

Structural stability in dynamical systems and singularity theory

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IHES

Definition of structural stability

Let $f \in \text{End}^r(M)$ which preserves a compact subset K of M ($f(K) \subset K$). The subset is C^r -**structurally stable** if for every $f' \stackrel{C^r}{\approx} f$, there exists a compact subset K' homeomorphic to K via a homeo $h \in \text{Hom}^0(K, K')$, $h \stackrel{C^0}{\approx} id$ s.t. f' preserves K' and the following diagram commutes:

$$\begin{array}{ccccc}
 & & f' & & \\
 & K' & \rightarrow & K' & \\
 h & \uparrow & & \uparrow & h \\
 & K & \rightarrow & K & \\
 & & f & &
 \end{array}$$

The endomorphism f is C^r -structurally stable when K can be chosen equal to M .

Definition of structural stability in Singularity theory

Let $f \in C^\infty(M, N)$ is C^r -**equivalently, structurally stable** if for every $f' \stackrel{C^\infty}{\approx} f$, there exist diffeos $h \stackrel{C^r}{\approx} id_M$, $h' \stackrel{C^r}{\approx} id_N$ such that the following diagram commutes:

$$\begin{array}{ccccc}
 & & f' & & \\
 & M & \rightarrow & N & \\
 h & \uparrow & & \uparrow & h' \\
 & M & \rightarrow & N & \\
 & & f & &
 \end{array} .$$

Morse example

Definition

A smooth real function is **Morse** if it has a finite number of critical points which are non-degenerate.

Theorem

A Morse function whose critical values are all different is C^∞ -equivalently stable.

Definition

A diffeomorphism f of M is **Morse-Smale** if its non-wandering set is a finite union of hyperbolic periodic points and the stable manifold and unstable manifold of its points are transverse.

Theorem

Morse-Smale dynamics are C^1 -structurally stable.

Hyperbolic compact set

Definition

Let $f \in \text{Diff}^1(M)$ and let K be a compact subset f -invariant (i.e. $f(K) = K$). The subset K is **hyperbolic** if there exist two Tf -invariant sub-bundles E^s and E^u , s.t. :

- $TM|_K = E^s \oplus E^u$,
- $\exists \lambda < 1, \exists C > 0, \forall x \in K, \forall n \geq 0$:

$$\|T_x f|_{E^s}^n\| < C \cdot \lambda^n \quad \text{and} \quad \|T_x f|_{E^u}^{-n}\| < C \cdot \lambda^n$$

f is **Anosov** if $K = M$.

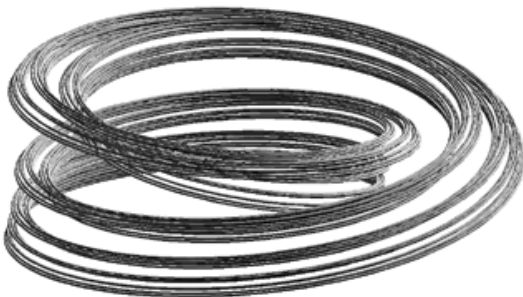


Figure: Smale's Solenoid

Theorem

hyperbolic compact sets are structurally stable.

Shub's theorem

Definition

Let $f \in \text{End}^1(M)$ and let K be a compact subset f -stable (i.e. $f(K) \subset K$).

The subset K is **expanded** if there exists $n \geq 1$ s.t., for every $x \in K$, $T_x f^n$ is invertible and with contracting inverse.

If $K = M$ f is **expanding**.

Theorem (Shub 60's)

Expanding endomorphisms are structurally stable.

Infinitesimal stability

Definition

Let $f \in \text{Diff}^1(M)$ is C^0 -infinitesimally stable if the following map is surjective:

$$\sigma \in \chi^0 \mapsto Tf \circ \sigma - \sigma \circ f \in \chi^0(f)$$

Theorem (Robin-Robinson-Mañé)

The C^0 -infinitesimally stable diffeos are the C^1 -equivalently stable maps.

Infinitesimal equivalent stability

Definition

Let $f \in C^\infty(M, N)$ is C^∞ -**equivalently infinitesimally stable** if the following map is surjective:

$$(\sigma, \xi) \in \chi^\infty(M) \times \chi^\infty(N) \mapsto Tf \circ \sigma - \xi \circ f \in \chi^\infty(f)$$

Theorem (Mather)

The C^∞ - infinitesimally equivalently stable maps are the C^∞ -equivalently stable maps.

Diffeomorphisms that satisfy Axiom A and the strong transversality condition

Definition

A C^1 -diffeomorphism f satisfies **Axiom A and the strong transversality condition (AS)** if:

- the non-wandering set Ω is hyperbolic,
- $cl(Per(f)) = \Omega$,
- $\forall (x, y) \in \Omega^2, W^s(x) \pitchfork W^u(y)$.

Theorem (Smale, Palis, Mañé, de Melo, Robbin, Robinson)

The C^1 -structurally stable diffeomorphisms of compact manifolds are exactly the AS diffeomorphisms.

Endomorphisms that satisfy Axiom A and the strong transversality condition

Definition

A C^1 -endomorphism f with empty critical set satisfies **Axiom A and the strong transversality condition (AS)** if:

- the non-wandering set Ω is the union of a hyperbolic set on which f acts bijectively and of a repulsive set,
- $cl(Per(f)) = \Omega$,
- $\forall (x, y_1, \dots, y_k) \in \Omega^2, W^s(x) \pitchfork W^u(y_1) \pitchfork \dots \pitchfork W^u(y_k)$.

Theorem (Mañe, Przytycki, Aoki-Moriyasu-Sumi, Iglesias-Portela-Rovella)

The C^1 -structurally stable endomorphisms of compact manifolds without singularity are AS. Moreover AS endomorphism without singularity that are structurally stable in dimension ?

- C^1 -structural stable diffeomorphisms are not dense, in dimension at least 2.
- C^2 - Ω stable diffeomorphisms are not dense in dimension at least 2,
- AS, conservative diffeomorphisms are Anosov.

AS compact set

Definition

A compact set K be invariant by a C^1 -diffeomorphism f ($f(K) = K$) is **AS** if:

- the non-wandering set Λ of $f|_K$ is hyperbolic,
- $cl(Per(f)) \supset \Lambda$,
- $\forall (x, y) \in K^2, W_\epsilon^s(x) \cap W_\epsilon^u(y) \subset K$.

Theorem (B.)

AS compact set of surface are structurally stable.

Persistence of Bundle normally AS

Theorem (B.)

Let $p : M \rightarrow S$ be a submersion of a compact manifold onto a surface. Let K be a compact subset of S . Let \mathcal{L} be the lamination structure on $p^{-1}(K)$ whose leaves are the fibers of p at K 's points.

Let $f \in \text{Diff}^1(M)$ s.t. :

- f preserves the fibration p ,
- the dynamics induced on the space of the leaves K of \mathcal{L} is AS,
- the \mathcal{L} -saturated of the non-wandering set of $f|_{\mathcal{L}}$ is a normally hyperbolic lamination,

Then the lamination \mathcal{L} is persistent.

Theorem (Thom-Mather)

C^0 equivalently structural stable maps are dense.

C^∞ -equivalently structural stable maps are not dense in
“not-nice” dimension.

Remark

Similar to the structural stability problem for rational function

Stratification

A **stratification** is a locally finite partition Σ of a compact subset A of a manifold M by locally closed subsets, called **strata**, satisfying:

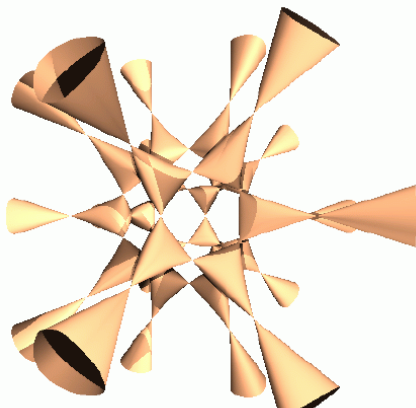
$$\forall (X, Y) \in \Sigma^2, \text{cl}(X) \cap Y \neq \emptyset \Rightarrow \text{cl}(X) \supset Y$$

We write then $X \geq Y$

Whitney Stratification

Theorem (Whitney-Thom-Lojasiewicz)

Any semi-analytic variety $S \subset \mathbb{R}^n$ splits into a stratification Σ by analytic manifold. Moreover this stratification is b -regular.



Thom-Mather Stratification

Theorem (Thom-Mather)

For every C^∞ -generic map from a compact manifold M into N , there exists a stratification on Σ_M of M and a stratification Σ_N on N such that:

- The strata of Σ_M and Σ_N are smooth submanifolds,
- f restricted to each stratum of Σ_M is a submersion onto a stratum of N ,
- This stratification is “stable”.

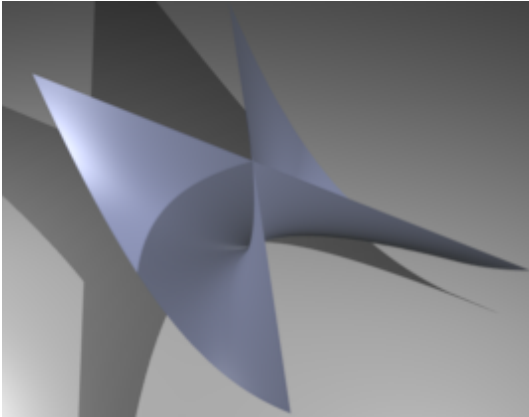


Figure: Whitney Umbrella

Stratification of laminations

A **lamination** of M is a locally compact subset \mathcal{L} of M , presented as a disjoint union of injectively immersed submanifolds (the leaves), whose tangent bundle $T\mathcal{L}$ is continuous on \mathcal{L} .

A **stratification of laminations** is a stratification whose strata have a structure of lamination.

Prop (B.)

For every AS-diffeomorphism f of M , the stable set of every basic piece Λ_i of f has a structure of lamination X_i whose leaves are submanifolds. The family $\Sigma_s := (X_i)_i$ forms a stratification of laminations such that $X_i \leq X_j$ iff $\Lambda_i \succeq \Lambda_j$ i.e. $W^u(\Lambda_i) \cap W^s(\Lambda_j) \neq \emptyset$.

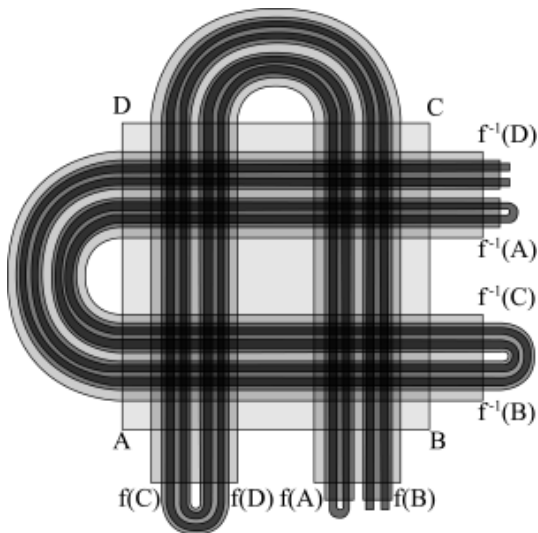


Figure: Smale's Horseshoe

Let $G := (V, A)$ be a finite oriented graph with a manifold M_i associated to each vertex $i \in V$, and with a smooth map $f_{ij} \in C^\infty(M_i, M_j)$ associated to each arrow $[i, j] \in A$ from i to j . For $k \geq 0$, such a graph is C^k -structurally stable if for every $(f'_{ij})_{[i,j] \in A} \approx^{C^\infty} (f_{ij})_{[i,j] \in A}$, there exists $(h_i)_{i \in V} \in \prod_{i \in V} \text{Diff}^k(M_i, M_i)$ s.t.:

$$\forall [i, j] \in A \quad \begin{array}{ccccc} & & f'_{ij} & & \\ & & \rightarrow & & \\ h_i & M_i & & M_j & \\ & \uparrow & f_{ij} & \uparrow & h_j \\ & M_i & \rightarrow & M_j & \end{array} \quad \text{commutes}$$

Theorem (Mather)

Let G be a graph of smooth proper maps, convergent and without cycle. The graph is C^∞ -structurally stable if the following map is surjective:

$$(\sigma_i)_{i \in V} \in \prod_{i \in V} \mathcal{X}^\infty(M_i) \mapsto (Tf_{ij} \circ \sigma_i - \sigma_j \circ f_{ij})_{[i,j] \in A} \in \prod_{[i,j] \in A} \mathcal{X}^\infty(f_{ij}).$$

Let f be a smooth endomorphism of a compact, non necessarily connected manifold. The endomorphism f is **attractor-repellor** if its non-wandering set is the disjoint union of two compact subsets R and A such that:

- the compact subset R is repulsive, but non necessarily transitive,
- the compact subset A is hyperbolic, the restriction of f to A is bijective, and the unstable manifolds of points of A are contained in A . However A is not necessarily transitive.

Theorem (B.)

Let f be an attractor-repellor, smooth endomorphism of a compact, non necessarily connected manifold M . If the following conditions are satisfied, then f is C^∞ -structurally stable:

- (i) the periodic points of f are dense in A ,
- (ii) the singularities S of f have their orbits that do not intersect the non-wandering set Ω ,
- (iii) the restriction of f to $M \setminus \hat{\Omega}$ is C^∞ -infinitesimally stable, with $\hat{\Omega} := cl(\cup_{n \geq 0} f^{-n}(\Omega))$,
- (iv) f is transverse to the stable manifold of A 's points: for any $y \in A$, for any point z in a local stable manifold W_y^s of y , for any $n \geq 0$, and for any $x \in f^{-n}(\{z\})$, we have:

$$Tf^n(T_x M) + T_z W_y^s = T_z M.$$

Step 1

If f is a C^∞ -endomorphism of a smooth manifold M satisfying the hypotheses of the main result, then the stable manifolds of the points of the attractor A form the leaves of a C^∞ -lamination (L^s, \mathcal{L}^s) .

Moreover the compact set $\hat{R} := \bigcup_{n \geq 0} f^{-n}(R)$ is canonically endowed with a compact lamination structure \mathcal{R} whose leaves are backward images by f of points of R . Also M is equal to the disjoint union of L^s with \hat{R} .

def.

A **lamination** is a space that can be locally modeled on the product of \mathbb{R}^d with a compact metric space T

def.

An **endomorphism of lamination** \mathcal{L} is a continuous map f of \mathcal{L} which is locally of the form:

$$(u, t) \in \mathbb{R}^d \times T \rightarrow (\phi_t(u), \psi(t)) \in \mathbb{R}^{d'} \times T',$$

where $(\phi_t)_{t \in T}$ is a continuous family of differentiable maps of \mathbb{R}^d into $\mathbb{R}^{d'}$ and ψ is a continuous map.

The endo f is **transversally bijective** if ψ is bijective for T same enough.

Step 2

Let (L, \mathcal{L}) be a C^∞ -lamination. Let C be a compact subset of L . Let f be a smooth endomorphism of (L, \mathcal{L}) satisfying:

- f is transversally bijective and proper,
- there exists $N > 0$ such that for every $x \in C$, an iterate $f^n(x)$ does not belong to C , for some $n \leq N$,
- the following map is surjective:

$$\chi^\infty(\mathcal{L}) \rightarrow \chi^\infty(f)$$

$$\sigma \mapsto \sigma \circ f - Tf \circ \sigma,$$

with $\chi^\infty(\mathcal{L})$ the space of C^∞ -sections of $T\mathcal{L}$ and $\chi^\infty(f^*)$ the space of C^∞ -sections of $f^*T\mathcal{L} \rightarrow L$.

Then for every f' close to f for the compact-open C^∞ -topology, there exists an isomorphism $I(f')$ of (L, \mathcal{L}) such that for every $x \in C$:

$$f' \circ I(f')(x) = I(f') \circ f(x)$$

Stability under deformation

Deformation

A **deformation** F of an endomorphism f is an application $\mathcal{L} \times \mathbb{R}$ of the form $F : (x, t) \mapsto (f_t(x), t)$ such that $f_0 = f$.

Trivial deformation

Let B be a neighborhood of $0 \in \mathbb{R}$. A deformation $F = (f_t)_t$ of f is **trivial relatively to $C \times B$** if there exists a deformation H of the identity of (L, \mathcal{L}) : $H : (x, t) \in L \times \mathbb{R} \mapsto (h_t(x), t)$, such that for all $(x, t) \in C \times B$: $h_t \circ f(x) = f_t \circ h_t(x)$.

Condition \mathcal{D}

We say that f is **stable under deformations relatively to C** if for any bounded ball B centered at 0, for any deformation $F : (x, t) \mapsto (f_t(x), t)$ of f \mathcal{W}^∞ -close enough to $F_0 := (x, t) \mapsto (f(x), t)$ is trivial relatively to $C \times B$.

Proposition (B.)

Let f be a C^∞ endomorphism of a lamination (L, \mathcal{L}) . Let C be a compact subset of L such that for some $N > 0$ the intersection $\bigcap_{n=0}^N f^{-n}(C)$ is empty. If f is transversally bijective and if:

I) the following map is surjective:

$$\chi^\infty(\mathcal{L}) \rightarrow \chi^\infty(f)$$

$$\sigma \mapsto \sigma \circ f - Tf \circ \sigma,$$

Then:

D) The morphism f is stable under deformations relatively to C .

Toward Algebraization of the problem

Thom-Levine theorem adapted

Let B be a subset of \mathbb{R} and let W be a neighborhood of $C \times B$ in $L \times \mathbb{R}$. There exists a \mathcal{W}^∞ -neighborhood V_ξ of 0 in the space of smooth vector fields with \mathbb{R} -component equal to 0 $\chi^\infty(\mathcal{L} \times \mathbb{R})^0$. A deformation F of f is trivial relatively to $C \times B$, if there exists $\xi \in V_\xi$ such that on W :

$$\tau_F := TF \circ \frac{\partial}{\partial t}|_{L \times \mathbb{R}} - \frac{\partial}{\partial t} \circ F = TF \circ \xi - \xi \circ F.$$

Local existence of ξ

Proposition

Let f be an endomorphism of \mathcal{L} infinitesimally stable. Let P_1 be a small plaque of \mathcal{L} such that $P_{i+1} := f^{-i}(P_1)$ and P_1 have disjoint closure, for every $i > 0$.

Let $s > 0$ be such that P_s is non-empty. Let B be a precompact neighborhood of $0 \in \mathbb{R}$. Then there exists a family of disjoint open neighborhoods $(U_i)_{i=1}^s$ of $(P_i)_{i=1}^s$ s.t. for any deformation F C^0 -close to F_0 , there exists $\xi \in \mathcal{X}^\infty(\mathcal{L} \times \mathbb{R})^0$ which is \mathcal{W}^∞ -close to zero with \mathbb{R} -component equal to 0 and such that :

$$\tau_F(x, t) = TF \circ \xi(x, t) - \xi \circ F(x, t), \quad \forall x \in \prod_{i=2}^s U_i, \quad t \in B.$$

algebraic Setting

Let R'_i be the ring $C^\infty(P_i)$.

For $t \in \mathbb{R}$, let R_i^t be the ring $C_{P_i \times \{t\}}^\infty(P_i \times \mathbb{R})$ of germs at $P_i \times \{t\}$.

We have $R'_i \cong R_i^t / I_i^t$, where I_i^t is the ideal $\{\rho \in R_i^t : \rho|_{P_i \times \{t\}} = 0\}$.

$$\phi_i^t : \rho \in R_i^t \mapsto \rho \circ F_{0|P_{i+1} \times \mathbb{R}} \in R_{i+1}^t$$

is a ring morphism that satisfies

$$\phi_i^t(I_i^t) \subset I_{i+1}^t.$$

The action of ϕ_i^t on $R'_i \cong R_i^t / I_i^t \rightarrow R'_{i+1} \cong R_{i+1}^t / I_{i+1}^t$ is:

$$\phi'_i : \rho \in R'_i \mapsto \rho \circ f \in R'_{i+1}.$$

Let \tilde{R}_i be the ring $C^\infty(P_i \times \mathbb{R})$.

Let X be in the space of smooth deformations of $f|_{\coprod_{i=2}^s P_i}$.

Let R_i^X be the ring $C_{F_0}^0(X, C^\infty(P_i \times \mathbb{R}))$ of germs at $F_0 := f \times id$.

We have $\tilde{R}_i \cong R_i^X / I_i^X$ where I_i^X is the ideal

$\{\rho \in R_i^X : \rho(F_0) = 0\}$.

$$\phi_i^X : (\rho F)_{F \in X} \in R_i^X \mapsto (\rho F \circ F|_{P_{i+1} \times \mathbb{R}})_{F \in X} \in R_{i+1}^X$$

is a ring homomorphism that satisfies

$$\phi_i^X(I_i^X) \subset I_{i+1}^X.$$

The action of ϕ_i^X on $\tilde{R}_i \cong R_i^X / I_i^X \rightarrow \tilde{R}_{i+1} \cong R_{i+1}^X / I_{i+1}^X$ is:

$$\tilde{\phi}_i : \rho \in \tilde{R}_i \mapsto \rho \circ F_0|_{P_{i+1} \times \mathbb{R}} \in \tilde{R}_{i+1}$$

Algebraic formulation of the problem

Let N'_i the R'_i -module $\chi^\infty(P_i)$ of vector fields on P_i .

Let \tilde{N}_i the \tilde{R}_i -module $\chi^\infty(P_i \times \mathbb{R})^0$ of vector fields on $P_i \times \mathbb{R}$ with \mathbb{R} -component equal to zero.

Let N_i^t the R_i^t -module $\chi_{P_i \times \{t\}}^\infty(P_i \times \mathbb{R})^0$ of germs at $P_i \times \{t\}$.

Let N_i^X the R_i^X -module $C_{F_0}^0(X, \chi^\infty(P_i \times \mathbb{R})^0)$.

Let M'_i the R'_i -module $\chi^\infty(f_{|P_i})$ of vector fields along $f_{|P_i}$,

Let \tilde{M}_i be the \tilde{R}_i -module $\chi^\infty(F_{0|P_i \times \mathbb{R}})^0$.

Let M_i^t be the R_i^t -module $\chi_{P_i \times \{t\}}^\infty(F_{0|P_i \times \mathbb{R}})^0$.

Let M_i^X be the R_i^X -module of germs at x_0 of continuous sections σ of the trivial bundle $X \times \chi^\infty(P_i \times P_{i-1} \times \mathbb{R}) \rightarrow X$, such that $\sigma(F)$ belongs to $\chi^\infty(F_{|P_i \times \mathbb{R}})^0$.

For $i \in \{2, \dots, s\}$, the following maps are homomorphisms of respectively R'_i , R_i^t , \tilde{R}_i and R_i^X -modules:

$$\begin{array}{ll} \alpha'_{i,j} : N'_i \rightarrow M'_i & \alpha^t_{i,j} : N_i^t \rightarrow M_i^t \\ \xi \mapsto Tf \circ \xi & \xi \mapsto TF^0 \circ \xi \end{array}$$

$$\begin{array}{ll} \tilde{\alpha}_{i,j} : \tilde{N}_i \rightarrow \tilde{M}_i & \alpha^X_{i,j} : N_i^X \rightarrow M_i^X \\ \xi \mapsto TF^0 \circ \xi & (\xi_F)_F \mapsto (TF \circ \xi_F)_{F \in X} \end{array}$$

Also for $i \in \{1, \dots, s-1\}$, the following maps are module homomorphisms over ϕ'_i , ϕ_i^t , $\tilde{\phi}_i$ and ϕ_i^X respectively:

$$\begin{array}{ll} \alpha'_{i,i+1} : N'_i \rightarrow M'_{i+1} & \alpha^t_{i,i+1} : N_i^t \rightarrow M_{i+1}^t \\ \xi \mapsto \xi \circ f|_{P_{i+1}} & \xi \mapsto \xi \circ F^0|_{P_{i+1} \times \mathbb{R}} \end{array}$$

$$\begin{array}{ll} \tilde{\alpha}_{i,i+1} : \tilde{N}_i \rightarrow \tilde{M}_{i+1} & \alpha^X_{i,i+1} : N_i^X \rightarrow M_{i+1}^X \\ \xi \mapsto \xi \circ F^0|_{P_{i+1} \times \mathbb{R}} & (\xi_F)_F \mapsto (\xi_F \circ F|_{P_{i+1}})_{F \in X} \end{array}$$

For $\delta \in \{', t, \sim, X\}$, we denote by

$$M^\delta := \bigoplus_{i=2}^s M_i^\delta \quad \text{and} \quad N^\delta := \bigoplus_{i=1}^s N_i^\delta$$

the Abelian sum of the modules $(M_i^\delta)_{i=2}^s$ and $(N_i^\delta)_{i=1}^s$ respectively. The modules M^δ and N^δ are modules over the rings $A := \bigoplus_{i=2}^s R_i^\delta$ and $B := \bigoplus_{i=1}^s R_i^\delta$ respectively. Let $I \cdot M^\delta$ (resp. $I \cdot N^\delta$) be the submodule of M^δ (resp. N^δ) spanned by elements of the form $i_k \cdot x_k$, with $i_k \in I_k^\delta$, $x_k \in M_k^\delta$ (resp. $x_k \in N_k^\delta$) and $k \in \{2, \dots, s\}$ (resp. $k \in \{1, \dots, s\}$). Let us consider the (additive) group morphism:

$$\alpha^\delta : N^\delta \rightarrow M^\delta$$

$$(\xi_i)_{i=1}^s \mapsto (\alpha_{i,j}^\delta(\xi_j) - \alpha_{i-1,j}^\delta(\xi_{j-1}))_{i=2}^s.$$

By infinitesimal stability, α' is onto. We want to prove that α^X sends $I^X \cdot N^X$ onto $I^X \cdot M^X$.

Let R and S be rings with units. Let I and J be **Jacobson ideals** (this means that for every $z \in J$, the element $1 + z$ is invertible) in R and S respectively.

Let $\phi : (R, I) \rightarrow (S, J)$ be a ring homomorphism.

ϕ is **adequate** if the following condition is satisfied:

Let A be a finitely generated R -module. Let B and C be S -modules, with C finitely generated over S . Let $\beta : B \rightarrow C$ be a homomorphism of S -modules. Let $\alpha : A \rightarrow C$ be a homomorphism over ϕ (i.e. $\alpha(a + b) = \alpha(a) + \alpha(b)$ and $\alpha(r \cdot a) = \phi(r) \cdot \alpha(a)$, for $a, b \in A$ and $r \in R$). Suppose that:

$$\alpha(A) + \beta(B) + J \cdot C = C.$$

Then we can conclude

$$\alpha(A) + \beta(B) = C \quad \text{and} \quad \alpha(I \cdot A) + \beta(J \cdot B) = J \cdot C.$$

Theorem (Mather)

For any $i \in \{2, \dots, s\}$, the ring homomorphisms:

$$\phi_i^t : (R_i^t, I_i^t) \rightarrow (R_{i+1}^t, I_{i+1}^t)$$

and

$$\phi_i^X : (R_i^X, I_i^X) \rightarrow (R_{i+1}^X, I_{i+1}^X)$$

are adequate.

Theorem (Mather)

Let (R_i, I_i) , be a pair of a be rings with Jacobson ideal.

$$\text{Let } (R_1, I_1) \xrightarrow{\phi_1} (R_2, I_2) \xrightarrow{\phi_2} \dots \xrightarrow{\phi_{s-1}} (R_s, I_s)$$

be a sequence of adequate homomorphisms. For every i , let N_i and M_i be R_i -modules, finitely generated. Let $\phi_{ij} = \phi_j \circ \dots \circ \phi_i$ and let $\alpha_{ij} : N_i \rightarrow M_j$ be a module-homomorphism over ϕ_{ij} .

Let $N := \bigoplus_{i=1}^s N_i$ and $M := \bigoplus_{i=1}^s M_i$ and let

$$\alpha : N \rightarrow M$$

$$(\xi_i)_i \mapsto (\alpha_{11}(\xi_1), \alpha_{12}(\xi_1) + \alpha_{22}(\xi_2), \dots, \alpha_{1s}(\xi_1) + \dots + \alpha_{ss}(\xi_s)).$$

Suppose that $\alpha(N) + \sum_{i=1}^s I_i M_i = M$. Then α sends N onto M and sends $\sum_{i=1}^s I_i \cdot N_i$ onto $\sum_{i=1}^s I_i \cdot M_i$.



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Persistence of stratifications of normally expanded laminations.

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