

Persistence of laminations

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Note on line: www.ihes.fr/~pberger/enseignement.html

Submanifold

definition

A **Submanifold** N of M is a subset such that there exists a family of maps $(\pi_i : U_i \mapsto \mathbb{R}^d)_i$, where:

- (U_i) is an open covering of N ,
- the differential of π_i is onto,
- $\pi_i^{-1}(\{0\}) = N \cap U_i$.

Bundle

definition

A **bundle** is a continuous map $\pi : M \rightarrow B$ of metric spaces such that for every $x \in B$ has an open neighborhood U such that $\pi^{-1}(U) \approx F_x \times U$, and for this identification, π is the first coordinate projection, with $F_x := \pi^{-1}(\{x\})$ the **fiber** of x .

example

- $M \times B \rightarrow B$ is the trivial bundle,
- the tangent space of a manifold is a bundle,
- If $f : M \rightarrow B$ is a C^r -map of compact manifolds with $T_x f$ onto for every $x \in M$ then $M \rightarrow B$ is a C^r bundle,
- Moebius strip.

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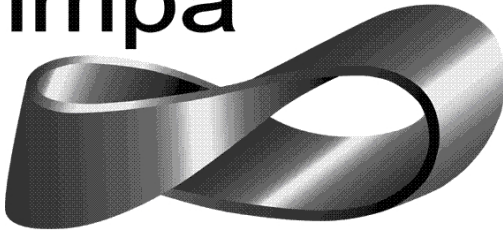


Figure: Moebius strip

Foliation

example

- trajectory of a smooth, non zero vector field,
- smooth bundle,
- Reeb Foliation,
- product of foliations.

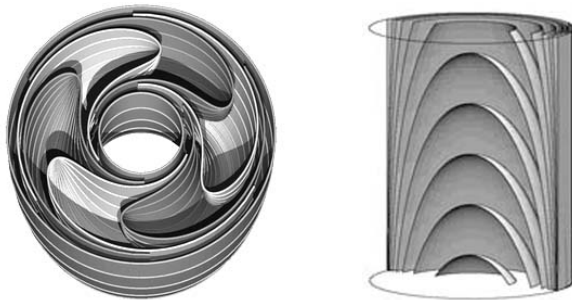


Figure: Reeb foliation (images from I. Voyager and E. Ghys)

definition

A C^r -foliation of a manifold the data of C^r -Bundle $\pi_i : U_i \mapsto \mathbb{R}^d$ where $(U_i)_i$ is a covering of M , such that the fibers are **coherent**:

If $\pi_i^{-1}(\{x\}) \cap \pi_j^{-1}(\{y\}) \neq \emptyset$ then $\pi_i^{-1}(\{x\}) \cap \pi_j^{-1}(\{y\})$ is an open set of $\pi_i^{-1}(\{x\})$.

Lamination

example

- Hyperbolic attractors (Solenoid, DA, Plykin attractors),
- Submanifold,
- product of laminations.



Figure: Smale's Solenoid

definition

A C^r -lamination \mathcal{L} in a manifold the data of a family of C^0 -bundles $\pi_i : U_i \cap \mathcal{L} \mapsto B_i$ where $(U_i)_i$ is a covering of \mathcal{L} , such that:

- the horizontal fibers are **coherent**,
- the **plaque** $\pi_i^{-1}(\{x\})$ is C^r -submanifold (equal to a connected open subset of \mathbb{R}^d) which depends continuously on $x \in B_i$ for every $x \in B_i$ and i .

definition

A **leaf** is a smallest set of \mathcal{L} which intersects all the plaques which intersect it.

Summary of Topological Categories

Submanifold \subset Bundle

\cap

Foliation \subset Lamination

Hyperbolicity implies structural stability

Definition

Let $f \in \text{Diff}^1(M)$ and K be a compact subset f -invariant (i.e. $f(K) = K$). The subset K is **hyperbolic** if there exist two Tf -invariant sub-bundles E^s and E^u , s.t. :

- $TM|_K = E^s \oplus E^u$,
- $\exists \lambda < 1, \forall x \in K :$

$$\|T_x f|_{E^s}\| < \lambda \quad \text{and} \quad \|T_x f|_{E^u}^{-1}\| < \lambda$$

Theorem (60')

Hyperbolic compact subsets are structurally stable.

Def

A compact set K is structurally stable if for $f' \stackrel{C^1}{\approx} f$ there exists a C^0 -embedding i of K s.t.

$$f' \circ i = i \circ f$$

In particular $f'(K') = K'$ with $K' = i(K)$.

Persistence of submanifolds

Definition

Let $f \in \text{Diff}^1(M)$. A C^1 -submanifold N , of class C^1 , which is f -invariant (i.e. $f(N) = N$), is **normally hyperbolic** if there exist two sub-bundles, Tf -invariant, E^s and E^u of $TM|_N$, satisfying:

$$TM|_N = E^s \oplus E^u \oplus TN,$$

such that E^s is uniformly contracted and more than TN and E^u is uniformly expanded and more than TN .

Theorem (Mañé)

The compact submanifolds which are persistent and uniformly locally maximal, for a diffeo, are exactly the compact and normally hyperbolic submanifolds.

Definition

An **embedding** $i : \mathcal{L} \rightarrow M$ is a homeomorphism onto its image, whose restriction to each leaf of \mathcal{L} is an immersion and whose differential along the leaves is continuous on \mathcal{L} (even transversally to the plaques).

fact

The image by an embedding i of a lamination \mathcal{L} of structure $(\pi_j)_j$ is a Lamination with structure $(\pi_j \circ i^{-1})_j$.

Definitions

$f \in \text{Diff}^1(M)$ **preserves** an embedded lamination \mathcal{L} if every leaf of \mathcal{L} is sent by f into a leaf of \mathcal{L} .

The lamination \mathcal{L} is **persistent** if for $f' \stackrel{C^1}{\approx} f$, there exists an embedding $i' \approx i_{can}$ s.t. f' preserves the lamination embedded by i' , and s.t. the dynamics induced by f and f' onto the space of the leaves are the same.

Theorem (Hirsh-Pugh-Shub) '70

Compact invariant laminations, which are normally hyperbolic and plaque-expansive, are persistent.

- Let N be a normally contracted submanifold, canonically embedded in $M = N \times E^s$. Let U be the η nghd of N . Let C be the ϵ nghd of TN in the Grassmannian. $\eta \ll \epsilon \ll 1$
We notice that $\overline{f(U)} \subset U$ and $\overline{Tf(C|U)} \subset C$.

- Put:

$$\Gamma := \{N' : N' \subset U \text{ and } TN' \subset C\}.$$

$$F : \begin{array}{ccc} \Gamma & \rightarrow & \Gamma \\ N' & \mapsto & f(N') \end{array}, \quad \text{if } f' \approx_{C^1} f, \text{ then } F' : \begin{array}{ccc} \Gamma & \rightarrow & \Gamma \\ N' & \mapsto & f'(N') \end{array},$$

$$\bar{\Gamma} \approx \{\sigma : N \rightarrow E'^s : \|\sigma\|_{C^0} \leq \eta, \text{Lip}(\sigma) \leq \epsilon\}, \quad E'^s \approx E^s \text{ smooth}$$

- The set $\bar{\Gamma}$ is a compact convex subset of a normed linear space of continuous sections of $N \times E^s \rightarrow N$.
By Schauder fixed-point theorem, F' has a fixed point N' .

If N' is differentiable but not C^1 , then there exists $(x_n)_n \rightarrow x$, such that $T_{x_n}N' \not\rightarrow T_xN'$:

$$\limsup \angle(T_{x_n}N', T_xN') > \delta > 0$$

By preservation and normal hyperbolicity:

$$\limsup \angle(T_{f'^{-1}(x_n)}N', T_{f'^{-1}(x)}N') = \limsup \angle(Tf'^{-1} \circ T_{x_n}N', Tf'^{-1} \circ T_{x_n}N')$$

And so on:

$$\limsup \angle(T_{f'^{-k}(x_n)}N', T_{f'^{-k}(x)}N') > \frac{\delta}{\lambda^k} > 2\epsilon, \text{ for } k \text{ large enough.}$$

By continuity, $(f'^{-k}(x_n))_n \rightarrow f'^{-k}(x)$, it comes $Lip(N') > \epsilon$.

A contradiction.

Let us show that if N' is Lipschitz then it is differentiable.

For $x \in N'$, $H(x) := \{C \cdot \lim_{\infty} \frac{x-x_n}{\|x-x_n\|} : x_n \rightarrow x, C \in \mathbb{R}\} \subset T_x M$.

Every line of $H(x)$ is contained in the union of planes ϵ -close to TN .

Let $CH(x)$ be the smallest convex set of the Grassmannian of $T_x M$ of $\dim(N)$ -plane such that:

$$\forall D \in H(x), \quad \exists P \in CH(x) : \quad D \subset P$$

Observe that:

$$\frac{\angle H(x)}{\lambda} \leq \angle H(f'(x)) \leq 2\epsilon$$

implies $\angle H(x) = 0$, and so that N' is differentiable.

Construction of the stable set of Normally hyperbolic submanifold

- Let $\epsilon > 0$ and then let E'^s and E'^u be smooth approximations of E^s and E^u .

Using a tubular neighborhood, suppose $M = N \times E'^s \times E'^u$.

$B := N \times B_{E'^s}(\eta) \times E^s(\eta)$ and $\partial^s B := \partial(N \times B_{E'^s}(\eta)) \times B_{E'^u}(\eta)$.

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- Take $\eta \ll \epsilon \ll 1$.

Let C be the ϵ -nghd of $TN \oplus E'^u$ in the Grassmannian.

$\overline{f(B)}$ stretches across B exactly at both sides of $\partial^s B$.

$$\overline{Tf(C|f^{-1}(B))} \subset C.$$

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- Then we proceed as before to show the existence of a $N^s \in \Gamma$ such that $f'(N^s) \supset N^s$.

Doing the same for f^{-1} , we get N^u . We remark that

$N' := N^u \pitchfork N^s$ is f' -invariant.

Definition

Let $f \in \text{Diff}^1(M)$ which preserves an embedded lamination \mathcal{L} . The lamination \mathcal{L} is **plaque-expansive** if, for every $\epsilon > 0$ small enough, for all pseudo-orbits $(x_n)_{n \in \mathbb{Z}}$ and $(y_n)_{n \in \mathbb{Z}}$ ϵ -close and satisfying that $f(x_n)$ (resp. $f(y_n)$) belongs to a plaque containing x_{n+1} (resp. y_{n+1}) and with diameter less than ϵ , then the points x_0 and y_0 belong to a same small plaque.