^{\frac{2\pi j}{n}}, r \in [r_n, 1]\}$. If the r_n's remain bounded away from 1, as in the construction of the sets E_n, then it would follow that the intervals comprising V_n would have lengths decaying exponentially.

We know this to be false, but on the other hand if $r_n \to 0$, then the sequence of slit-disk maps must be converging to the identity map. Now apply Pfluger's theorem for small r as in the proof of Lemma 4. In this case, if we are far enough out in the sequence the image of the circle $\{|z| = r\}$ is within the annulus $\{r(1-\epsilon) < |z| < r(1+\epsilon)\}$, so that

$$\operatorname{mod}(\Gamma_r(V_n)) \geq \frac{2\pi}{\log \frac{1}{r(1-\epsilon)}}.$$

Then

$$\begin{aligned} \operatorname{cap}(V_n) &\geq \lim_{r \to 0} \frac{1}{\sqrt{r}} \exp\left[-\frac{\log \frac{1}{r(1-\epsilon)}}{2}\right] \\ &= \sqrt{1-\epsilon}. \end{aligned}$$

This means that $\gamma(V_n)$, and thereby $\gamma(V_n \cup E_n)$, becomes arbitrarily small as $n \to \infty$.

Let μ_n be the equilibrium distribution for the set $V_n \cup E_n$. Then by definition $\mu_n(E_n) = \omega(0, E_n, \mathbb{C} \setminus \{V_n \cup E_n\})$. By adding the boundary component $\{|z| = 2\}$ we decrease the harmonic measure of E_n from the origin in $\{\mathbb{C} \setminus \{V_n \cup E_n\}\}$, so that $\omega(0, E_n, D(0,2) \setminus \{E_n \cup V_n\}) \leq \mu_n(E_n)$. We show that in fact $\mu_n(E_n) \to 0$.

Suppose that $\mu_n(E_n) > \epsilon$ for all n and consider the energy integral $I(\mu_n)$.

$$
\begin{aligned}
\gamma(V_n \cup E_n) &= \int\int \log \frac{1}{|x-y|} d\mu_n(x) d\mu_n(y) \\
&= \int_{V_n}\int_{V_n} \log \frac{1}{|x-y|} d\mu_n(x) d\mu_n(y) + \\
&\quad \int_{E_n}\int_{E_n} \log \frac{1}{|x-y|} d\mu_n(x) d\mu_n(y) + \\
&\quad 2\int_{V_n}\int_{E_n} \log \frac{1}{|x-y|} d\mu_n(x) d\mu_n(y) \qquad (1.7)
\end{aligned}
$$

Since $\frac{\mu_n(z)}{\mu_n(E_n)}$ and $\frac{\mu_n(z)}{\mu_n(V_n)}$ are probability distributions on E_n and V_n, respectively, we have that $\int_{E_n}\int_{E_n} \log \frac{1}{|x-y|} d\mu_n(x) d\mu_n(y) \geq (\mu_n(E_n))^2\gamma(E_n)$ and $\int_{V_n}\int_{V_n} \log \frac{1}{|x-y|} d\mu_n(x) d\mu_n(y) \geq (\mu_n(V_n))^2\gamma(V_n)$. Using our assumption that μ_n gives mass at least ϵ to E_n, we can write Equation 1.7 as

$$
\gamma(V_n \cup E_n) \geq (1-\epsilon)^2\gamma(V_n) + (\epsilon)^2\gamma(E_n) + 2\int_{E_n}\int_{V_n} \log \frac{1}{|x-y|} d\mu_n(x) d\mu_n(y) \quad (1.8)
$$

We know that the left hand side as well as the first term of the right-hand side of the preceding equation are approaching zero as n becomes large. As demonstrated at the beginning of the section, $\gamma(E_n) \geq \gamma_0 > 0$, so unless the last term on the right-hand side cancels out the contribution of $\epsilon^2\gamma(E_n)$, it will be necessary that $\epsilon \to 0$ as n becomes large. In fact the last term of the right-hand side becomes arbitrarily small as $n \to \infty$. We conclude this by observing that

$$\int_{E_n}\int_{V_n} \log \frac{1}{|x-y|} d\mu_n(x) d\mu_n(y) = \int_{E_n}\int_{V_n} \log \frac{1}{2\sin\frac{\theta}{2}} d\mu_n(x) d\mu_n(y)$$

where θ is as shown in Figure 1.6 below.

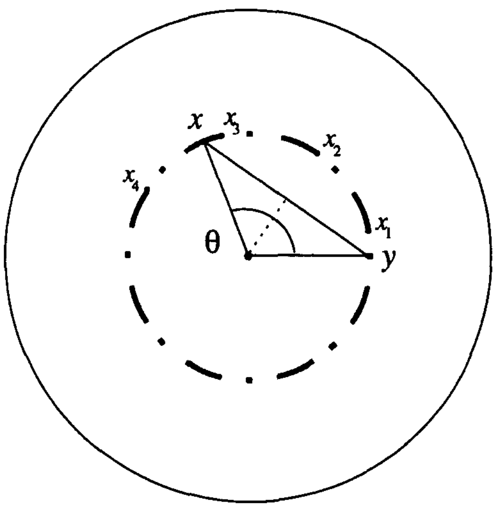

Figure 1.6: Definition of θ

Fix $y \in E_n$ and a small $\theta_0 > 0$. Then the integrals $\int_{V_n} \log \frac{1}{2\sin\frac{\theta}{2}} d\mu_n(x)$ are bounded below, respectively, by a sequence of Riemann sums approximating the integral $\int_{\theta_0}^{\pi} \log \frac{1}{2\sin\frac{\theta}{2}} d\theta$.

$$\int_{V_n} \log \frac{1}{2\sin\frac{\theta}{2}} d\mu_n(x) \leq 2 \sum_{j=1}^{\lceil n/2 \rceil} \frac{1}{n} \log \frac{1}{2\sin\frac{\theta_j}{2}}$$

where θ_j is the angle corresponding to y_j as in Figure 1.6.

Since $\int_{\theta_0}^{\pi} \log \frac{1}{2\sin\frac{\theta}{2}} d\theta \to 0$ as $\theta_0 \to 0$, the last term in Equation 1.8 cannot cancel out the positive contribution of $\epsilon^2\gamma(E_n)$ to the energy integral of $E_n \cup V_n$. The mass $\mu_n(E_n)$ therefore cannot have a positive lower bound.

So for any $\epsilon > 0$ we can choose n large enough that $\omega(0, E_n, D(0,2) \setminus \{E_n \cup V_n\}) < \epsilon$. Then as shown in Equation 1.6 the constant c in Lemma 9 cannot remain bounded below 1. □

1.4 Proof of Theorem 1

Theorem 1 follows from Lemma 11 below by Lemma 1.

Lemma 11. *Let K and $\{\phi_n\}$ be as in Theorem 1, and fix $\delta > 0$. Then for any ϵ there is an n such that the set $E_n^\delta = \{x \in \partial\mathbb{D} : |\tilde{\phi}_n(x) - \tilde{\phi}(x)| > \delta\}$ has $\mathrm{cap}_K E_n^\delta < \epsilon$.*

Suppose Lemma 11 to be true. Fix $\epsilon > 0$. Choose a sequence $\{g_{\epsilon,j}\}$ positive real numbers as in Lemma 1. Choose a subsequence $\{\phi_{1_1}, \phi_{1_2}, \ldots\}$ of $\{\phi_n\}$ so that $\gamma_K(E_{1_j}^{\frac{1}{2}}) \geq g_{\epsilon,j}$. Choose another sequence $\{g_{\frac{\epsilon}{2},j}\}$, and extract from $\{\phi_{1,j}\}$ subsequence $\{\phi_{2_1}, \phi_{2_2}, \ldots\}$ of $\{\phi_{1_1}, \phi_{1_2}, \ldots\}$ so that $\gamma_K(E_{2_j}^{\frac{1}{4}}) \geq g_{\frac{\epsilon}{2},j}$. Continue the process inductively to create a sequence of nested subsequences having the property that $\gamma_K(E_{i_j}^{2^{-i}}) \geq g_{\frac{\epsilon}{2^i},j}$. Then the diagonal subsequence $\{\phi_{j_j}\}$ has boundary values $\{\tilde{\phi}_{j_j}\}$ which cannot fail to converge on a set of positive K-capacity.

We assume that the E_n^δ's for our sequence have logarithmic capacity uniformly bounded below since Lemma 11 would be trivial otherwise. We claim that the following lemma is sufficient to prove Lemma 11.

Lemma 12. *For all $k \in \mathbb{N}$ there is n_k sufficiently large so that $E_{n_k}^\delta$ can be written as*

$$\cup_{j=1}^k E_{n_k,j}^\delta, \text{ where } E_{n_k,j}^\delta = E_{n_k}^\delta \cap [e^{i\frac{2\pi(j-1)}{k}}, e^{i\frac{2\pi j}{k}}]$$

and such that

$$\gamma(E_{n_k,j}^\delta) \geq \frac{1}{\rho_j} c(\delta), \text{ where } \sum_{j=1}^k \rho_j \leq 1.$$

Define $m(k) = \inf_{x,y \in [e^{i\frac{2\pi(j-1)}{k}}, e^{i\frac{2\pi j}{k}}]} \left| \frac{K|x-y|}{\log|x-y|} \right|$. Suppose Lemma 12 is true. Let ν be the minimizing probability distribution for $\gamma_K(E^\delta_{n_k})$. Define $\sigma_j = \nu(E^\delta_{n_k,j})$, so that $\sum_{j=1}^{n} \sigma_j = 1$. Then $\frac{1}{\sigma_j} \cdot \nu|_{E^\delta_{n_k,j}} = \nu_j$, then is a probability distribution on $E^\delta_{n_k,j}$. Let μ_j be the equilibrium distribution for $E^\delta_{n_k,j}$. Then the energy integral for $E^\delta_{n_k}$ can be written as

$$
\begin{aligned}
\gamma_K(E^\delta_{n_k}) &= \iint K(|x-y|)d\nu(x)d\nu(y) \\
&= \sum_{j=1}^{k} \iint K(|x-y|)d\nu(x)d\nu(y) + \sum_{j \neq l} \iint K(|x-y|)d\nu(x)d\nu(y) \\
&\geq \sum_{j=1}^{k} \sigma_j^2 \iint K(|x-y|)d\nu_j(x)d\nu_j(y) \\
&\geq m(k) \sum_{j=1}^{k} \sigma_j^2 \iint \log \frac{1}{|x-y|} d\nu_j(x)d\nu_j(y) \\
&\geq m(k) \sum_{j=1}^{k} \sigma_j^2 \iint \log \frac{1}{|x-y|} d\mu_j(x)d\mu_j(y) \\
&\geq m(k) \sum_{j=1}^{k} \sigma_j^2 \gamma(E^\delta_{n_k,j}) \\
&\geq m(k) \sum_{j=1}^{k} \sigma_j^2 \frac{1}{\rho_j} c(\delta) \\
&\geq m(k)c(\delta)
\end{aligned}
$$

The last inequality uses the fact that if $\sum_{j=1}^{k} \sigma_j = 1$ and $\sum_{j=1}^{k} \rho_j \leq 1$ then $\sum_{j=1}^{k} \frac{\sigma_j^2}{\rho_j} \geq 1$. This can be proven by induction. By choosing k large enough, we can ensure that $m(k)c(\delta) \geq \log \frac{1}{\epsilon}$, so that $\mathrm{cap}_K(E^\delta_{n_k}) \leq \epsilon$.

So proving Lemma 12 is the main issue in the proof of Lemma 11. To get a sense of how the proof will work in the general case, we prove the lemma for the particular case of the sequence of slit-disk maps in Section 1.3.

1.4.1 Proof of Lemma 12 for Slit-Disk Maps

The geometry of the slit-disk domains in Section 1.3 makes the decomposition described in Lemma 12 very natural.

From the symmetry of these sets it must be true that each of the n component intervals $E^{\delta}_{n,j}$ of E^{δ}_{n} has equal energy $\gamma(E^{\delta}_{n,j})$, so to prove Lemma 12 in this particular example we show that we can set $\rho_j = \frac{1}{n}$ for all j, or in other words, that

$$\gamma(E^{\delta}_{n,j}) \geq n \cdot c(\delta).$$

We will use the reformulation of Pfluger's theorem, Corollary 1. Pick R very close to 1 so that $|\phi(x) - \phi(Rx)| \leq \frac{\delta}{4}$ for all $x \in \partial\mathbb{D}$. Let N be such that for $n \geq N$, $|\phi(Rx) - \phi_n(Rx)| \leq \frac{\delta}{4}$.

Note first that for such an n, the set E^{δ}_{n} is contained in the set

$$\tilde{E}^{\delta}_{n} = \{x \in \partial\mathbb{D} \text{ such that } |\phi_n(x) - \phi_n(Rx)| \geq \frac{\delta}{2}\}$$

We will actually show that the K-capacity of $\tilde{E}^{\delta}_{n}$ is smaller than ϵ by proving Lemma 12 for $\tilde{E}^{\delta}_{n}$.

We place an upper bound on the modulus of $\Gamma_{\tilde{E}^{\delta}_{n,j}}(R)$ as follows.

Let $\Gamma'(R)$ be the image of $\Gamma_{\tilde{E}^{\delta}_{n,j}}(R)$ in the domain Ω_n Consider the metric $\rho(z) = \frac{2}{\delta}$ for $z \in \{\frac{2\pi}{n}(j-1) < \arg(z) < \frac{2\pi}{n}j, \frac{1}{2} + \frac{\delta}{2} < |z| < \frac{1}{2} + \delta\}$, and $\rho(z) = 0$

otherwise. This metric is admissible for the family Γ' connecting the curve $\phi_n(|z|=R)$ to $\phi_n(\tilde{E}^{\delta}_{n,j})$ as shown in Figure 1.7 below.

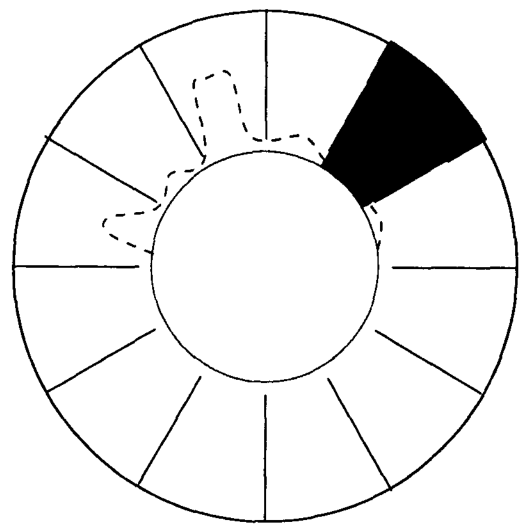

Figure 1.7: The metric $\rho(z)$ is supported in the shaded region.

This gives the bound

$$\operatorname{mod}(\Gamma'(R)) \leq \int\int \rho^2 dxdy \leq \left(\frac{2}{\delta}\right)^2 \operatorname{area}(\operatorname{supp}(\rho)) \leq \left(\frac{2}{\delta}\right)^2 \frac{\pi}{n}(\frac{1}{2}\delta + \frac{3}{4}\delta^2).$$

So that

$$\gamma(\tilde{E}^{\delta}_{n,j}) \geq \frac{n}{4}\left(\frac{1}{2\delta} + \frac{3}{4}\right)^{-1} - \log\frac{1}{\sqrt{R}},$$

which for n sufficiently large proves Lemma 12 with $c(\delta) = \frac{\delta^2}{2\delta+3\delta^2}$

1.4.2 Proof of Lemma 12

A key observation from the case of the slit-disk maps is that what enabled us to prove that

$$\gamma(E^{\delta}_{n_k,j}) \geq \frac{1}{\rho_j}c(\delta), \text{ where } \sum_{j=1}^{k}\rho_j \leq 1.$$

was the fact that the moduli of the curve families $\Gamma_{\tilde{E}^{\delta}_{n_k,j}}(R)$ could be computed using a constant metric $\rho = \frac{2}{\delta}$ in a set of n disjoint subsets of a finite-area region. The n regions were naturally defined by the geometry of the slit disks. In the general case we cannot assume that n points $\{z_j = e^{i\frac{2\pi j}{n}}\}_{j=0}^{n-1}$ will divide the region between $\phi_n(\{|z| = R\})$ and $\phi_n(\{|z| = 1\})$ into the appropriate subregions, but as we show in the next lemma, we can take arbitrarily small neighborhoods $\{U_j\}$ of the points $\{z_1, \ldots, z_n\}$ and be sure that there is a subsequence $\{\phi_{n_k}\}$ so that for each j some $x \in U_j$ has $|\phi_n(x) - \phi_n(Rx)|$ very small. In other words, we want to have a picture like the one below.

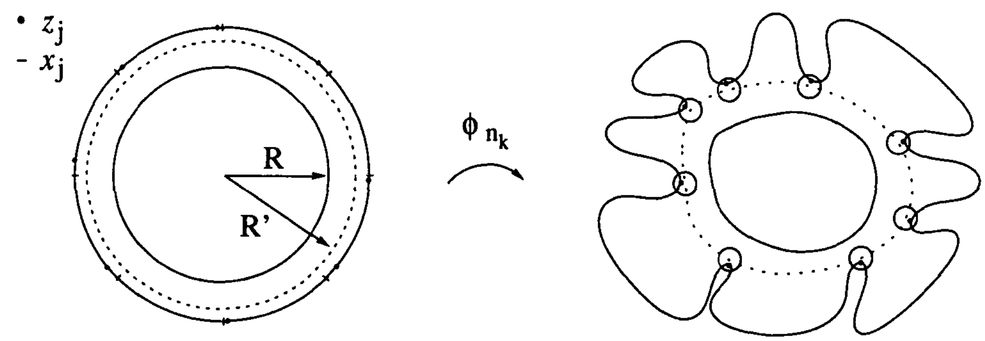

Figure 1.8: The goal of Lemma 13

Lemma 13. *Let $\{\phi_n\}$ be a uniformly convergent sequence of conformal maps of $\mathbb{D}$ and let $U = \cup U_j$ be a collection of neighborhoods of the points $\{z_j\}$ in $\partial\mathbb{D}$. Then for any $\eta > 0$ and $R < 1$ there is $R' \in [R, 1)$ and a subsequence $\{\phi_{n_k}\}$ so that for all n_k and each j there is at least one $x_j \in U_j$ satisfying $|\phi_{n_k}(R'x_j) - \phi_{n_k}(x_j)| \leq \eta$.*

Proof:

We show that the lemma holds if U is just one interval. The complete result follows by taking a sequence of n nested subsequences. We assume without loss of generality that $\frac{2\pi}{|U|} = M \in \mathbb{N}$, so that $\partial\mathbb{D}$ can be expressed as the disjoint union of $U_1, \ldots, U_M$, each a copy of U.

The proof of this lemma follows almost directly from Corollary 1. The part we will use is

$$\omega(0, U, \mathbb{D}) \leq \text{cap}(U) \leq \frac{1}{\sqrt{R}} e^{-\frac{\pi}{\text{mod}(\Gamma_U(R))}}$$

Suppose there is $N \in \mathbb{N}$ such that for all $n > N$ there is no appropriate x in U. Then

$$\text{mod}(\Gamma_R(U)) \leq \frac{a_n(R)}{\eta^2}$$

where $a_n(R)$ is the area of the annular region $\phi_n(\{R < |z| < 1\})$. The measure of U is forced to be small by making $a_n(R)$ small. If there is some $R' \geq R$ for which $\lim_{n\to\infty} a_n(R') = 0$, the proof is completed. If we define $a(R) = \liminf_{n\to\infty} a_n(R)$, and if $\lim_{R\to 1} a(R) = 0$, then the proof is again complete.

Now consider the case in which $a(R') \geq a_0 > 0$ for all $R' > R$. Then as above, we have that

$$\text{mod}(\Gamma_R(U)) \leq \frac{a_0}{\eta^2}.$$

However, we can also write $\Gamma_R(\partial\mathbb{D}) = \cup_{m=1}^{M} \Gamma_R(U_m)$, so that

$$\begin{aligned} \text{mod}(\Gamma_R(\partial\mathbb{D})) &\leq \sum_{m=1}^{M} \text{mod}(\Gamma_R(U_m)) \\ \frac{2\pi}{\log \frac{1}{R}} &\leq M \cdot \text{mod}(\Gamma_R(U)) \end{aligned}$$

Combining this inequality with the one in Equation 1.4.2 we have

$$\frac{1}{M}\frac{2\pi}{\log\frac{1}{R}} \leq \mathrm{mod}(\Gamma_R(U)) \leq \frac{a_0}{\eta^2}.$$

Since $\frac{2\pi}{M} = |U|$, Equation 1.4.2 is equivalent to $|U| \leq \frac{a_0}{\eta^2}\log\frac{1}{R}$. This is a contradiction for R sufficiently close to 1, completing the proof.

□

We would like to say that each of the fingers in the region $\phi_{n_k}(\{R' < |z| < 1\})$ supports an admissible constant metric $\rho = c(\delta)$ for the family Γ_j of curves from $\{|z| = R'\}$ to $E^{\delta}_{n_k,j}$. We must show that such a ρ is admissible. We first prove this under the assumption that $\partial\Omega_n$ is locally connected for all n, or equivalently, that all of the maps ϕ_n have continuous extension to the boundary, $\tilde{\phi}_n$, on all of $\partial\mathbb{D}$.

Let $B_j = D(\phi_n(R'x_j), \eta)$. Consider the components of $\Omega_n \setminus \{\cup_j B_j \cup \phi_n(D(0,R')\}$.

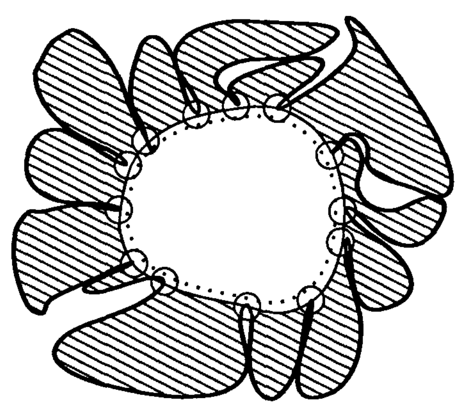

Figure 1.9: The regions $U_1, \ldots, U_n$

Sort these components into n disjoint sets $U_1, \ldots, U_n$, where a component U is included in U_j if its preimage in the disk has part of its boundary lying on the arc $[x_j, x_{j+1}]$.

By the continuity of $\tilde{\phi}$ we can choose n so that if

$$|x_j - x_{j+1}| < \frac{2\pi}{k} + 2^{-k^2} \to |\phi(x_j) - \phi(x_{j+1})| < \frac{\delta}{100}.$$

By the local connectivity of Ω, we choose R sufficiently close to 1 so that

$$|\phi(x) - \phi(R'x)| < \frac{\delta}{100} \text{ for all } z \in \partial\mathbb{D} \text{ and all } R' > R.$$

Now choose $\eta > 0$ so that if z_1, z_2 are two points of $\phi(\{|z| = R\})$ and $|z_1 - z_2| \leq \eta$ then there is a continuum from z_1 to z_2 in $\phi(\{|z| = R\})$ of diameter smaller than $\frac{\delta}{100}$. Assume without loss of generality that $\eta \leq \frac{\delta}{100}$.

Apply Lemma 13 as described above to extract a subsequence $\{\phi_{n_k}\}$ and an $R' > R$ so that to each k there is associated a set of n points $x_1, \ldots, x_n \in \partial\mathbb{D}$ with $|x_j - x_{j+1}| \leq \frac{2\pi}{k} + 2^{-k^2}$, and so that

$$|\phi_{n_k}(R'x_j) - \phi_{n_k}(x_j)| \leq \eta \text{ for all } j.$$

By choosing k large, we can be sure that

$$|\phi(R'x) - \phi_{n_k}(R'x)| \leq \eta.$$

Now if a point z is in the set $E^{\delta}_{n_k,j}$ then $|\phi_{n_k}(z) - \phi(z)| \geq \delta$, and $|\phi_{n_k}(x) - \phi_{n_k}(R'x)| \geq \frac{99\delta}{100}$. What the list of inequalities above gives us is that *any* point $R'x$ on the arc $\phi_{n_k}([R'x_j, R'x_{j+1}])$ is within 5η of either $\phi_{n_k}(R'x_j)$ or $\phi_{n_k}(R'x_j)$. Let C be a member of the curve family Γ_j. If C starts at a point on $\phi_{n_k}([R'x_j, R'x_{j+1}])$ it must travel a distance of at least $\delta - 5\eta$ through U_j to reach $\tilde{E}^{\delta}_{n_k,j}$. The same is true if C passes through either of the balls B_j or B_{j+1}. It will be troublesome, however, if there is a $C \in \Gamma_j$ which does not fall into one of these two cases. This would be possible if there is a portion of $\phi_{n_k}(R')$ which makes a loop into U_j, offering a shortcut to curves on their way to $E^{\delta}_{n_k,j}$, as in Figure 1.10.

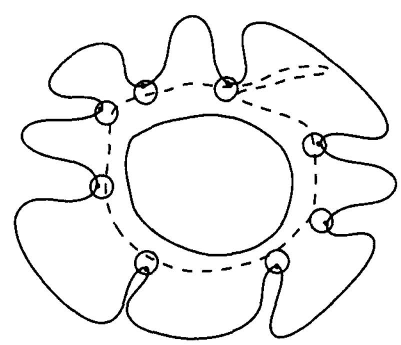

Figure 1.10: The dashed Line is $\phi_{n_k}(R')$

The value of η above was chosen so that if a loop begins and ends in an η-ball, it cannot have diameter greater than $\frac{\delta}{100}$. We showed this for a loop of $\phi(\{|z| = R\})$, and the other inequalities show that the same holds true for $\phi_{n_k}(R')$ if we replace $\frac{\delta}{100}$ with $\frac{3\delta}{100}$.

So each U_j supports an admissible metric $\rho = \frac{1}{\delta - 5\eta}$. Since the $U_j s$ are disjoint and contained in a δ-neighborhood of $\partial\Omega$, we can write $\text{mod}\Gamma_j \leq p_j \frac{\text{area}_\phi(\delta)}{(\delta-\eta)^2}$, where $\text{area}_\phi(\delta)$ is the area of the above mentioned δ-neighborhood of $\partial\Omega_n$ and $\sum_{j=1}^{n} p_j \leq 1$.

We therefore have by Pfluger's theorem that for each j

$$\gamma(E_j) \geq \frac{\pi(\frac{9\delta}{10})^2}{\rho_j \text{area}_\phi(\delta)} - \log \frac{1+r}{\sqrt{R}}$$

with $\sum_{j=1}^{n} p_j \le 1$.

Removing the Assumption of Local Connectivity

Let ϵ' be the smaller of $\frac{\epsilon}{2}$ and $\frac{1}{10}(\frac{1}{k} + 2^{-k^2})$. For each n, let A_n be an open subset of $\partial\mathbb{D}$ with $\text{cap}(A_n) \le \epsilon' 2^{-n-1}$ such that ϕ_n extends continuously to $\partial\mathbb{D}$ off of A_n. By Lemma 1, we can define the set $A = \cup A_n$ with $\text{cap}(A) \le \frac{\epsilon'}{2}$ so that all maps in the sequence $\{\phi_n\}$ have continuous extension to the boundary off of the set A with $\text{cap}(A) \le \epsilon'$. We will assume that A is contained in each set E_n^δ and show that the part of E_n^δ on $\partial\mathbb{D} \setminus A$ has capacity smaller than $\frac{\epsilon}{2}$.

Choose R sufficiently close to 1 so that on the compact set $\partial\mathbb{D} \setminus A$ we have $|\phi(x) - \phi(Rx)| < \eta$. In the previous section we applied Lemma 13 inductively to the sets U_j which were $2^{-(k^2+1)}$-neighborhoods of the points $\{e^{j\frac{2\pi}{k}}\}$, respectively, to generate a subsequence $\{\phi_{n_k}\}$ with the property that for each k there is a set of points $\{x_j\}$ with $x_j \in U_j$. In this case we apply the same lemma to the sets $\tilde{U}_j = U_j \setminus A$ with the same results. For n sufficiently large there are $R' \ge R$ and points $x_1, \ldots, x_m$ on $\partial\mathbb{D}$ none of which are in A but which may change with n, so that $|x_j - x_{j+1}| \le \frac{1}{k} + 2^{-k^2}$ and so that $|\phi_{n_k}(R'x_j) - \phi_{n_k}(x_j)| \le \eta$. We again assume that all values of n_k are large enough to ensure that $|\phi_{n_k}(R'x) - \phi(R'x)| < \eta$.

Let $I_1, \ldots, I_S$ be the components of the open set A. For each I_j consider the circular arc $\tilde{I}_j(\theta)$ lying in $\mathbb{D}$ having the same endpoints as I_j and meeting $\partial\mathbb{D}$ at an angle θ. Fix θ_0 so that the arc of angle $2\theta_0$ lies in the annulus $\{R' < |z| < 1\}$ for each I_j, and set $\tilde{I}_j = \tilde{I}_j(\theta_0)$. Let D be the domain bounded by the arcs $\tilde{I}_j$ and $\partial\mathbb{D} \setminus A$.

Let the sets E_j be the intersection of $\tilde{E}_n^\delta \setminus A$ with the arc between x_j and

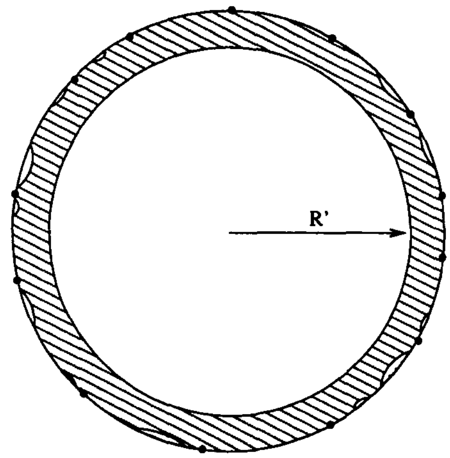

Figure 1.11: The domain D and the points $x_1, \ldots, x_m$

x_{j+1}. We can then compute the modulus of $\Gamma_{E_j}(R')$ in $\phi_n(D)$ just as in the preceding section. To apply Pfluger's Theorem as before, however, we must account for the fact that these moduli were computed in the image of the restricted domain D. Let V_j be the crescent cut out of $\mathbb{D}$ by the arc $\tilde{I}_j$. We know that $\rho \circ \phi_n(z)$ is an admissible metric for the curve family connecting $D(0, R')$ to E_j in the restricted domain D. We can extend this metric to V_j as $\rho'(z) = \rho_j \circ \phi_n \circ \tau(z)$, where $\tau(z)$ is the Möbius transformation reflecting V_j across $\tilde{I}_j$ onto the crescent bounded by $\tilde{I}_j$ and $\tilde{I}_j(2\theta_0)$.

Figure 1.12: Extending the metric $\rho \circ \phi_n(z)$

Since the reflected regions are all disjoint, we at worst double the area attributed to the curve family by ρ', and so at worst halve the contribution to the energy integral for each family. See Figure 1.12. Therefore, for the non-locally connected case, instead of Equation 1.4.2 we have

$$\gamma(E_j) \geq \frac{\pi(\frac{9\delta}{10})^2}{\rho_j 2\text{area}_\phi(\delta)} - \log\frac{1+r}{\sqrt{R}},$$

proving Lemma 12.

$\square$

Chapter 2

Computer-generated Quasiarcs

Let γ be a quasiarc in the complex plane, that is, $\gamma = \phi([0,1])$ for a K-quasiconformal homeomorphism ϕ of $\mathbb{C}$ onto itself. Denote the endpoints of γ by $a = \phi(0)$ and $b = \phi(1)$. If f is the Riemann mapping of the exterior of the unit disk to $\hat{\mathbb{C}} \setminus \gamma$ taking -1 to a and 1 to b, we associate to γ a quasisymmetric self-map h of $\partial\mathbb{D}$ defined as follows:

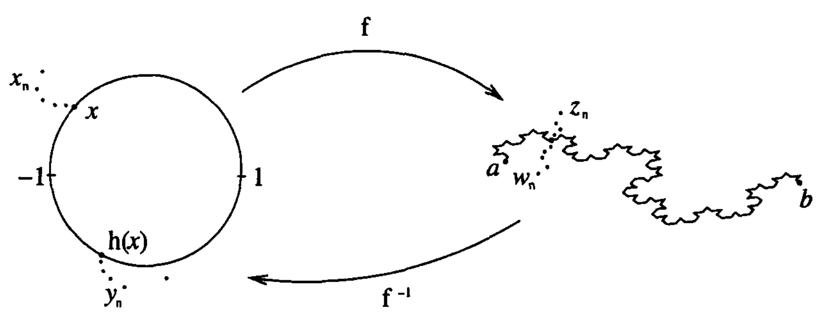

Figure 2.1: A quasiarc and its associated quasisymmetric map of $\partial\mathbb{D}$

For $x \in \partial\mathbb{D}$ let $\{x_n\}$ be any sequence of points in $\mathbb{D}^*$ converging to x. Then $\{z_n\}$ with $z_n = f(x_n)$ will converge to $w \in \gamma$, as will w_n where $w_n = \phi \circ \overline{\phi(z_n)}$. Let $y_n = f^{-1}(w_n)$, and define $h(x) \in \partial\mathbb{D}$ to be the limit point of the sequence y_n.

Observe that h is orientation-reversing, fixes ± 1, and that $h \circ h(z) = \text{id}$. We will use the letter h to refer exclusively to such maps.

Conversely, given any h, there exists a corresponding quasiarc which we will denote γ_h and a conformal map f_h from $\mathbb{D}^*$ to $\mathbb{C} \setminus \gamma_h$ so that x and $h(x)$ are identified under the continuous extension of f_h to $\partial\mathbb{D}$. (see again [Bis03]) We give a new proof of this fact, Theorem 7, by constructing a sequence of maps $\{f_n\}$ converging to f_h.

Our approximating maps $\{f_n\}$ are explicitly constructed from finitely many "pinching" maps of the form $P_{\{a,b\}}(z) = \frac{-i}{log(\frac{z-a}{z-b})}$ taking the exterior of the segment connecting a and b conformally onto the exterior of a pair of disks of radius $\frac{\pi}{2}$ tangent at the origin.

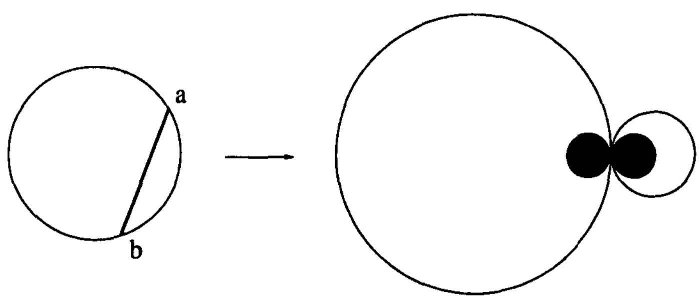

Figure 2.2: The basic pinching map

Due to the explicit construction of the maps $\{f_n\}$ it is not difficult to write computer programs to generate pictures of quasiarcs corresponding to a given h. See the last section for examples.

For distinct points x and y lying on the boundary of a disk D, the function $P_{\{x,y\}}(z)$ maps the complement of D conformally onto the exterior of a pair of tangent disks, identifying the points x and y at the point of tangency. Fix an h. Let $x_0, \ldots, x_{n+1}$ be equally spaced points on the upper unit semicircle,

$x_0 = 1$, $x_{n+1} = -1$. We define f_n to be the composition of the n maps which pinch together the pairs $\{x_1, h(x_1)\}, \ldots, \{x_n, h(x_n)\}$ in succession, $f_n = \tau_n \circ P_{\{x_n,h(x_n)\}} \circ \ldots \circ P_{\{x_1,h(x_1)\}}(z)$. The map $\tau_n(z)$ is a linear normalization ensuring that f_n fixes ± 1. We refer to the collection of pairs $\{\{x_j, h(x_j)\}\}_1^n$ as the 'pinching data' for f_n.

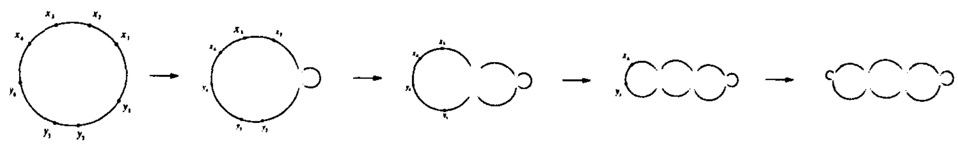

Figure 2.3: A composition of five pinching maps

Each f_n maps the exterior of the unit disk conformally onto the exterior of a chain of $n+1$ closed analytic curves, the two leftmost of which are circles. Each "pinch", or point of tangency, corresponds to the identification of a point $x \in \partial\mathbb{D}$ with its image, $h(x)$. Our idea is to show that as n becomes large, these chains converge to a quasiarc γ_h with the properties described above, so that the sequence of maps $\{f_n\}$ converges to f_h.

Theorem 7. *Let $h : \partial\mathbb{D} \to \partial\mathbb{D}$ be an orientation-reversing quasisymmetric map , fixing ± 1, and satisfying $h \circ h(z) = z$. Then there exists a quasiarc γ_h and a map f of $\mathbb{D}^*$ to $\hat{\mathbb{C}} \setminus \gamma_h$, extending continuously to the boundary $\partial\mathbb{D}$ such that $f(z_1) = f(z_2)$ if and only if $z_2 = h(z_1)$. The quasiconformal map ϕ of $\mathbb{C}$ with $\gamma_h = \phi([0,1])$ has constant of quasiconformaility determined by the constant of quasisymmetry for h.*

2.1 Background and Definitions

We recall the definition of quasisymmetry as presented in [Pom92]. A map h of $\partial\mathbb{D}$ is called quasisymmetric if it is one-to-one and if there is an increasing continuous function $\lambda(x)$ defined for positive x such that $\lambda(0) = 0$ and

$$\left|\frac{h(z_1) - h(z_2)}{h(z_2) - h(z_3)}\right| \leq \lambda\left(\left|\frac{z_1 - z_2}{z_2 - z_3}\right|\right), \text{ for } z_1, z_2, z_3 \in \partial\mathbb{D}. \tag{2.1}$$

For self-maps of $\partial\mathbb{D}$, we may use the simpler but equivalent condition that

$$|z_1 - z_2| = |z_2 - z_3| \implies |h(z_1) - h(z_2)| \leq \lambda(1)|h(z_2) - h(z_3)|. \tag{2.2}$$

The second condition has the benefit that it is not necessary to refer to a function $\lambda(x)$, but we retain the notation $\lambda(1)$ for the constant in Equation 2.2 so that we can employ both characterizations of quasisymmetry in the proof of a Lemma 16. We will refer to $\lambda(1)$ as the constant of quasisymmetry for h.

Crucial to our proof of Theorem 7 is the fact that any quasisymmetric self-map of $\partial\mathbb{D}$ is the boundary value function for a K-quasiconformal self-map H of $\mathbb{D}$. For a given h, the constant of quasisymmetry and the constant K of its quasiconformal extension do not in general agree, but there are several theorems outlining a relationship between them. For instance, if $\{h_n\}$ is a sequence of quasisymmetric self-maps of $\partial\mathbb{D}$ with constants $\{\lambda_n(1)\} \to 1$, then the corresponding quasiconformal maps $\{H_n\}$ of $\mathbb{D}$ to itself will likewise satisfy $\{K_n\} \to 1$. More specifically, we have the following quantitative relationship between the two constants (see [Leh87], pgs. 16, 38).

$$\lambda(1) \leq K \leq 725^{\lambda(1)-1}. \tag{2.3}$$

Quasiconformal maps have a reflection property like that of conformal maps. In particular, we have the following lemma.

Lemma 14. *Let Ω_1, Ω_2 be domains with circular boundary components S_1 and S_2, respectively. Let ϕ be a K-quasiconformal mapping of Ω_1 onto Ω_2 such that $\phi(S_1) = S_2$. Denote by $\tilde{\Omega_i}$ the reflection of Ω_i in the circle S_i. The map ϕ can be extended to a K-quasiconformal map between $\Omega_1 \bigcup S_1 \bigcup \tilde{\Omega_1}$ and $\Omega_2 \bigcup S_2 \bigcup \tilde{\Omega_2}$.*

Proof: We assume without loss of generality that each S_i is the boundary of the unit disk at the origin. First observe that the map $\tilde{\phi} = \frac{1}{\phi(\frac{1}{\bar{z}})}$ takes $\tilde{\Omega_1}$ onto $\tilde{\Omega_2}$. Being the composition of a K-quasiconformal map with two conformal maps, it is itself K-quasiconformal [Ahl66]. Clearly $\tilde{\phi}$ is ACL in all of $\Omega_1 \bigcup S_1 \bigcup \tilde{\Omega_1}$. The dilatation of $\tilde{\phi}$ is bounded a.e. by $\frac{K-1}{K+1}$, since the boundary curve S_1 has zero area. The map $\tilde{\phi}$ is therefore K-quasiconformal. $\square$

We need to consider the particular case of a K-quasiconformal map $\phi : \Omega_1 \to \Omega_2$ where Ω_i is the complement of a pair disks tangent at the origin. By the same argument as above we extend ϕ to a K-quasiconformal self-map of $\hat{\mathbb{C}} \setminus \{0\}$ by repeated reflections across circular boundary components. A point is removable for quasiconformal maps, so the map ϕ can be extended to the whole of $\hat{\mathbb{C}}$.

2.2 Main Theorem

We claim it is sufficient to prove that the maps $\{f_n\}$ for $h(z) = \overline{z}$ converge to the map $f(z) = \frac{1}{2}(z + \frac{1}{z})$, taking the exterior of the unit disk to the exterior of the segment $[-1, 1]$.

Lemma 15. *Let g_n map $\mathbb{D}^*$ onto the exterior of an n-chain generated from pinching data for the conjugation map. Then $g_n \to g = \frac{1}{2}(z + \frac{1}{z})$.*

We postpone the proof of Lemma 15 and first explain its sufficiency in proving Theorem 7. Suppose for the moment that it is so. Lemma 16 below shows that for any given h, the pinching data for the conjugation map are related to the pinching data for h by a K-quasiconformal self-map of the unit disk with K bounded above by a function of the constant of quasisymmetry for h.

Lemma 16. *Let $h : \partial\mathbb{D} \to \partial\mathbb{D}$ be an orientation-reversing quasisymmetric map fixing ± 1, with constant of quasisymmetry $\lambda(1)$. Then there exists a K-quasiconformal map $H : \mathbb{D} \to \mathbb{D}$ with boundary value function given by $H(z) = z$ for $\mathrm{Im}(z) > 0$ and $H(z) = h(\overline{z})$ for $\mathrm{Im}(z) < 0$. The constant K satisfies $K \leq 725^{\tilde{\lambda}-1}$, where $\tilde{\lambda} = \max\{2\lambda^2(1), \frac{1}{\lambda(\frac{1}{\lambda})}\}$.*

The proof of this lemma is the content of Section 2.2.1.

Let H_0 denote the quasiconformal mapping of the plane taking the pinching data for the conjugation map onto the pinching data for h. The figure below shows the successive pinchings comprising f_n along the top row, and the corresponding steps comprising g_n along the bottom row.

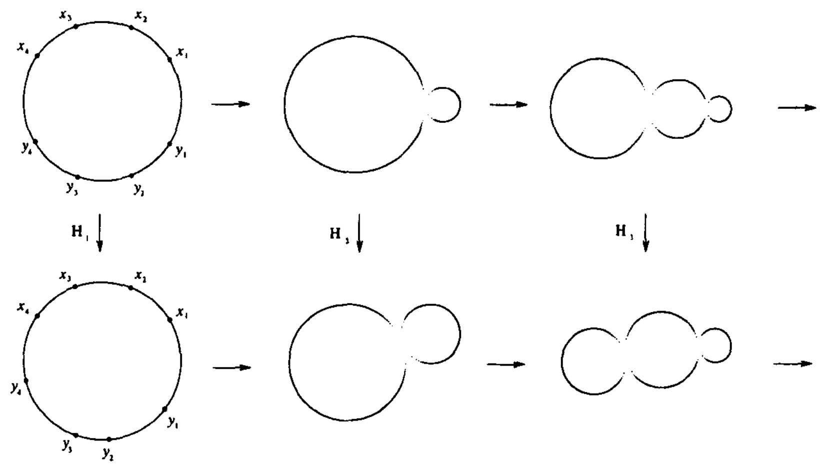

Figure 2.4: The sequence of quasiconformal maps $\{H_n\}$

Consider the map $P_{x_2,y_2} \circ H_0 \circ P^{-1}_{x_2,\overline{x_2}}(z)$ between the domains in stage one. it is a K-quasiconformal map from the exterior of a pair of tangent disks to the exterior of another such pair. As shown in Lemma 14, this map can be extended to a K-quasiconformal map from $\mathbb{C}$ to itself. Call this map H_1. Now consider the map between the domains in stage two, $P_{x_3,y_3} \circ H_1 \circ P^{-1}_{x_3,\overline{x_3}}(z)$. By the same argument we can extend this map to all of $\mathbb{C}$. We call the extended map H_2. Continuing this process, we have an n-chain of analytic curves generated by the pinching data for our map h expressed as the K-quasiconformal image of an n-chain generated by the pinching data for the conjugation map. We denote this map by ϕ_n. The sequence $\{\phi_n\}$ is normal, and so will converge to a K-quasiconformal map of $\mathbb{C}$ to itself, taking the interval $[-1, 1]$ onto the quasiarc γ_h.

2.2.1 Proof of Lemma 16

Proof: We show that the self-map η of $\partial\mathbb{D}$ given by the boundary values of H is quasisymmetric and then use Equation 2.3 to bound K.

If x, y, z are all contained in $\{\text{Im}(z) \geq 0\}$ (or $\{\text{Im}(z) \leq 0\}$) then the condition 2.2 is satisfied by the quasisymmetry of h.

The general case follows from the case in which $y = -1$. Choose x with $\text{Im}(x) > 0$, and set $z = \overline{x}$. Suppose first that $\eta(z) = h(x)$ is closer than z to -1. Since η fixes x and y, it is obviously true that

$$|\eta(y) - \eta(z)| \leq |\eta(x) - \eta(y)| \leq \lambda|\eta(x) - \eta(y)|$$

where λ is the constant of quasisymmetry for h. To prove the inequality in the opposite direction, we use the fact that h is an involution to write

$$\begin{aligned} |\eta(x) - \eta(y)| &= |x - y| \\ &= |h(h(x)) - h(y)| \\ &\leq \lambda|h(\overline{h(x)}) - h(y)| \end{aligned}$$

where the last line uses the quasisymmetry of h applied to the intervals $(-1, h(x))$ and $(-1, h(\overline{h(x)}))$. By assumption $h(x)$ lies in the interval between $y = -1$ and $z = \overline{x}$, so it must also be that

$$|h(\overline{h(x)}) - h(y)| < |h(x) - h(y)|.$$

By definition of η and the preceding inequality, this gives

$$|\eta(x) - \eta(y)| \leq \lambda |h(x) - h(y)| = \lambda |\eta(z) - \eta(y)|.$$

A similar argument demonstrates quasisymmetry of η in the case when $\eta(z) = h(x)$ is not contained in the interval between $y = -1$ and $z = \overline{x}$. The inequality

$$|\eta(x) - \eta(y)| \leq |\eta(y) - \eta(z)| \leq \lambda |\eta(y) - \eta(z)|$$

follows immediately. In the other direction,

$$\begin{aligned} |\eta(y) - \eta(z)| &= |h(y) - h(x)| \\ &= |h(y) - h(\overline{h(x)})| \\ &\leq \lambda |h(h(x)) - h(y)| \\ &\leq \lambda |x - y| \\ &\leq \lambda |\eta(x) - \eta(y)| \end{aligned}$$

Note that we have shown that $|I|$ is comparable to $|\eta(I)|$ with constant λ for any interval I on $\partial\mathbb{D}$ with one endpoint in $\{\mathrm{Im}(>)0\}$. From this it follows easily that the quasisymmetry condition will hold with constant λ whenever both x and y are in the upper halfplane.

Now suppose that we are given x, y, z with $|x - y| = |y - z|$, and both y and z in the lower halfplane. First we find an upper bound for $\frac{|\eta(y)-\eta(z)|}{|\eta(x)-\eta(y)|}$. Since $|z - (-1)| \leq 2|y - z|$, then $|\eta(y) - \eta(z)| < |\eta(z) - \eta(-1)| \leq \lambda |z - (-1)| \leq$

$2\lambda|y-z|$. For the denominator, $|\eta(x)-\eta(y)| \geq \frac{1}{\lambda}|x-y|$, by applying the observation in the preceding paragraph to both $|x-(-1)|$ and $|y-(-1)|$. Therefore, $\frac{|\eta(y)-\eta(z)|}{|\eta(x)-\eta(y)|} \leq 2\lambda^2$.

To determine a lower bound for $\frac{|\eta(y)-\eta(z)|}{|\eta(x)-\eta(y)|}$, we first rewrite the expression in terms of h, and use the fact that h is an involution:

$$\frac{|\eta(y)-\eta(z)|}{|\eta(x)-\eta(y)|} = \frac{|h(\overline{z})-h(\overline{y})|}{|x-h(\overline{y})|} = \frac{|h(\overline{z})-h(\overline{y})|}{|h \circ h(x)-h(\overline{y})|}.$$

In other words, we are finding a lower bound on the ratio of the images of the adjacent intervals $(\overline{z},\overline{y})$ and $(\overline{y},h(x))$. But from preceding arguments, $|h(x)-\overline{y}| \leq \lambda|\overline{x}-\overline{y}|$. Therefore since $\frac{|\overline{z}-\overline{y}|}{|h(x)-\overline{y}|} \geq \frac{1}{\lambda}\frac{|\overline{z}-\overline{y}|}{|\overline{x}-\overline{y}|} = \frac{1}{\lambda}$, we have that

$$\frac{|\eta(y)-\eta(z)|}{|\eta(x)-\eta(y)|} = \frac{|h(\overline{z})-h(\overline{y})|}{|h \circ h(x)-h(\overline{y})|} \geq \lambda(\frac{1}{\lambda(1)})$$

by the quasisymmetry of h. We conclude that for any adjacent intervals (x,y) and (y,z) on ∂ with $|x-y| = |y-z|$, $\frac{1}{\lambda'} \leq \frac{|\eta(x)-\eta(y)|}{|\eta(y)-\eta(z)|} \leq \lambda'$, where λ' is the maximum of $2\lambda^2$ and $\frac{1}{\lambda(\frac{1}{\lambda})}$.

□

2.2.2 Proof of Lemma 15

It now remains to be shown that the sequence of n-chains generated by pinching data for the conjugation map converges to the segment $[-1,1]$. This relies upon the following lemma.

Lemma 17. *Let Ω be the complement of an n-chain constructed as above. Label the n closed curves comprising the chain $b_1,\ldots,b_n$. Then for each*

$j = 1, \ldots, n$ there is an annular region A_j having b_j as interior boundary component and intersecting the boundary of Ω only on adjacent curves b_{j-1} and b_{j+1}, and such that the family Γ_j of curves separating the boundary components of A_j satisfies $\operatorname{mod}(\Gamma_j) \geq M_0$

We will show that for all n, any pair of adjacent curves b_j and b_{j+1} in an n-chain contain balls B_j, B_{j+1} centered on the real line of comparable diameters. The constant of comparability, C_0, is independent of n. This will imply the existence of the annuli described in Lemma 17 by observing first that the curves $\{b_j\}$ lie in disjoint vertical strips in the plane S_j, and that the annular region $\{S_{j-1} \cup S_j \cup S_{j+1}\} \setminus B_j$ has modulus at least M_0.

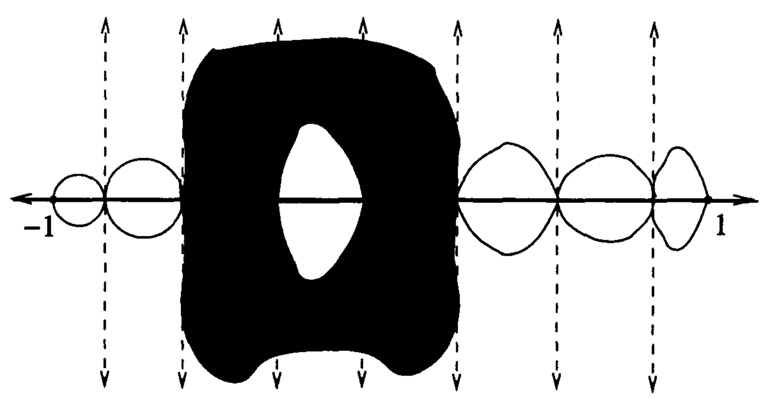

Figure 2.5: Blobs are contained in disjoint vertical strips

Suppose there is an ϵ such that for arbitrarily large n, the corresponding n-chain is not contained in the rectangle $\{z||\operatorname{Re}(z)| \leq 2, |\operatorname{Im}(z)| \leq \epsilon\}$. Then for n arbitrarily large, at least one of the blobs in an n-chain intersects the line $\operatorname{Im}(z) = \epsilon$. Let $z_0 \in B_j^n$ be one such point of intersection. From the preceding argument, there is a disk of size at least $r_0 = 2\epsilon e^{\frac{-2\pi}{M_0}}$, centered at z_0, and contained in the union of B_j^n and annular region about B_j^n described in

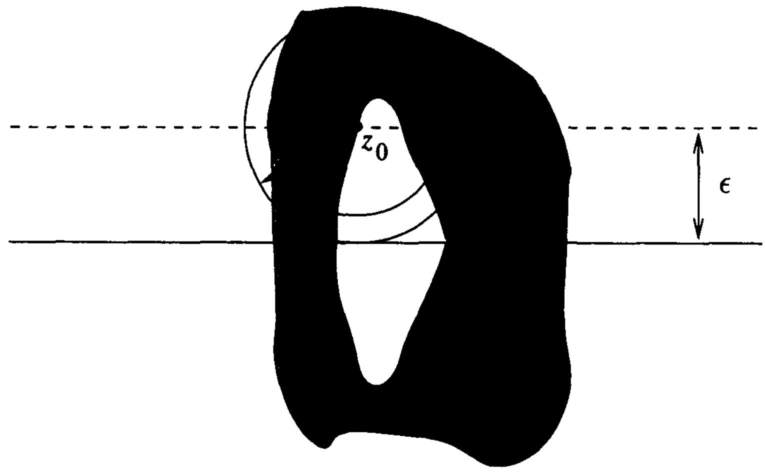

Figure 2.6: The annulus $\mathbb{A}(z_0, r, \epsilon)$

Lemma 17.

This follows from the fact that any element of the family of curves separating the boundary components of the annulus $\mathbb{A}(z_0, r, \epsilon)$ NOT contained in A_j contains an element from the family of curves connecting the boundary components of the annulus A_j about b_j.

The metric ρ which minimizes the area integral for $\tilde{\Gamma}_j$ in A_j is admissible for the family Γ separating the boundary components of $\mathbb{A}(z_0, r, \epsilon)$, and so

$$\mathrm{mod}(\Gamma) \leq \int\int |_{\mathbb{A}(z_0,r,\epsilon)}\rho(x)\rho(y)dxdy \leq \mathrm{mod}(\tilde{\Gamma}_j)$$

since the area of A_j is greater than that of $A_j \cap \mathbb{A}(z_0, r, \epsilon)$. This is equivalent to

$$\frac{2\pi}{\log \epsilon r_0} \geq M_0$$

Which gives the correct bound on r. Note now that the harmonic measure of the disk $D(z_0, r_0)$ intersected with the n-chain has a lower bound ω_0

independent of n.

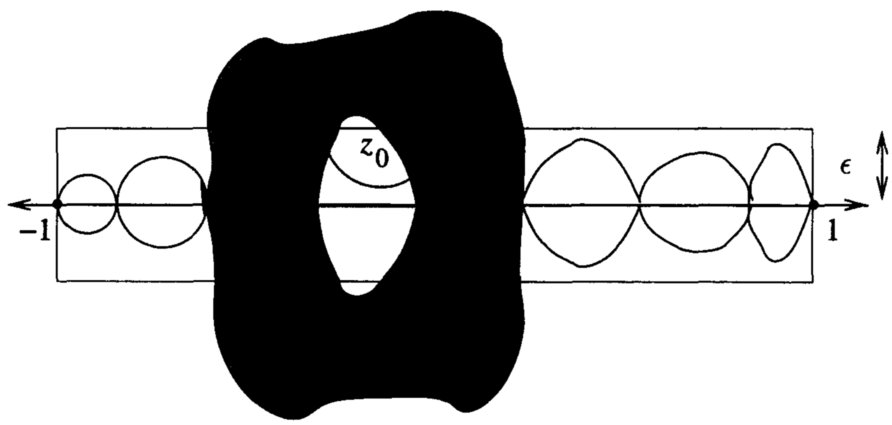

Figure 2.7: $D(z_0, r_0)$ has harmonic measure bounded below independent of n

Since at most three blobs intersect the disk, one or more blobs will have harmonic measure at least $\frac{\omega_0}{3}$. This is surely false, since any given blob in an n-chain has harmonic measure exactly $\frac{1}{n}$. We see therefore that for the map $h(z) = \overline{z}$, the sequence of n-chains is converging to the segment $[-1, 1]$.

2.2.3 Proof of Lemma 17

We begin by making a few observations about the blobs in our n-chains. First observe that each blob is convex. The proof of this fact is left for the reader (Grant-that means you). Coupling this with the conformality of the maps off of the segment connecting x and x', so that each blob must meet the real axis at a right angle, we see that the blobs are contained in disjoint vertical strips in the plane.

We must show that as the pinching sequence is executed, each series of three consecutive blobs has the property that there is an annulus of modulus at least M_0 with the center blob as interior boundary component and not intersecting the chain other than in the two adjacent blobs.

Given a point x on a disk D positioned as shown in Figure 2.8, the diameters of the circles bounding the image domain are functions of the angle labelled θ. Neither of the resulting disks can have diameter smaller than $\frac{1}{\pi}$. The disks both have size $\frac{2}{\pi}$ when $\theta = \frac{\pi}{2}$. The derivative of a function $P_{\{x',\overline{x'}\}}(z)$ is $(P_{\{x',\overline{x'}\}}(z))^2 \frac{2\mathrm{Im}(x')}{|z-x'|^2}$, for $z \in \mathbb{R}$. It's norm is minimal at the origin and increases with z.

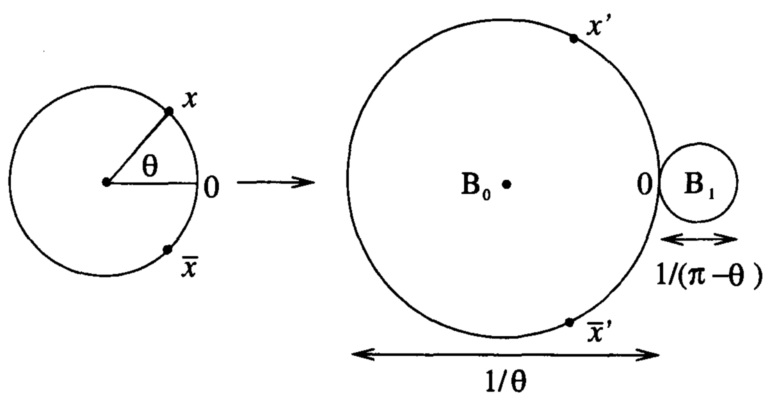

Figure 2.8: The pinching map

Note that it will always be the case that $\theta' > \theta$. This follows from the fact that the smaller of the disks and the arc between 0 and x' must have the same harmonic measure. Let θ_j be the angle analogous to θ' for the jth step in the length n pinching sequence. We will use the fact that the sequence $\theta_1, \theta_2, \ldots, \theta_n$ is increasing.

We split the process of generating an n-chain into two periods characterized by the size of θ_j for the current pinching map. Fix θ_0 small and positive (the exact value to be determined in a few paragraphs). Then we break our indices $j = 1, \ldots, n$ into three sets. The set $\{j = 1, 2, \ldots, N_1\}$ consists of the indices for which $\theta_j \leq \theta_0$. The second set $\{j = N_1 + 1, N_1 + 2, \ldots, n\}$ consists of indices for which $\theta_0 < \theta_j$. By choosing n large, we can insure that neither of these sets is empty. In each period we have bounds on the size of the the new disks being formed and on the distortion of the diameter of "older" curves in the chain under the formation of the new disks. We assume n to be very large in each case.

First Stage: Fix θ_0 very small. And suppose x_{j-1}, x_j, x_{j+1} all have $\theta_j \leq \theta_0$ Performing the pinching map $P_{x_{j-1}}$ will yield two tangent circles: one of radius at least $\frac{1}{\theta_0}$ on the left, and one (called b_{j-1}) of radius between $\frac{1}{\pi}$ and $\frac{1}{\pi-\theta_0}$ on the right. We want to estimate how the size of b_{j-1} will change as we continue pinching the left circle with P_{x_j}. On one hand, we know that the imaginary component of x_j, the next point to be pinched, is at least as great as the radius of the disk bounded by b_j.

On the other hand, if the diameter of b_j is sufficiently small in comparison to the diameter of B_0, (in fact the ratio is $\frac{\pi-\theta_0}{\theta_0}$), then the imaginary component of x_j cannot be too many times greater than the radius of the disk bounded by b_j.

If it were, then the boundary of the disk b_j would have harmonic measure smaller than that of the arc between zero and x' on the boundary of B_0. This situation becomes more extreme as the ratio R of the diameter of B_0 to the diameter of B_1 increases, so there is some fixed M such that if $\theta_j \leq \theta_0$, then $\text{Im}(x') \in (\frac{1}{2}\text{diam}(b_{j-1}), M\text{diam}(b_{j-1}))$.

These observations imply the existence of bounds on the derivative of $P_{\{x',\overline{x'}\}}(z)$,

$$P'_{\{x',\overline{x'}\}}(z) = (P_{\{x',\overline{x'}\}}(z))^2 \frac{2\text{Im}(x')}{|z-x'|^2}$$

at $z \in \mathbb{R}$.

$$\frac{2}{\pi^3}|x_j - z|^{-2} \leq P'_{x_j}(z) \leq 16M\pi \left(P_{x_j}(z)\right)^2$$

The same bounds will hold for $P'_{x_{j+1}}(z)$, so since each ball has diameter in the range $(\frac{1}{\pi}, \frac{1}{\pi-\theta_0})$ when initially generated, after applying the composition $P_{x_{j+1}} \circ P_{x_j}(z)$ there will be $c_1, c_2, c_3, d > 0$ so that b_j is contained in an annulus of inner radius d and outer radius $c_1 d$, and b_{j-1}, b_{j+1} each contained in annuli of inner radius $c_2 d$ and outer radius $c_3 d$. This is sufficient to demonstrate the existence of the annulus described in Lemma 17.

Consider the union of the three vertical strips containing b_{j-1}, b_j, and b_{j+1}. This region will have width at least $(1 + 2c_2)d$. Truncate the region above and below the real axis at a distance of $2c_1 d$. The annular region A obtained by removing the interior of b_j from the rectangle just described will have area no greater than $(1 + 2c_2)d \cdot (2c_1 d) - (c_1 d)^2$ and any curve connecting b_j to the boundary of the rectangle must have length at least $c_2 d$. This yields a lower bound on the modulus of the family of curves separating the boundary components of A, so $\mathrm{mod}(A) \geq \frac{c_2^2}{c_1^2 - 1 + 2c_1 c_3}$.

Second Stage: We now consider the period during which $\theta_j \geq \theta_0$. First note that there is a $\tilde{\theta} < \pi$ bounding the argument of x_j above. This is because for θ very close to π, the resulting balls will have $R << 1$, yet the smaller ball must have harmonic measure at least as great as the large one. This must be false for reasons similar to those used in the first paragraph of this proof. It holds therefore that during this period any newly generated disk will have diameter in the interval $(\frac{1}{\pi-\theta_0}, \frac{1}{\pi-\tilde{\theta}})$. Again, the distortion of disks under subsequent pinching maps is bounded, so that any three consecutive b_js generated during this period will be contained in annuli whose radii have bounded ratio. As in the first stage, this will correspond to a lower bound on the appropriate annular region.

Finally we must consider the case where a chain of three consecutive blobs comes partly from each of the two stages. For n sufficiently large, we can insure that whenever this is the case, the three corresponding θ_js are all in an ϵ-neighborhood of θ_0. We can therefore apply the same estimates as above with arbitrarily small adjustments. $\square$

2.3 Computer-generated Quasiarcs

We now provide several computer images illustrating application of this process. In the first two examples, the function used to generate the pinching data is shown at the top of the figure. The figures show the result of pinching for data sets with 10, 50, 125, 250, and 500 pairs of points.

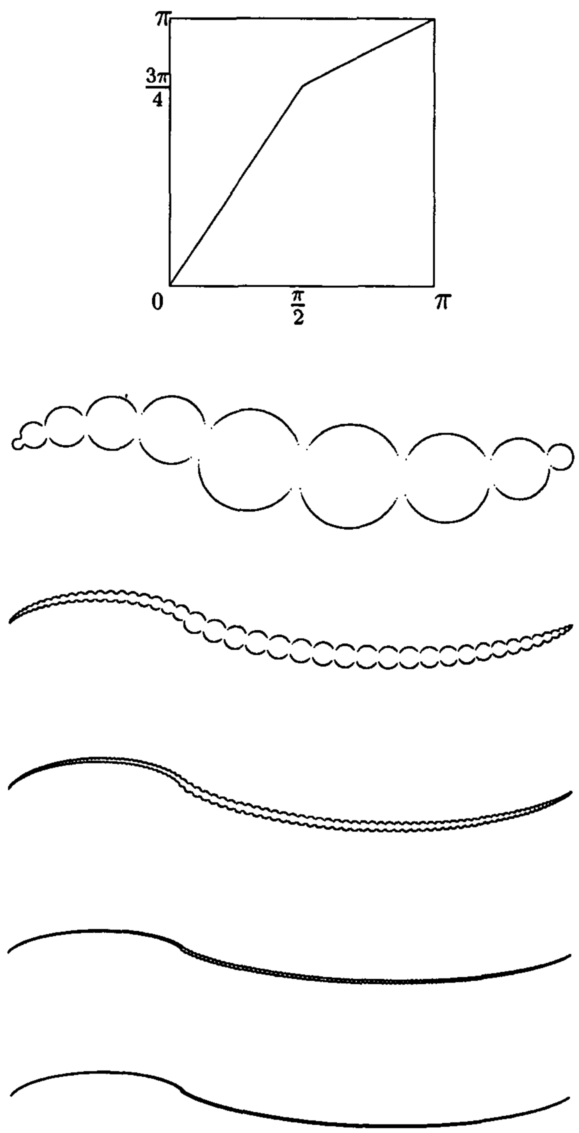

Figure 2.9: A piecewise-linear Function

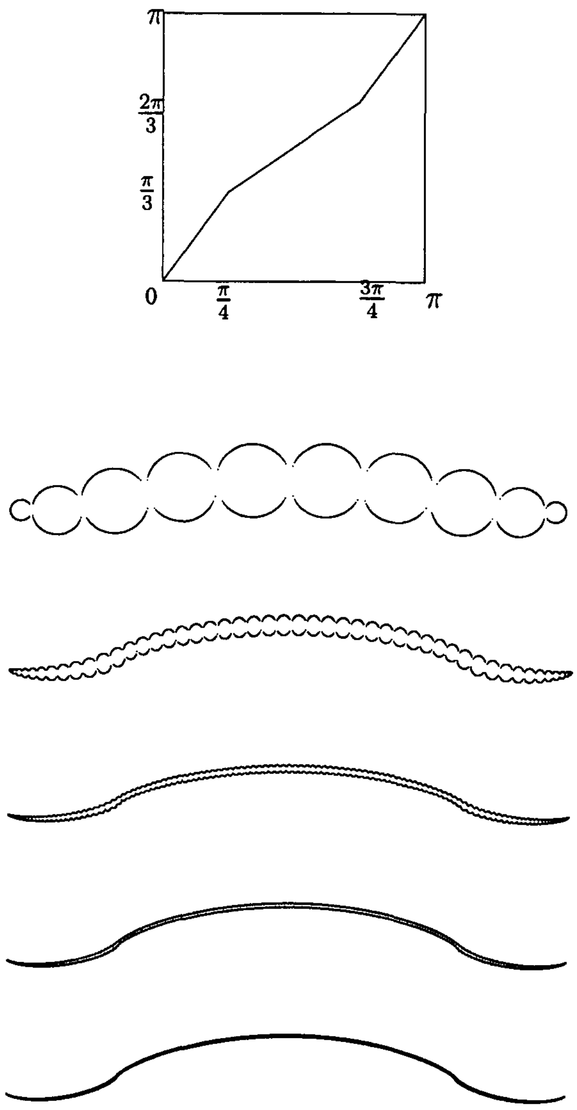

Figure 2.10: Another piecewise-linear function

In several cases we have made use of Donald Marshall's numerical conformal mapping software ZIPPER [Mar] to generate appropriate pinching data without the need for an explicit h. The ZIPPER software creates approximate maps which are the composition of the extensions to the boundary of Riemann map ϕ from the interior of the unit disk to the interior of a quasicircle with the conformal map of the exterior of the same quasicircle to the exterior of the unit disk. We then normalize to insure that ± 1 are fixed, and reflect the points across the real axis to get an appropriate involutive quasisymmetric map. In the first example (see Figure 2.11) we began with an approximate Von Koch snowflake Ω_{768} (of 768 vertices). The lefthand column shows the output of pinching $11, 31, 52, 103, 307$ and 1530 points. The righthand column shows just the points of tangency connected with straight segments for the same pinching data.

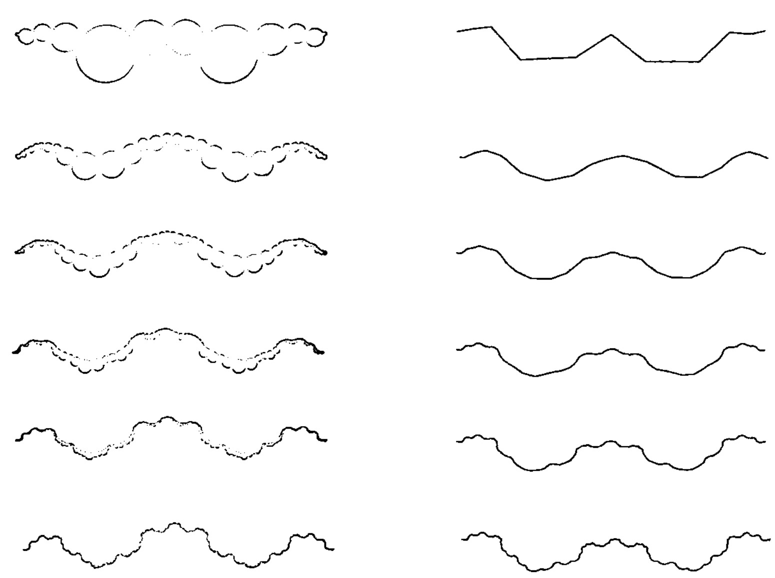

Figure 2.11: The welding map for the 768-sided snowflake curve

Figure 2.12 shows the quasiarc resulting from a map created as in the preceding example, but corresponding to a square instead of a snowflake.

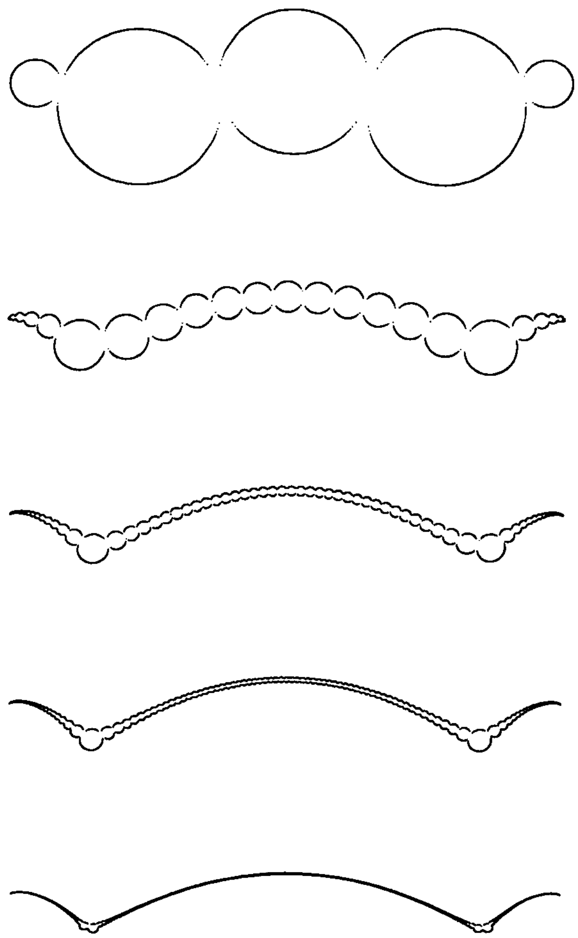

Figure 2.12: The welding map for a square

The next example shows the result of using a non-quasisymmetric map, the $h(\theta) = \theta^2$. In the zoomed box one can see that the curve is spiraling inwards. The is because a length of curve must have harmonic measure ω on one side and ω^2 on the other side. Since the derivative of $h(\theta) = \theta^2$ is zero at the $z = 1$, the curve spirals inward. The chains shown are for pinching data sets of 10, 50, 125, 250, and 500 pairs of points.

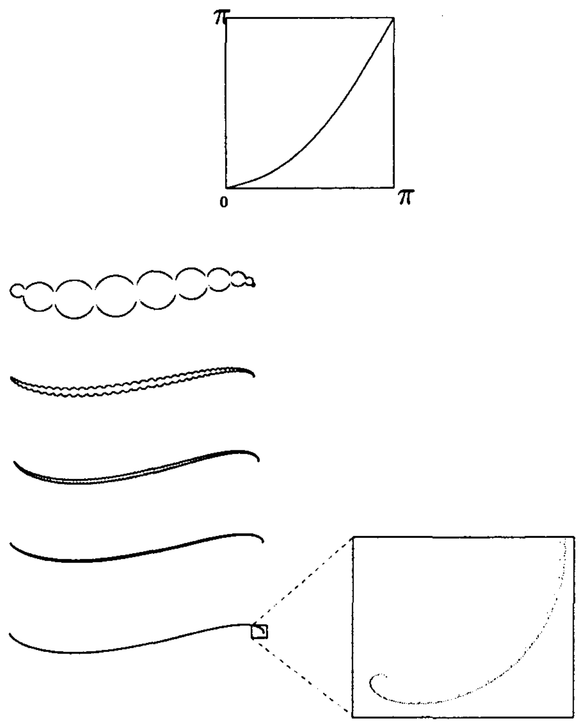

Figure 2.13: The resulting curve for a non-quasisymmetric function

Chapter 3

A Proof of Koebe's Theorem for Finitely Connected Domains

3.1 Introduction

We give a proof of Koebe's well-known circle domain theorem:

Theorem 8 (Koebe). *Let Ω_0 be a domain in the complex plane with n boundary components, where $n < \infty$. Then Ω_0 is conformally equivalent to a circle domain with the same number of boundary components.*

In the case where Ω_0 has just one boundary component, this is the Riemann Mapping Theorem. We use induction on the number of boundary components, n, to prove it in the general case.

We assume all of the boundary components to be nondegenerate, since the inductive process is trivial otherwise. If z_0 is a degenerate boundary component of Ω_0, then by hypothesis there is a conformal map f from $\Omega_0 \cup z_0$ to a circle domain Ω_0' with $n-1$ boundary components. The image of z_0 under such a map is a point z_0', so that $f|_{\Omega_0}$ is a map from Ω_0 onto the circle domain

$\Omega_0' - \{z_0'\}$.

Under our inductive hypothesis, we may assume that all but one of the boundary components of Ω_0 are circles, denoted $C_1^0, \ldots, C_{n-1}^0$. We denote the non-circular boundary component by γ_0, and assume that γ_0 bounds the unbounded component of $\hat{\mathbb{C}} \setminus \bar{\Omega}_0$. We may also assume that C_1^0 is the boundary of the disk of radius ρ_0 centered at the origin.

Let f_1 be a conformal map of $\Omega_0 \cup \bar{D}(0, \rho_0)$ into $\mathbb{D}$ such that $C_2^0, \ldots, C_n^0$ map to circles, and such that γ_0 maps onto $\mathbb{T}$. Let Ω_1 be the image of Ω_0 under the composition of f_1 and the map $\frac{r_1}{z}$, where the contraction factor r_1 is chosen so that $\mathrm{dist}(0, \gamma_1) = 1$, where γ_1 denotes the exterior boundary component of Ω_1. Let $C_1^1, \ldots, C_{n-1}^1$ denote the interior boundary components. Define $g_1(z) : \Omega_0 \to \Omega_1, g_1(z) = \frac{r_1}{f_1(z)}$.

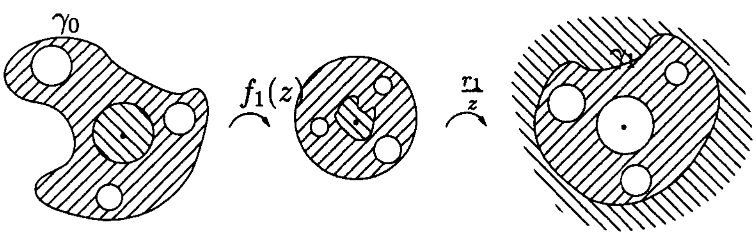

Figure 3.1: γ_1 is closer to being a circle than γ_0

We define $g_k(z) : \Omega_0 \to \Omega_k$ inductively. Let $g_k(z) = \left(\frac{r_k}{f_k(z)}\right) \circ g_{k-1}(z)$, where f_k is a map of Ω_{k-1} into $\mathbb{D}$ analogous to f_1 above. Let γ_k be the image of γ_0 under the map g_k.

We claim that the sequence $\{g_k\}$ has as a limit the desired mapping of Ω_0 onto a circle domain. This results from γ_{k+1} being closer than γ_k is to a circle in the following way:

Lemma 18. *Define R_k to be the maximum value of $|z|$ for $z \in \gamma_k$. Then*

$$1 \leq R_{k+1} \leq (R_k)^{C(R_k)},$$

where $C(R_k) < 1$ whenever $R_k > 1$.

Note that γ_k is a circle centered at the origin iff $R_k = 1$.

Application of the Schwarz-Pick Inequality for multiply connected domains [HS93] to f_k from Ω_k into $\mathbb{D}$ shows easily that $R_{k+1} \leq R_k$ (this is the content of the next section). The sequence $\{g_k\}$, with $|g_k(z)| \leq R_0$, is therefore a normal family, possessing a limit function g to which a subsequence $\{g_{n_k}\}$ converges uniformly on compact subsets of Ω_0. Evidently g is holomorphic and non-constant, so it must be univalent. The crux of this proof of Theorem 8 consists of showing that $R_k \to 1$ for any subsequence of $\{g_k\}$, as Lemma 18 provides.

Sketch of proof of Lemma 18: To prove Lemma 18 we consider the harmonic function $\log\left|\frac{f_{k+1}(z)}{z}\right|$ in the domain $\Omega_k \cup \bar{D}(0, \rho_k)$ and show that there is a lower bound, $c(R_k)$, on the harmonic measure (from a point in C_1^k) of the portion of γ_k which is closer than $\sqrt{R_k}$ to the origin.The Schwarz-Pick Lemma will show that $\frac{1}{R_k} \leq \left|\frac{f_{k+1}(z)}{z}\right|$ on each of the interior boundary components. This will yield, for $z \in C_1^k$,

$$
\begin{aligned}
\log\left|\frac{f_{k+1}(z)}{z}\right| &= \int_{\gamma_k\cap D(0,\sqrt{R_k})} \log\left|\frac{f_{k+1}(\zeta)}{\zeta}\right| d\omega(\zeta) \\
&\quad + \int_{\partial\Omega_k\setminus\{\gamma_k\cap D(0,\sqrt{R_k})\}} \log\left|\frac{f_{k+1}(\zeta)}{\zeta}\right| d\omega(\zeta) \\
&\geq c(R_k)\log\frac{1}{\sqrt{R_k}} + (1-c(R_k))\log\frac{1}{R_k} \\
&= C(R_k)\log\frac{1}{R_k} \qquad (3.1)
\end{aligned}
$$

where $C(R_k) = (1 - \frac{c(R_k)}{2}) < 1$.

Now let z_1, z_2 be points in C_1^k. For $i = 1, 2$, we have

$$
R_k^{-C(R_k)} \leq \left|\frac{f_{k+1}(z_i)}{z_i}\right| < 1.
$$

Taking a ratio of such inequalities, and using the fact that $|z_1| = |z_2|$, we have

$$
R_k^{-C(R_k)} \leq \left|\frac{f(z_1)}{f(z_2)}\right| \leq R_k^{C(R_k)},
$$

So that $R_{k+1} = \sup_{z_1,z_2\in C_1^k}\left|\frac{f_{k+1}(z_1)}{f_{k+1}(z_2)}\right|$ must satisfy

$$
1 \leq R_{k+1} \leq R_k^{C(R_k)}
$$

This will prove Lemma 18, showing that any uniformly convergent subsequence of iterates of the process in Fig. 3.1 must have $R_{n_k} \to 1$. Existence of $c(R_k)$ is demonstrated in a subsequent section.

3.2 Schwarz-Pick Lemma for Multiply-Connected Domains

Let $\rho_{\mathbb{D}}$ denote the hyperbolic metric in the unit disk.

Theorem 9 (He, Schramm). *Let U be a domain in the complex plane which contains $\mathbb{D}$, and let U_0 be obtained from U by deletion of n disjoint disks. If $U_1 \subset \mathbb{D}$ is the image of U_0 under a conformal homeomorphism f such that the image of any circular boundary component is again circular, then $\rho_{\mathbb{D}}(x, y) > \rho_{\mathbb{D}}(f(x), f(y))$ for x, y in $\mathbb{D} \cap U_0$.*

We apply this theorem to the functions $f_k : U_0 \to U_1$ defined above, where $U_0 = \Omega_{k-1} \cup \bar{D}(0, \rho_{k-1})$ In particular, we observe that each f_k reduces radial distances, so that the holomorphic function $\frac{f_k(x)}{x}$ will satisfy $\left|\frac{f_k(x)}{x}\right| < 1$ on $\Omega_{k-1} \cap \mathbb{D}$ and $U_1 = f_k(U_0)$. Clearly the same inequality holds in $\Omega_{k-1} \setminus \mathbb{D}$, so that $\left|\frac{f_k(x)}{x}\right| \leq 1$ on all of Ω_{k-1}. We apply the same theorem to $\frac{f_k^{-1}(y)}{R_{k-1}}$ and obtain $\frac{1}{R_{k-1}} \leq \left|\frac{f_k(x)}{x}\right|$ for $z \in \Omega_{k-1}$.

Now for arbitrary $z_1, z_2 \in C_1^{k-1}$ (so that $|z_1| = |z_2|$) we have $1 \leq \left|\frac{f_k(z_1)}{f_k(z_2)}\right| \leq R_{k-1}$. Using the fact that the inversion and normalization comprising the second step in Figure 3.1 preserve this ratio, we see that $R_k = sup_{z_1, z_2 \in C_1^{k-1}} \left|\frac{f_k(z_1)}{f_k(z_2)}\right| \leq R_{k-1}$.

3.3 Finding Lower Bound on Harmonic Measure of $D(0, \sqrt{R_k}) \cap \gamma_k$

Before demonstrating the existence of a lower bound on the harmonic measure of $D(0, \sqrt{R_k}) \cap \gamma_k$ we must point out a few facts about the domains Ω_k. We will need to show that two interior boundary components of Ω_k cannot get too close together through the iterative process described above. We will also use the fact that the interior boundary components cannot get too close to the exterior boundary component γ_k. These are lemmas 19 and 20 below.

Lemma 19. *For $i = 0, 1$, let A_i be an annular region in Ω_i with outer boundary component γ_i. Let $A_k = g_k(A_0)$ for k odd, and let $A_k = g_k \circ g_1^{-1}(A_1)$ for k even. Then there exists a minimum distance $d > 0$, depending only on the modulus of A_0, between the two boundary components of any A_k.*

Proof: Fix $i = 0$ and assume k odd. Suppose a ball of radius r_1 with center in A_k intersects both boundary components of A_k. Define $A_k{}'$ to be the annular region between the disk of radius r_1 and a disk of radius $r_2 > r_1$. Then for $\frac{1}{2} < r_2 < 1$, any member of the family of curves separating the boundary components of $A_k{}'$ will contain a member of the family of curves connecting the boundary components of A_k. (Note that we can choose r_2 to be at least $\frac{1}{2}$, since γ_k always lies outside of the unit disk and $0 \notin \Omega_0$.) If M_0 is the modulus of the family of curves separating the boundary components of A_0, comparison of moduli shows that

$$\frac{1}{2\pi} \log(\frac{r_2}{r_1}) \leq M_0$$

so that

$$r_1 \geq r_2 e^{-2\pi M_0} \geq \frac{1}{2} e^{-2\pi M_0}.$$

Therefore, if M_0 is the modulus of any such annulus in Ω_0, the above estimate provides a lower bound, $d_0 = \frac{1}{2} e^{-2\pi M_0}$, on the distance between γ_k and the other boundary components of Ω_k. Similarly, we find a lower bound for the distance d_1 between the boundary components of A_k for k even and set $d = \min\{d_0, d_1\}$ □

The fact that there is a lower bound on the distance between any two interior boundary components of Ω_k relies on the limit function, g, being a conformal homeomorphism. We show this now, following the corresponding section of Ahlfors's proof of the Riemann Mapping Theorem [Ahl73].

Let g_{n_k} be the subsequence which converges uniformly on compact subsets of Ω_0 to the limit function g. For any point z_1 in Ω_0, define the sequence $\tilde{g}_{n_k}$, where $\tilde{g}_{n_k}(z) = g_{n_k}(z) - g_{n_k}(z_1)$. The sequence $\tilde{g}_{n_k}$ will be a normal family, with $|\tilde{g}_{n_k}| < 2R_0$, and $\tilde{g}_{n_k} \neq 0$ in $\Omega_0 \setminus z_1$. According to Hurwitz's Theorem, any limit function of the sequence $\tilde{g}_{n_k}$, in particular $g(z) - g(z_1)$, is either identically zero, or nowhere zero in $\Omega_0 \setminus z_1$. By the argument in the preceding paragraph, the image of Ω_0 under any $\tilde{g}_{n_k}$ must contain an annulus with minimum distance d between its boundary components, so that the function $g(z) - g(z_1)$ cannot be constant. Therefore $g(z) \neq g(z_1)$ for any $z \in \Omega_0 \setminus z_1$, so g is univalent.

Lemma 20. *There exists $\epsilon > 0$ such that dist(C_i^k, C_j^k)) $> \epsilon$ for any two distinct boundary components C_i^k and C_j^k of Ω_k.*

Proof: For each boundary component C_j^0 of Ω_0, let A_j and M_j be, respectively, the maximal round annulus in Ω_0 with interior boundary C_j^0, and the modulus of the family of curves, Γ_j connecting the boundary components of this annulus. Suppose there is a sequence of conformal maps g_n of Ω_0, converging uniformly on compact sets to a conformal homeomorphism g, such that for some $i \neq j$, $\text{dist}(g_n(C_i^n), g_n(C_j^n)) \to 0$. Then for all $\epsilon > 0$, there is a disk of radius ϵ with center in some A_j^n, intersecting both boundary components of A_j^n. So long as the disk of radius $r_2 > \epsilon$ does not contain C_j^n, any member of the family of curves separating the boundary components of the annulus $\{\epsilon < |z| < r_2\}$ contains an element of the family $g_n(\Gamma_j)$. Comparison of the moduli of these two families shows

$$\frac{1}{2\pi}\log\left(\frac{r_2}{\epsilon}\right) \leq M_j \tag{3.2}$$

So that r_2 can be no greater than $\epsilon e^{2\pi M_j}$. However, any $r_2 < \text{diam}(C_j^n)$ is in fact admissible, so that $\text{diam}(C_j^n) \geq \epsilon e^{2\pi M_j}$ would contradict Equation 3.2. Since the same argument can be applied to C_i^n, we see that $\text{dist}(g_n(C_i^n), g_n(C_j^n)) \to 0$ would imply that Ω_0 is conformally equivalent to a domain in which the pair of boundary components C_j^0, C_i^0 is replaced with one degenerate boundary component. Therefore there is an $\epsilon > 0$ below which $\text{dist}(g_n(C_i), g_n(C_j))$ cannot shrink, for any $n, i \neq j$. $\square$

We now demonstrate the existence of a lower bound $c(R_k)$ on the harmonic measure of $D(0, \sqrt{R_k}) \cap \gamma_k$. Consider a disk, D, of radius $\sqrt{R_k} - 1$ centered at $z_0 \in \gamma$, where $|z_0| = 1$. Assume for the time being that $d > \sqrt{R_k} - 1$ (with d as given in Lemma 19), so that D does not intersect any boundary components other than γ_k. Let D_0 be a concentric disk of radius $\frac{\sqrt{R_k}-1}{64}$.

We use the following inequality [GM05] to bound the harmonic measure of the portion of $\partial\Omega$ lying outside of D: Let Ω be a Jordan domain and let $E \subset \partial\Omega$. Then,

$$\omega(z, E, \Omega) \leq \frac{8}{\pi} e^{-\left(\frac{\pi}{\text{mod}\Gamma_{\sigma,E}}\right)}, \tag{3.3}$$

where σ is any path in Ω from z to $\partial\Omega \setminus E$, and $\Gamma_{\sigma,E}$ is the family of curves connecting σ to E in $\Omega \setminus \sigma$.

For any $z \in D_0 \cap \Omega_k$ we choose a path σ contained in $D_0 \cap \Omega_k$. Each member of the curve family $\Gamma_{\sigma,\partial\Omega_k \setminus D}$ will then contain a curve connecting the two boundary components of the annulus $D \setminus \bar{D}_0$, so that $\text{mod}(\Gamma_{\sigma,\partial\Omega_k \setminus D}) \leq \frac{2\pi}{\log(64)}$.

Application of Equation (3.3) then gives

$$\omega(z, \partial\Omega_k \setminus D, \Omega_k) \leq \frac{1}{\pi}.$$

so that whenever $z \in D_0 \cap \Omega_k$, $\omega(z, D \cap \partial\Omega_k, \Omega_k) \geq \frac{\pi-1}{\pi}$.

In other words, there is a lower bound of $\frac{\pi-1}{\pi}$ on the harmonic measure of $D \cap \Omega_k$ if we are in a sufficiently small neighborhood, $D_0 \cap \Omega_k$, of z_0. We now show that there is a lower bound on the harmonic measure of $\partial D_0 \cap \Omega_k$ viewed from any point $z \in C_1^k$. This is where we use the existence of a minimum value $\epsilon > 0$ for the distance between points $z_1 \in C_j^k, z_2 \in C_i^k, i \neq j$, for all Ω_k (Lemma 20).

For any $z \in C_1^k$, consider a path p_0 from z to z_0 which follows the line segment s connecting z to z_0 until s meets one of the circular boundary components C_i^k, after which it curves around the shorter arc of $C_i^k \setminus s$ until it can continue along s once again. Since the segment s has length less than 2, the path p_0 can have length at most $2 \cdot \frac{\pi}{2} = \pi$. By altering p_0 slightly, we can find a path p from z to z_0 satisfying the condition that any ball of radius $\frac{\epsilon}{2}$ centered on p is contained in Ω. A safe upper bound on the length of the path p is 2π.

Cover the path p with finitely many balls $\{B_i\}_0^N$ of radius $\frac{\epsilon}{2}$ and centers $b_i \in p$ as follows. Let b_0 be the point $z \in C_1$, where p begins. The location of b_i is determined inductively so that b_i coincides with the first point "after" b_{i-1} where the circle $|z - b_{i-1}| = \frac{\epsilon}{2}$ intersects p.

To bound the harmonic measure of $\partial D_0 \cap \Omega$ from a point $z \in C_1^k$, or in other words, bound the probability that a Brownian path starting at z will pass through D_0 when first exiting Ω, we find a lower bound on the probability that such a path will stay inside $\cup_0^N \{B_i\}$ until it exits Ω. We do so by taking the product of the probabilities that a path will enter and leave each ball B_i through specified arcs on its boundary.

Let E_i be the open subarc of ∂B_i symmetric about b_{i+1} having angular measure $\frac{\pi}{3}$. Then for any $z \in E_i$, $\omega(z, E_{i+1}, B_{i+1}) \geq \left(1 - \frac{\pi}{6}\right) \frac{\pi\epsilon}{4}$, where $1 - \frac{\pi}{6}$ is a lower bound on the distortion of the length of E_{i+1} under the Möbius automorphism of B_{i+1} taking z to b_{i+1}. Therefore if N is the number of prescribed balls needed to cover p, the probability that a Brownian path from a point $z \in C_1^k$ will reach D_0 before exiting Ω_k is

$$\begin{aligned}
\omega(z, D \cap \partial\Omega_k, \Omega_k) &\geq \omega(z, E_0, B_0) \left(\prod_{i=1}^{N-2} \left(1 - \tfrac{\pi}{6}\right) \omega(b_i, E_i, B_i)\right) \\
&\qquad \cdot\ \omega(b_{N-1}, \partial D_0 \cap \Omega_k, B_{N-1}) \\
&\geq \tfrac{1}{6}\left(\left(1 - \tfrac{\pi}{6}\right)\tfrac{1}{6}\right)^{N-2} \tfrac{(\sqrt{R_k}-1)}{\pi\epsilon}.
\end{aligned}$$

Since we certainly have that $N \leq \frac{4\pi}{\epsilon}$, our bound is

$$\omega(z, D \cap \partial\Omega_k, \Omega_k) \geq \frac{\sqrt{R_k} - 1}{6\pi\epsilon} \left(\left(1 - \frac{\pi}{6}\right)\frac{1}{6}\right)^{\frac{4\pi}{\epsilon}}$$

Multiplying the above inequality by the probability that a path beginning in D_0 will exit Ω_k through $D \cap \partial\Omega_k$ we have

$$c(R_k) = \omega(z, D \cap \partial\Omega_k, \Omega_k) \geq \frac{\sqrt{R_k} - 1}{6\pi\epsilon} \left(\left(1 - \frac{\pi}{6}\right) \frac{1}{6} \right)^{\frac{4\pi}{\epsilon}} \frac{(\pi - 1)}{\pi} \qquad (3.4)$$

This is the desired lower bound on the harmonic measure of $\partial\Omega_k \cap D(z_0, \sqrt{R_k} - 1)$. By the argument in Equation 3.1, we have proven Lemma 18, with

$$R_{k+1} \leq R_k{}^{(1-M(\sqrt{R_k}-1))},$$

where M is a constant depending only on Ω_0. The sequence $\{R_k\}$ is therefore decreasing and bounded below by 1. If it were to converge to $\bar{R} > 1$ then $\bar{R} \leq R_k{}^{(1-M(\sqrt{R_k}-1))}$ for all k. But this would imply that

$$\bar{R}^{\frac{1}{(1-M(\sqrt{\bar{R}}-1))}} \leq R_k{}^{\frac{(1-M(\sqrt{R_k}-1))}{(1-M(\sqrt{\bar{R}}-1))}} \leq R_k,$$

for every k, so that $\{R_k\}$ is bounded away from $\bar{R}$. Therefore $\{R_k\} \to 1$. This completes the proof of Theorem 8.

If we are not in the case where $d > \sqrt{R_k} - 1$, we let the disk have radius d instead of $\sqrt{R_k} - 1$, and the estimate in Equation 3.4 goes through as before with $C = C(d) < 1$. After a finite number of iterations, we will be in the case $d > \sqrt{R_k} - 1$. We also assumed above that $\epsilon \leq d$. If this is not the case, replace ϵ with d.

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