

Differential equations: Dynamics and Chaos

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Chapter 1

Introduction

1.1 What is a dynamical system?

A dynamical system is a phase space which parametrizes the states of the system together with an evolution rule which predicts the short term evolution. The aim of the dynamical system theory is to understand the long term evolution.

Example 1. *If the system consist of an object falling down, the phase space is formed by 3 coordinates for the position p and 3 coordinates for the velocity v , that is $\mathbb{R}^3 \times \mathbb{R}^3$.*

The short time evolution is given by the Newton rule:

$$\begin{cases} \frac{\partial}{\partial t} \vec{p}(t) = \vec{v}(t) \\ \frac{\partial}{\partial t} \vec{v}(t) = \vec{g}_0 \end{cases}$$

The orbits of the system are the trajectories.

Example 2. *In Canada, there are rabbits and Lynxes. Lynxes eat rabbits. Let x_n be the number of Lynxes at the year n . Let c be the coefficient of reproduction of Lynxes. This coefficient is the half mean number of descendant of Lynxes minus the probability to death, such that $x_{n+1} = c \cdot x_n$.*

The more Lynxes there are, the less rabbit there are, the less food there are, the weaker c is. Thus c is a decreasing function of x . Therefore we can approximate c as a decreasing affine maps of x :

$$c = c(x) = ax + b, \quad \text{with } a < 0 \text{ and } b \text{ constant.}$$

Consequently:

$$x_{n+1} = ax_n^2 + bx_n = f(x_n), \quad \text{with } f(x) = ax^2 + bx.$$

Here the phase space is the number of lynxes, that is \mathbb{R} and the short times evolution is f . The orbit of the system are $(f^n(x))_{n \geq 0}$.

Notations 3. Most of the times, f^n denote the n^{th} iterate of f , that is:

$$\underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}.$$

Example 4. For many sciences such as biology, economy, \dots , all phenomena are assumed linear. Then the phase space is \mathbb{R}^n and the evolution rules is the linear map induced by some matrix $A \in \mathcal{M}_n(\mathbb{R})$. Then the orbit of the point $x \in \mathbb{R}^n$ is $(A^n x)_{n \geq 0}$.

Exercise 5. Compute A^n when A is:

- diagonal,
- diagonalizable,
- trigonal,
- trigonalizable.

Find a strategy, like a plan for a computer program, that return for any $A \in \mathcal{M}_n(\mathbb{R})$, (a very good approximation of) A^n in function of n .

1.2 Plan

In the first chapter we will study the “deterministic” dynamical system, that is when every point is attracted by a point a , and we will show that when the dynamics is a C^1 -diffeomorphism f of \mathbb{R} , the dynamics is identifiable to the one induced by the differential $d_a f$ of f at a .

Beyond determinism, the most important and standard dynamics are related to two mains categories:

- the quasi-periodic behaviors, whose paradigm is a rotation on a circle;
- the hyperbolic (or chaotic) behavior, whose paradigm is the doubling angle map on a circle.

$$\alpha \in \mathbb{R}/\mathbb{Z} \mapsto 2\alpha \in \mathbb{R}/\mathbb{Z}.$$

For chaotic dynamics, Fractal geometry occurs. That is why in chapter 2 we will study them.

In chapter 3, we will see that some chaotic dynamics are very well understood since they are typically identifiable to the dynamics on the space $\{0, 1\}^{\mathbb{N}}$ of infinite words in the alphabet $\{0, 1\}$, and their dynamics are typically identifiable to the shift map σ :

$$\sigma : (x_i)_{i \geq 0} \in \{0, 1\}^{\mathbb{N}} \mapsto (x_{i+1})_{i \geq 0}$$

In the 4th chapter we will see that any perturbation of some chaotic dynamics are identifiable to the initial dynamics.

In the 5^{ve} chapter, we will study the quasi-periodic dynamics by asking when does a map of the circle is identifiable to a rotation.

1.3 Phase spaces, regularity of the dynamics and basic tools

1.3.1 Phase spaces

Our frame work, as a phase space will be either:

- the *euclidean space* \mathbb{R}^n , for some $n \geq 1$,
- the *torus* $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$.

The euclidean space is canonically endowed with the *euclidean metric*:

$$d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2},$$

for any $x, y \in \mathbb{R}^n$, with $x = (x_i)_i$ and $y = (y_i)_i$.

Let us now describe the torus \mathbb{T}^n . We notice that the binary relation \sim on \mathbb{R}^n defined by:

$$(x_i)_i \sim (y_i)_i \Leftrightarrow (x_i - y_i) \in \mathbb{Z}, \forall i \in \{1, \dots, n\}$$

is an equivalence relation.

The equivalence class of $x \in \mathbb{R}^n$, that is the set of the point $y \in \mathbb{R}^n$ equivalent to x , is equal to $x + \mathbb{Z}^n := \{(x_i + n_i)_i; (n_i)_i \in \mathbb{Z}^n\}$, with $x =: (x_i)_i$.

Let \mathbb{T}^n be the space of the equivalence classes. Such a space is endowed with the following canonical metric:

$$d(x + \mathbb{Z}^n, y + \mathbb{Z}^n) = \min_{(m_i)_i, (n_i)_i \in \mathbb{Z}^n} d(x + (m_i)_i, y + (n_i)_i) = \min_{(n_i)_i \in \mathbb{Z}^n} d(x, y + (n_i)_i).$$

For instance, \mathbb{T}^1 is a circle. Thus \mathbb{T}^2 is the product of two circles, or a donut. And so, \mathbb{T}^n is the product of n -circles.

Let $\pi : x \in \mathbb{R}^n \mapsto \pi(x) := x + \mathbb{Z}^n \in \mathbb{T}^n$.

We notice that for all $p \in \mathbb{T}^n$ and $x \in \mathbb{R}^n$ such that $\pi(x)$ is equal to p , the subset $V_x := \{x' \in \mathbb{R}^n; d(x', x) < 1/2\}$ is sent isometrically onto $\{p' \in \mathbb{T}^n; d(p', p) < 1/2\}$ by π . Thus the spaces \mathbb{R}^n and \mathbb{T}^n are locally the same.

Finally we notice that \mathbb{T}^n has a structure of group $+$ since $(\mathbb{Z}^n, +)$ is a subgroup of the commutative group $(\mathbb{R}^n, +)$.

1.3.2 Regularity of the maps of these spaces

Let E be the space \mathbb{R}^n or \mathbb{T}^n . Let F be the space \mathbb{R}^k or \mathbb{T}^k .

A *continuous* function f from E into F is a map such that for any $x \in E$, for any $\epsilon > 0$ there exists $\delta > 0$ such that for every y δ -distant to x the point $f(y)$ is ϵ -distant to $f(x)$.

An *homeomorphism* of E is a bijective, continuous map of E whose inverse is continuous.

A map f from \mathbb{R}^n into \mathbb{R}^k is *differentiable* if for any $x \in \mathbb{R}^n$, there exists a linear map $d_x f$ from \mathbb{R}^n into \mathbb{R}^k such that:

$$f(y) = f(x) + d_x f(y - x) + o(d(x, y)),$$

where $o(d(x, y))$ is a quantity which is small compare to $d(x, y)$ when y approaches x . In other words:

$$\frac{\|o(d(x, y))\|}{d(x, y)} \rightarrow 0, \quad \text{as } y \rightarrow x.$$

A map f from \mathbb{T}^n into \mathbb{R}^k is differentiable if for any $p \in \mathbb{T}^n$, there exists a linear map $d_p f$ from \mathbb{R}^n into \mathbb{R}^k such that:

$$f(p') = f(p) + d_p f \circ \pi_{|V_x}^{-1}(p' - p) + o(d(p, p')),$$

with $x \in \pi^{-1}(p)$ and with $\pi_{|V_x}$ the restriction of π to the neighborhood V_x defined above.

A map f from \mathbb{T}^n into \mathbb{T}^k is differentiable if for any $p \in \mathbb{T}^n$, there exists a linear map $d_p f$ from \mathbb{R}^n into \mathbb{R}^k such that:

$$f(p') = f(p) + \pi \circ d_p f \circ \pi_{|V_x}^{-1}(p' - p) + o(d(p, p')),$$

with $x \in \pi^{-1}(p)$.

A map f from \mathbb{R}^n into \mathbb{T}^k is differentiable if for any $p \in \mathbb{R}^n$, there exists a linear map $d_p f$ from \mathbb{R}^n into \mathbb{R}^k such that:

$$f(p') = f(p) + \pi \circ d_p f(p' - p) + o(d(p, p')).$$

One can show easily that $d_x f$ is then unique. Also, the differentiability of a map implies the continuity of the same map.

A map f from E into F is *continuously differentiable* or of *class C^1* , if it is differentiable and its differential:

$$df : x \in \mathbb{R}^n \mapsto d_x f$$

is continuous.

A C^1 -diffeomorphism of \mathbb{R}^n (resp. \mathbb{T}^n) is a bijective, C^1 -map of \mathbb{R}^n (resp. \mathbb{T}^n) whose inverse is of class C^1 .

Exercise 6. Show that a function f is continuous (resp. C^1) from \mathbb{T}^n into \mathbb{R} iff there exists a continuous (resp. C^1) map g from \mathbb{R}^n into \mathbb{R} such that:

$$g = f \circ \pi.$$

In other words the continuous or C^1 -functions on \mathbb{T}^n are the \mathbb{Z}^n -periodic functions on \mathbb{R}^n .

Exercise 7. Is the following map from \mathbb{T}^{k^2} into \mathbb{R} differentiable?

$$(x, y) \in \mathbb{T}^{k^2} \mapsto d(x, y)^2$$

What is the differential of $x \in \mathbb{R}^k \mapsto \|x\|^2$?

1.3.3 Orbits and dynamical system

As we saw we restrict our study to the phase space M that are Euclidean space or tori.

We will restrict our study to:

- the homeomorphisms of M ,
- the continuous endomorphisms of a M (that a continuous map from the M into itself),
- the diffeomorphisms of M ,
- the C^1 -endomorphisms of M .

When the dynamics f is non-invertible, the orbit of a point $x \in M$ is the sequence $(f^n(x))_{n \geq 0}$. When the dynamics f is invertible, the (resp. forward, back ward) orbit of a point $x \in M$ is the sequence $(f^n(x))_{n \in \mathbb{Z}}$ (resp. $(f^n(x))_{n \geq 0}$, resp. $(f^n(x))_{n \leq 0}$).

1.3.4 Some basic theorems

In this section, we just remind some basic differential theorems of the euclidean space (see ... for a proof). As a torus is locally an euclidean space, and as these result are local, the same statement hold for tori.

Proposition 8 (Chain Rule). *Let E, F, G be euclidean spaces or tori. Let f be a differentiable map from E to F and g be a differentiable map from F to G . Then we have for every $x \in E$:*

$$d_x(g \circ f) = d_{f(x)}g \circ d_x f.$$

Exercise 9. *Does the following map is differentiable : $(x, y) \in \mathbb{R}^{k^2} \mapsto d(x, y)$?*

Theorem 10 (Means values inequality). *Let f be a C^1 map from \mathbb{R}^k to \mathbb{R}^n . Let x and y in \mathbb{R}^k . Then we have:*

$$d(f(x), f(y)) \leq \max_{z \in [x, y]} \|d_z f\| \cdot d(x, y),$$

where $\|d_z f\|$ is the norm of $d_z f$ subordinated to the Euclidean norms.

Theorem 11 (Local inversion theorem). *Let E and F be a torus or/and an euclidean space. Let f be a C^1 map from E to F . If for any $x \in E$ such that $d_x f$ is invertible, then there exists $r > 0$, r_1 such that f is a diffeomorphism from $\{x' \in \mathbb{R}^n : d(x, x') < r\}$ onto its image that contains $\{y \in \mathbb{R}^n : d(f(x), y) < r'\}$.*

Theorem 12 (Global inversion theorem). *Let f be a bijective, C^1 map from \mathbb{R}^k onto itself. If for any $x \in E$ such that $d_x f$ is invertible, then f is a diffeomorphism of \mathbb{R}^k .*

Chapter 2

Local theory in Dynamical system

A basic idea to understand a dynamical system, is to consider its *fixed points*, that is a point sends into itself by the dynamics. This can be generalized to the *periodic points*, that is a point sends into itself by some iterate of the dynamics. These particular points allow us to understand some times the dynamics around them by using the tools explain below.

Let f be a C^1 -function of \mathbb{R}^n into \mathbb{R}^n which fixes the point 0: $f(0) = 0$. We want to compare the dynamics of the differential d_0f of f at 0 with the dynamics of f . Suppose that there exists a *homeomorphism* h of \mathbb{R}^n (that is a continuous, bijective map h of \mathbb{R}^n whose inverse is continuous) such that:

$$f \circ h = h \circ d_0f.$$

Then the orbits of d_0f are sent by h to the orbits of f . As d_0f is a linear map, we understand very well its dynamics. Since such maps f are very rare, we are more interested in the following concept:

Definition 13. *Let f be a map of \mathbb{R}^n into itself which fixes a point $a \in \mathbb{R}^n$. f is linearizable at a if there exists a homeomorphism h of \mathbb{R}^n such that, for any x small enough:*

$$f \circ h(x) = h \circ d_a f(x).$$

Remarks :

- We notice that the maps h sends the (forward/backward) orbits of $d_a f$ which lands close to a to the orbits of f .

- Not all the maps fixing a are linearizable. For instance $f : x \mapsto x - x^3$ is not linearizable at 0. Its derivative at 0 is the identity, the orbits of the identity are single points, but the forward orbit of any point close to zero is a sequence close to zero and converging to zero:

$$|f(x)| = |x||1 - x^2| < |x|, \text{ if } x \neq 0.$$

Nevertheless we have the following theorem:

Theorem 14. *Let f be a C^1 -diffeomorphism which fixes $a \in \mathbb{R}$ and satisfies that $d_a f$ is contracting: there exists $\lambda < 1$ such that for any $u \in \mathbb{R}$:*

$$\|d_a f(u)\| \leq \lambda \|u\|.$$

Then f is linearizable at a .

Remark: The contraction of $d_a f$ is equivalent to the existence $\lambda < 1$ such that

$$\|d_a f(u) - d_a f(v)\| \leq \lambda \|u - v\|, \quad \forall u, v \in \mathbb{R}^n.$$

Example 15. *Let $f : x \mapsto x^3 + 1/3 \cdot x$ such a map is linearizable at 0*

Example 16. *Let $f : (x, y) \in \mathbb{R}^2 \mapsto R_{x^2+y^2}(x/2, y/2)$, where R_α is the rotation centered at 0 and with angle α . Such a map is linearizable at 0. In other words, there exists a homeomorphism h of \mathbb{R}^2 such that for x close to 0:*

$$f \circ h(x) = h \circ H_{1/2}(x),$$

with $H_{1/2}(x, y) = (x/2, y/2)$. Figures ... and ... show the orbits of the maps f and $H_{1/2}$.

fig

The homeomorphism h sends the orbits of $H_{1/2}$ starting at points close to zero to orbits of f .

Actually the above theorem is a special case of the following admitted theorem:

Theorem 17. *Let f be a C^1 -diffeomorphism which fixes $a \in \mathbb{R}$ and satisfies that $d_a f$ has no eigenvalue of modulus 1, then f is linearizable at a .*

Proof of theorem 14 when $n = 1$, $a = 0$ and $f'(0) > 0$. Let $\lambda < 1$ be greater than the derivative $f'(0)$. By the mean value theorem, for $r > 0$ small enough, any point $x \in [-r, r]$ satisfies $|f(x)| < \lambda \cdot |x|$.

Thus, f sends $[-r, r]$ into $(-r, r)$. Also, for all $x \in [-r, r]$ and $n \geq 0$, the norm $|f^n(x)|$ is less than $\lambda^n |x|$ and so $(f^n(x))_n$ converges to 0. Thus $\cup_{n \geq 0} f^n([-r, r] \setminus f([-r, r]))$ is equal to $[-r, r]$ punctured of 0.

Let h_0 be a homeomorphism from $[-r, r] \setminus f((-r, r))$ onto $[-r, r] \setminus (-f'(0) \cdot r, f'(0) \cdot r)$, such that $h_0(r) = r$ and $h_0(-r) = -r$. For instance one can choose the map:

$$x \mapsto \begin{cases} \frac{r-f'(0) \cdot r}{r-f(r)}(x-r) + r & \text{if } x > 0 \\ \frac{r-f'(0) \cdot r}{r+f(-r)}(x+r) - r & \text{if } x < 0 \end{cases}$$

Let h be the map of \mathbb{R} defined by :

$$h(x) = \begin{cases} x & \text{if } |x| > r \\ 0 & \text{if } x = 0 \\ d_0 f^n \circ h_0 \circ f^{-n}(x) & \text{if } x \in f^n([-r, r] \setminus f([-r, r])) \end{cases}$$

We notice that h is a homeomorphism of \mathbb{R} . Moreover, h satisfies for any $x \in f^n([-r, r] \setminus f([-r, r]))$ with $n > 1$:

$$d_0 f \circ h(x) = d_0 f^{n+1}(0) \circ h_0 \circ f^{-n}(x) = d_0 f'^{n+1} \circ h_0 \circ f^{-n-1} \circ f(x) = h \circ f(x)$$

as $f(0) = h(0) = 0$, for any $x \in f([-r, r])$,

$$d_0 f(0) \circ h(x) = h \circ f(x).$$

□

Proof of theorem 14 when $n \geq 1$. First of all we may assume that $a = 0$ by considering the map $x \mapsto f(x - a)$ instead of f . Then if the map h is convenient for $f(x - a)$, the map $h(x + a)$ is convenient for f .

Let us denote by B_r the ball centered at 0 and with radius r :

$$B_r := \{x \in \mathbb{R}^n; \|x\| < r\}.$$

Let us denote by S_r the sphere centered at 0 and with radius r :

$$S_r := \{x \in \mathbb{R}^n; \|x\| = r\}.$$

The idea of the proof is the same as in the case $n = 1$ unless that the homeomorphism $h_0 : B_r \setminus f(B_r) \rightarrow B_r \setminus d_0f(B_r)$ for $r > 0$ small is harder to construct since:

1. the existence of such a homeomorphism is not obvious,
2. h_0 has to glue well with $d_0f \circ h_0 \circ f^{-1}$ at $f(S_r)$:

$$h_0(x) = d_0f \circ h_0 \circ f^{-1}(x), \quad \forall x \in f(S_r).$$

As f is contracting, there exists λ such that for r sufficiently small, f sends B_r into $B_{\lambda r}$.

Let ρ be a C^1 -function equal to 1 at $[0, \lambda]$ and equal to 0 on $[1, \infty)$.

Let $h_0(x) = x + \rho(\|x\|^2/r^2) \cdot (d_0f \circ f^{-1}(x) - x)$, for $r > 0$ small and with f^{-1} the inverse function of f .

We will show below that, for $r > 0$ small enough, h_0 is a homeomorphism from $B_r \setminus f(B_r)$ onto $B_r \setminus d_0f(B_r)$ such that condition 2 above is satisfied. Then, we can proceed as in the proof of the theorem for $n = 1$ to conclude.

To show that h_0 is a diffeomorphism, we will prove that h_0 is equal to the identity id of \mathbb{R}^n minus a $1/2$ -contracting maps¹ ϵ of \mathbb{R}^n , for $r > 0$ small enough. Therefore the differential of h_0 is invertible for every $x \in \mathbb{R}^n$. To use the global inversion theorem (th. 12), we shall prove that h_0 is a bijection onto \mathbb{R}^n .

Let $x, y \in \mathbb{R}^n$ sent by h_0 to a same point. We have:

$$h_0(x) = h_0(y) \Leftrightarrow x + \epsilon(x) = y + \epsilon(y) \Leftrightarrow x - y = \epsilon(y) - \epsilon(x)$$

Thus by contraction of ϵ , the point x and y are the same. Consequently, h_0 is injective.

¹for all x, y , $d(\epsilon(x) - \epsilon(y)) \leq 1/2d(x, y)$

As h_0 is equal to id on $\mathbb{R}^n \setminus B_r$, the image of h_0 contains $\mathbb{R}^n \setminus B_r$. As h_0 is equal to $d_0f \circ f^{-1}(x)$ on $f(B_r)$, h_0 sends $f(B_r)$ onto $d_0f(B_r)$. Consequently, by injectivity, h_0 sends $B_r \setminus f(B_r)$ into $B_r \setminus d_0f(B_r)$. It remains to prove that h_0 sends $B_r \setminus f(B_r)$ onto $B_r \setminus d_0f(B_r)$. Let $y \in B_r \setminus d_0f(B_r)$. The line $(0, y)$ intersects $S_r \cup B_r \setminus d_0f(B_r)$ into segments. Let $[y_1, y_2]$ be the one that contains y , with y_1 in S_r and y_2 in $d_0f(S_r)$. Let t be the supremum of the $t' \in [0, 1]$ such that $[y_1, y_1 + t(y_2 - y_1)]$ is included in the image of h_0 . If $t = 1$, then y is in the image of h_0 and so we have shown the surjectivity of h_0 . Let us suppose that $t < 1$. Let us show that $y_0 := y_1 + t(y_2 - y_1)$ belongs to the image of h_0 . By definition of t , there exists $(x_n)_{n \geq 0} \in h^{-1}([y_1, y_1 + t(y_2 - y_1)]) \subset B_r \setminus d_0f(B_r)$ such that $(h_0(x_n))_n$ converges to y_0 . As $B_r \setminus d_0f(B_r)$ is bounded, we can find a subsequence $(x_{n_i})_{i \geq 0}$ which converges in \mathbb{R}^n (see section 3.2 for more details). Let x its limit. We notice that x belongs to $S_r \cup B_r \setminus f(B_r)$. By continuity, h_0 sends x to y_0 . By the local inversion theorem, there exists a $r' > 0$, such that $\{y \in \mathbb{R}^n : d(y_0, y) < r'\}$ is contained in the image of h_0 . This is a contradiction with the fact that t is the supremum.

Let us show that h_0 is equal to $id - \epsilon$ with ϵ $1/2$ -contracting and fixing 0 when r is small enough. Let $\epsilon(x) = x - h_0(x) = -\rho(\|x\|^2/r^2) \cdot (d_0f \circ f^{-1}(x) - x)$. We notice that ϵ fixes 0 since h_0 and id fix 0. Moreover

$$d_x \epsilon = -d_{\|x\|^2/r^2} \rho \cdot \frac{\langle 2x, \cdot \rangle}{r} \cdot \frac{d_0f \circ f^{-1}(x) - x}{r} - \rho(\|x\|^2/r^2) \cdot (d_0f \circ d_x f^{-1} - id).$$

As f is of class C^1 , the differential $d_x f$ is close to d_0f for x close to 0. Thus $d_0f \circ d_x f^{-1} - id$ is close to 0 when x is close to 0. Thus, for r small enough, for any $x \in B_r$ and $u \in \mathbb{R}^n$:

$$\begin{aligned} \|(d_0f \circ d_x f^{-1} - id)(u)\| &\leq 1/4 \|u\| \\ \Rightarrow \|\rho(\|x\|^2/r^2) \cdot (d_0f \circ d_x f^{-1} - id)(u)\| &\leq 1/4 \|u\| \end{aligned}$$

By definition of the derivative, $\|f(x) - d_0f(x)\|/\|x\|$ is small when x is small. Thus for r small and any $x \in B_r$, $\frac{d_0f \circ f^{-1}(x) - x}{r}$ is small. Consequently, for r small enough, for any $x \in B_r$ and $u \in \mathbb{R}^n$:

$$\left\| -d_{\|x\|^2/r^2} \rho \cdot \frac{\langle 2x, u \rangle}{r} \frac{d_0f \circ f^{-1}(x) - x}{r} \right\| \leq \frac{\|u\|}{4}.$$

Therefore, $\|d_x \epsilon(u)\| \leq \|u\|/2$, for any r small enough, $x \in B_r$ and $u \in \mathbb{R}^n$. By the mean values theorem ϵ is well $1/2$ -contracting. \square

Exercise 18. Find a diffeomorphism of \mathbb{R}^2 fixing 0, whose differential at 0 is rotation with irrational angle but which is not linearizable at 0.

Exercise 19. linearization of $z \mapsto z^2$, regularity in dimension 1.

Chapter 3

A geometry of chaos: Fractals

3.1 Motivations

This field of dynamical system was initiated by Benoit Mandelbrot in the 80's to model many natural geometry such as:

- the geometry of geography phenomena like the design of a coast(fig ..), a mountains (fig..), rivers (fig. ...),
- the geometry of many organism like a cauliflower (fig. ..., a tree (fig..), a lung (fig..),
- the geometry of a many object in chaotic dynamics that we will see further (fig ...).

One can define informally the fractal by satisfying some of these properties:

- It has a fine structure at arbitrarily small scales.
- It is too irregular to be easily described in traditional Euclidean geometric language.
- It is self-similar (at least approximately or stochastically).
- It has a a length, surface, volume, equal to infinity or 0 ; more precisely its (Hausdorff) dimension is not an integer.

We propose to give a dynamical construction of most of these objects. For this we need some basic elements of topology.

3.2 Compact subsets of \mathbb{R}^n

Let us fix $n \geq 1$. We remind that \mathbb{R}^n is canonically equipped with the following norm:

$$u = (u_i)_{1 \leq i \leq n} \in \mathbb{R}^n \mapsto \|u\| := \sqrt{\sum_{i=1}^n u_i^2}$$

This is a norm define the distance:

$$d : (u, v) \in (\mathbb{R}^n)^2 \mapsto \|u - v\| \in \mathbb{R}^+$$

To say that a function d is a *distance* means that it satisfies:

Symmetry property: $\forall u, v \in \mathbb{R}^n, d(u, v) = d(v, u).$

Definition property: $\forall u, v \in \mathbb{R}^n, d(u, v) = 0 \iff u = v.$

Triangular inequality: $\forall u, v, w \in \mathbb{R}^n, d(u, w) \leq d(u, v) + d(v, w).$

We remind that a *ball centered at $x \in \mathbb{R}^n$ with radius $r \geq 0$* is the subset:

$$B(x, r) := \{y \in \mathbb{R}^n; d(x, y) < r\}.$$

A subset O of \mathbb{R}^n is said to be *open* if its is empty or satisfy that for any $x \in O$ there exists $r > 0$ such that $B(x, r)$ is included in O .

Example 20. \mathbb{R} is open, a ball is open, a union of ball is open, $\mathbb{R}^{n-1} \times (0, \infty)$ is open.

Property 21. Any (possibly infinite) union of open subsets of \mathbb{R}^n is open. Any finite intersection of open subsets is open.

Proof. Let $(O_i)_i$ be a possibly infinite family of open subsets of \mathbb{R}^n . Let $x \in \cup_i O_i$. Thus x belongs to the subset O_j , for some j . As O_j is open there exists $r > 0$ such that $B(x, r)$ is contains into O_j , and so in $\cup_i O_i$.

Let $(O_i)_{1 \leq i \leq k}$ be a finite family of open subsets of \mathbb{R}^n . Let $x \in \cap_i O_i$. Thus x belongs to each subset O_i . As O_i is open there exists $r_i > 0$ such that $B(x, r_i)$ is contains into O_i . Thus $B(x, r)$ is contains in $\cap_i O_i$, with $r := \min_i r_i > 0$. □

A subset C of \mathbb{R}^n is said to be *closed* if its complement $F^c := \{x \in \mathbb{R} : x \notin F\}$ is open.

Property 22. Any (possibly infinite) intersection of closed subsets of \mathbb{R}^n is closed. Any finite union of closed subsets is closed.

Proof. Exercise. □

Remark: A subset can be neither closed, neither open. For instance: $[0, 1) \subset \mathbb{R}$.

Exercise 23. Say for each of these subsets if they are closed, open or nothing in \mathbb{R}^2 :

1. $B(x, r) \setminus \{0\}$,
2. $\{y \in \mathbb{R}^2 : 0 < \|y\| \leq 1\}$,
3. $\{y \in \mathbb{R}^2 : 0 < \|y\| < 1\}$,
4. $(0, 1] \times \{0\} \cup \{0\} \times \mathbb{R}$.

Property 24. Let F be a subset of \mathbb{R}^n . Then F is closed if and only if for any sequence $(x_n)_n \in F^{\mathbb{N}}$ converging to some $x \in \mathbb{R}^n$ has its limit x in F .

Proof. \Rightarrow) We suppose F closed. Let $(x_n)_n$ be a sequence of $F^{\mathbb{N}}$ converging to some $x \in \mathbb{R}^n$. Suppose for the sake of a contradiction that x belongs to $\mathbb{R}^n \setminus F$. As $\mathbb{R}^n \setminus F$ is open, there exists $r > 0$ s.t. $B(x, r)$ is included in $\mathbb{R}^n \setminus F$. As $(x_n)_n$ converges to x , for n sufficiently large, x_n belongs to $B(x, r)$ and so to $\mathbb{R}^n \setminus F$. Contradiction.

\Leftarrow) for any sequence $(x_n)_n \in F^{\mathbb{N}}$ converging to some $x \in \mathbb{R}^n$ has its limit x in F . Let us suppose by the sake of a contradiction that F is not closed that is $\mathbb{R}^n \setminus F$ is not open. Therefore there exists $x \in \mathbb{R}^n \setminus F$ s.t. for any $r > 0$ the ball $B(x, r)$ is not included in $\mathbb{R}^n \setminus F$. Thus there exists $x_n \in B(x, 1/n) \cap F$ for any $n \geq 0$. Such a sequence $(x_n)_n \in F^{\mathbb{N}}$ converges to $x \in \mathbb{R}^n \setminus F$. Contradiction □

The *diameter* of a subset E of \mathbb{R}^n is the positive number:

$$\text{diam } E := \sup_{x, y \in E} d(x, y).$$

for instance the diameter of $[-1, 1]$ is 2, the diameter of $[-1, 1]^2$ is $\sqrt{2}$, the diameter of $[-1, 1]^n$ is \sqrt{n} , the diameter of $B(x, r)$ is $2r$. A subset B of \mathbb{R}^n is *bounded* if it has finite diameter. A subset K of \mathbb{R}^n is *compact* if it is bounded and closed.

Exercise 25. Prove that the subset K of $[0, 1]$ formed by the points $[0, 1]$ without 4, 5 and 6 in their decimal expression (for instance $0, 13278290137$ but not $0, 0, 132782690137$) is a compact subset of \mathbb{R} .

Felix Hausdorff (1968–1942) has defined a very useful distance on the set of nonempty compact subset of \mathbb{R}^n . The *Hausdorff distance* d_{HD} between two compact subsets K, K' is defined by :

$$d_{HD}(K, K') := \sup_{x \in K} d(x, K') + \sup_{x' \in K'} d(x', K)$$

with $d(x, K') := \inf_{x' \in K'} d(x, x')$ and $d(x', K) := \inf_{x \in K} d(x, x')$. d_{HD} is a distance on the set \mathcal{K}_n of nonempty compact subsets of \mathbb{R}^n since it satisfies the triangular inequality, the symmetry and definition properties explain above.

Proposition 26. Let $(K_p)_{p \geq 0}$ be sequence of compact subsets of \mathbb{R}^n . If $\bigcap_{p=0}^N K_p$ is not empty for any $N \geq 0$ then $\bigcap_{p \geq 0} K_p$ is empty .

Proof. For any $N \geq 0$, let x_N be in $\bigcap_{p=0}^N K_p$. The idea of the proof is to find an increasing sequence $(N_i)_i \in \mathbb{N}^{\mathbb{N}}$ such that $(x_{N_i})_i$ converges to some $x \in \mathbb{R}$. As each K_p is closed and $(x_{N_i})_{i \geq p}$ lands in K_p , the point x belongs to K_p and so to $\bigcap_p K_p$. Consequently this intersection is not empty.

Let us construct $(n_i)_i \in \mathbb{N}^{\mathbb{N}}$. As K_1 is compact it is include in $[-M, M]^n$ for some $M \geq 0$. Thus $\bigcap_{p=0}^N K_p$ is included in $[-M, M]^n$ for any $N \geq 0$. Let $N_i \geq 0$ and $(a_{ik})_{1 \leq k \leq n} \in \{-2^i, -2^i + 1, \dots, 0, \dots, 2^i\}$ such that $\prod_{k=1}^n [-M \cdot a_{ik}/2^i, M \cdot (a_{ik} + 1)/2^i]$ contains x_{N_i} and infinitely many elements of $(x_N)_N$. As we can share $\prod_{k=1}^n [-M \cdot a_{ik}/2^i, M \cdot (a_{ik} + 1)/2^i]$ in finitely many (2^n) pieces of the form $\prod_{k=1}^n [-M \cdot b_{ik}/2^{i+1}, M \cdot (b_{ik} + 1)/2^{i+1}]$ with $b_{ik} \in \{2a_{ik}, 2a_{ik} + 1\}$, there exists $a_{i+1 k} \in \{2a_{ik}, 2a_{ik} + 1\}$ such that $\prod_{k=1}^n [-M \cdot a_{i+1 k}/2^{i+1}, M \cdot (a_{i+1 k} + 1)/2^{i+1}]$ contains infinitely many $(x_n)_n$. Let $n_{i+1} > n_i$ be in $\prod_{k=1}^n [-M \cdot a_{i+1 k}/2^{i+1}, M \cdot (a_{i+1 k} + 1)/2^{i+1}]$.

We notice that by induction $(N_i)_i$ is an increasing sequence. Moreover, the sequence $(x_{N_i})_i$ converges since its i first digits of the binary expression of each coordinate of x_{N_i} is the same as the one of x_{N_j} for $j \geq i$. \square

Exercise 27. Prove that d_{HD} is a distance, that is for any $K, K', K'' \in \mathcal{K}_n$ we have:

$$d(K, K') = 0 \Leftrightarrow K = K', \quad d(K, K') = d(K', K), \quad d(K, K'') \leq d(K, K') + d(K', K'').$$

Definition 28 (Property). *Let E be a subsets of \mathbb{R}^n . The closure of E is the smallest closed subset of \mathbb{R}^n containing E , that is the intersection of all closed subset of \mathbb{R}^n containing E and consists of the limits of all converging sequences of E .*

Proof. Let C_1 be the intersection of all closed subset of \mathbb{R}^n containing E . By property ... , the subset C_1 is closed. As E is included in C_1 by property ..., C_1 contains the subset C_2 consisting of the limits of all converging sequences of E . To show that C_1 and C_2 are equal, it is sufficient to show that C_2 contains C_1 and so that C_2 is closed. Let $(x_k)_k$ be a sequence of points in C_2 converging to some point x . Let us show that x belongs to C_2 . For any $p \geq 1$, there exists k_p such that $d(x_{k_p}, x)$ is less than $1/p$. As x_{k_p} belongs to C_2 , there exists $y_p \in E$ $1/p$ -close to x_{k_p} . Thus $y_p \in E$ and x are $2/p$ -close. Therefore $(y_p)_p \in E^{\mathbb{N}}$ converge to x . \square

Exercise 29. *Find the closure of the subsets of exercise 23.*

3.3 Construction of mathematical Fractals

Let $(f_i)_{1 \leq i \leq k}$ be a family of contracting homeomorphism of \mathbb{R}^n . That is for any $i \in \{1, \dots, k\}$, there exists $\lambda_i < 1$ such that for all $(x, y) \in \mathbb{R}^n$:

$$d(f_i(x), f_i(y)) \leq \lambda_i \cdot d(x, y).$$

Theorem 30. *For $K \in \mathcal{K}_n$, let $F(K)$ be the subset $\cup_{i=1}^k f_i(K)$ of \mathbb{R}^n . Then $F(K)$ is compact and the map:*

$$F : K \in \mathcal{K}_n \rightarrow F(K) \in \mathcal{K}_n,$$

has a unique fixed point.

Example 31. - *snow flake, fractal cube, trees + function*

Exercise 32. *Find for each of these following fractals some functions (f_i) making them.*

Proof of theorem 30. 1. $F(K)$ is compact.

Let $K \in \mathcal{K}_n$. We will prove that each subset $f_i(K)$ is a nonempty compact. Then as a finite union of bounded subset is bounded, $F(K)$ is bounded. And as a finite union of closed subset is closed (prop ..), the $F(K)$ is closed and so compact.

Let us prove that $f_i(K)$ is compact. First of all, as f_i is contracting, the diameter of $f_i(K)$ is less than the diameter of K . Thus $f_i(K)$ is bounded.

To show that $f_i(K)$ is closed, we have to prove that its complementary $f_i(K)^c$ is open. As f_i is bijective, $f_i(K)^c$ is equal to the image $f_i(K^c)$ of the complementary of K by f_i . Let $y = f_i(x)$ be in $f_i(K^c)$. As K is close, its complementary K^c is open. Thus there exists $r > 0$ such that the ball $B(x, r)$ is included in K^c . As f_i^{-1} is continuous, the points close to y are sent by f_i^{-1} to points close to x . Thus, for $\delta > 0$ small enough $B(y, \delta)$ is sent by f_i^{-1} into $B(x, r)$. Consequently $B(y, \delta)$ is included in $f_i^{-1}(B(x, r)) = f_i(B(x, r))$. As $B(x, r)$ is contained into K^c , the ball $B(y, \delta)$ is included in $f_i(K^c) = f_i(K)^c$. As this is true for any $y \in f_i(K)^c$, $f_i(K)^c$ is open and $f_i(K)$ is closed.

2. F is contracting for d_{HD} .

We want to show that for any $K, K' \in \mathcal{K}_n$, we have:

$$d_{HD}(F(K), F(K')) \leq \lambda \cdot d_{HD}(K, K'),$$

with $\lambda := \max(\lambda_i, i \in \{1, \dots, k\})$.

we remind that

$$d_{HD}(F(K), F(K')) = \sup_{y \in F(K)} d(y, F(K')) + \sup_{y' \in F(K')} d(y', F(K)).$$

Let us now evaluate $\sup_{y \in F(K)} d(y, F(K'))$:

$$\begin{aligned} \sup_{y \in F(K)} d(y, F(K')) &= \max_i \sup_{y \in f_i(K)} d(y, F(K')) = \max_i \min_j \sup_{y \in f_i(K)} d(y, f_j(K')) \\ &\leq \max_i \sup_{y \in f_i(K)} d(y, f_i(K')) = \max_i \sup_{x \in K} d(f_i(x), f_i(K')) \\ &\leq \max_i \lambda_i \cdot \sup_{x \in K} d(x, K') = \lambda \sup_{x \in K} d(x, K') \end{aligned}$$

Thus

$$\sup_{y \in F(K)} d(y, F(K')) \leq \lambda \sup_{x \in K} d(x, K')$$

And similarly:

$$\sup_{y' \in F(K')} d(y', F(K)) \leq \lambda \sup_{x' \in K'} d(x', K)$$

Consequently:

$$d_{HD}(F(K), F(K')) \leq \lambda \sup_{x \in K} d(x, K') + \lambda \sup_{x' \in K'} d(x', K) \leq \lambda \cdot d_{HD}(K, K').$$

3. F has a unique fixed point in \mathcal{K}_n

We will prove in, that any contracting maps of \mathcal{K}_n has a unique fix point. □

Exercise 33. Draw the shape of the fractal obtained by the following functions:

- $\mathbb{R}^n = \mathbb{C}$, $f_1 := z \mapsto (1/2)(z - i)$, $f_2 := z \mapsto \frac{e^{-i\pi/6}}{2}(z + i)$, $f_3 := z \mapsto \frac{e^{i\pi/6}}{2}(z + i)$,
- $\mathbb{R}^n = \mathbb{C}$, $f_1 := z \mapsto \frac{e^{i\pi/6}}{2}(z - 2)$, $f_2 := z \mapsto \frac{e^{i\pi/6}}{2}(z + 2)$

Picture obtained by computer programming are even better!

Exercise 34. show that the maps $f : x \in [-2, 2] \mapsto x^2 - 2$, $g : x \in \mathbb{R}/2\pi\mathbb{Z} \mapsto 2x$ and $h : x \in \mathbb{R}/2\pi\mathbb{Z} \mapsto \cos(x)$ satisfy:

$$f \circ h = h \circ g$$

conclude that the set of points of $[-2, 2]$ whose orbit does not intersect an open neighborhood of $\{0\}$ is (homeomorphic) to a fractal.

Chapter 4

Coding chaotic dynamics

4.1 Symbolic dynamics

Giving an integer $n \geq 0$, we denote by Σ_n the set $\{1, \dots, n\}^{\mathbb{N}}$ of sequences $(a_i)_{i \geq 0}$ of integers in $\{1, \dots, n\}$. On Σ_n , we define the shift σ dynamics:

$$\begin{aligned}\Sigma_n &\rightarrow \Sigma_n \\ (a_i)_{i \geq 0} &\rightarrow (a_{i+1})_i\end{aligned}$$

Such a dynamical system (σ, Σ_n) is quite simple and thus would be a good model to describe the expanding maps. Let us improve this family of models by considering some restrictions of these maps. Let $\mathcal{M}_n(\{0, 1\})$ be the set of matrices with entries equal to 0 or 1 and of dimension $n \times n$.

For $A \in \mathcal{M}_n(\{0, 1\})$, let Σ_A be the subset of Σ_n formed by the sequences $(a_i)_i$ such that for every i , the entry in the a_i^{th} column and the a_{i+1}^{th} line is positive.

For $n \geq 0$ and $A \in \mathcal{A}_n$, we endow Σ_A with the following distance:

$$d((a_i)_i, (a'_i)_i) = \sum_{i \geq 0} \frac{|a_i - a'_i|}{2^i}$$

We notice that for this distance, the balls are of the form:

$$\{(a'_i)_i \in \Sigma_A : a_i = a'_i, \forall i \leq N\},$$

where $(a_i)_i$ is some element in Σ_A and N is an integer.

We notice that these balls (which generate the topology) are open and closed.

Proposition 35. *The space Σ_A is totally disconnected: for any two distinct points $a, a' \in \Sigma_A$, there exist two open neighborhoods V_a and $V_{a'}$ of respectively a and a' , that are disjoint but cover Σ_A .*

Proof. Let k the first integer such that $a_k \neq a'_k$. Let V_a be the ball centered at a and with radius $1/2^k$. Let $V_{a'}$ be the complement of a . As V_a is open and closed, $V_{a'}$ is well open, moreover $V_{a'}$ contains a' . \square

We remark that the shift map σ is continuous for this distance.

Proposition 36. *The space Σ_A is compact.*

Proof. Let $(a^k)_k$ be a sequence in Σ_A . We want to construct a sequence of infinite subsets $(J_l)_l$ of \mathbb{N} such that for every $k, k' \in J_l$ the l first coordinates of a^k and $a^{k'}$ are equal. The construction of the $(J_l)_l$ is done by induction as in the proof of the completeness of \mathbb{R} . (for instance as the first digit of a_k belongs to the finite set $\{1, \dots, n\}$ and as there infinitely $(a^k)_k$, one of these digits $j_0 \in \{1, \dots, n\}$ is taken infinitely many times. Let J_0 be the infinite subset of \mathbb{N} formed by the integer k such that a_1 is equal to j_0).

Let us now define an increasing sequence of integers $(n_i)_i$ by induction: n_0 is the minimum of J_0 , n_{i+1} is the minimum of $J_{n+1} \cap [n_n + 1, \dots, \infty[$. We notice that the sequence $(a^{n_i})_i$ converges. \square

A matrix A of $\mathcal{M}_n(\{0, 1\})$ is said to be non nilpotent if, for every $k \geq 0$, the product A^k is nonzero. We denote by \mathcal{A}_n the subset of non nilpotent matrices.

Proposition 37. *For $A \in \mathcal{M}_n(\{0, 1\})$, the set Σ_A is non-empty if and only if the A is non nilpotent.*

Proof. If Σ_A is non empty there exists $(a_i)_i \in \Sigma_A$. We notice that if $A = [\alpha_{kl}]$ and $A^i := [\alpha_{kl}^{(i)}]$. We notice that $\alpha_{a_1, a_0} = \alpha_{a_1, a_0}^{(1)}$ is nonzero, by definition of Σ_A . Suppose by induction that $\alpha_{a_i a_0}^{(i)}$ is non zero for some $i \geq 0$. Also, we have:

$$\alpha_{kl}^{(i+1)} = \sum_{p=1}^n \alpha_{kp} \cdot \alpha_{pl}^{(i)}. \quad (4.1.1)$$

We notice that α_{a_{i+1}, a_i} is non zero. Thus by the above formula $\alpha_{a_{i+1}, a_0}^{(i+1)}$ is nonzero since $\alpha_{a_i a_0}^{(i)}$ is nonzero (the terms of the sum are non negative). Thus by induction, A^i is nonzero for every i . In other words, A is non nilpotent.

Suppose that A is non nilpotent. Let us show by induction on $i \geq 1$, if $\alpha_{kl}^{(i)}$ is nonzero, then there exists a family $(a_j^i)_{k=0}^i$ such that:

- $\alpha_{a_{j+1}^i, a_j^i}$ is non zero,
- a_i^i is equal to k .

This is obvious when $i = 1$. Let $i \geq 0$. By equation (4.1.1), if $\alpha_{kl}^{(i+1)}$ is nonzero, then α_{kp} and $\alpha_{pl}^{(i)}$ are non zero for some p . By induction hypothesis, there exists a family $(a_j^i)_{j=0}^i$ such that:

- $\alpha_{a_{j+1}^i, a_j^i}$ is non zero,
- a_i^i is equal to p .

We notice that $a_j^{i+1} := a_j^i$ if $j \leq i$ and $a_{i+1}^{i+1} = k$ satisfies the requested properties.

Let $a_j^i := 1$ for every $j > i$. Let $a^i := (a_j)_{j \geq 0}$. Now we can extract extract as in the proof of the compactness of Σ_A a subsequence $(a^{n_i})_i$ that converges to some a . We easy show that a belongs to Σ_A . \square

Definition 38. A matrix $A \in \mathcal{M}_n(\{0, 1\})$, is said to be irreducible if there exists $n \geq 0$ such that A^n has only positive entries.

Remark: If A is irreducible, then A is non nilpotent.

Exercise 39. Show that if A is irreducible then σ is transitive: it has a dense orbits.

Definition 40. Let f be a map from a metric space (E, d) into itself. The map f is said to be ϵ -expansive, if the following property holds:

For every points $x, y \in E$, if $f^n(x), f^n(y)$ are at most ϵ distant for every $n \geq 0$, then x and y are equal.

The map f is said to be expansive if f is ϵ -expansive for some $\epsilon > 0$.

Proposition 41. for all $n \geq 0$ and $A \in \mathcal{A}_n$, the shift σ is expansive on Σ_A .

Proof. Let us show that σ is $1/2$ -expansive. Let $x = (a_i)_i, y = (a'_i)_i \in \Sigma_A$, s.t. $f^n(x), f^n(y)$ are at most $1/2$ distant for every $n \geq 0$. Thus, for every $i \geq 0$, $|a_i - a'_i| \leq d(\sigma^i(x), \sigma^i(y)) \leq 1/2$. As a_i and a'_i are integer, they are equal, for every i . Thus, x and y are equal. \square

Parler de la decomposition en composante irreductible. Devaney §1.6

4.2 Expanded compact subset

In all that follow n is a positive integer and f is a C^1 map of \mathbb{R}^n (or \mathbb{T}^n).

Definition 42. A compact subset K of \mathbb{R}^n , f -invariant ($f^{-1}(K) = K$) is expanded if there exists $\lambda < 1$ such that for any $x \in K$, $d_x f$ is invertible and with inverse λ -contracting.

Example 43. Let $f : (x, y) \in \mathbb{C} \rightarrow z^2 \in \mathbb{C}^2$. Then the unit circle of the complex plan \mathbb{R}^2 is expanded.

Property 44 (Admitted). If f expands a compact subset K , then there exists $\epsilon > 0$ such that for any points $x \in K$, the restriction $f|_{B(x, \epsilon)}$ of f to the ball $B(x, \epsilon)$ is a diffeomorphism onto its image and its image contains $B(f(x), \epsilon)$. Moreover the inverse of $f|_{B(x, \epsilon)}$ is λ -contracting, with $\lambda < 1$ which does not depends on x .

Proposition 45. If f expands a compact subset K , then f is expansive on K .

4.3 Link between symbolic dynamics and expanded compact subsets

Actually there is a strong links between the symbolic (shift) dynamics and the dynamics over an expanded compact subset given by the following theorem:

Theorem 46. There exists $p \geq 1$ and $A \in M_p(\{0, 1\})$ such that the dynamics induced by f on K is semi-conjugated to the shift σ of $\Sigma_A := \{(x_i)_{i \geq 0} \in \{1, \dots, p\}^{\mathbb{N}} : A_{x_{i+1}x_i} = 1\}$. In other word, there exists a continuous maps $\pi : \Sigma_A \rightarrow K$ continue onto K and such that:

$$f \circ \pi = \pi \circ \sigma.$$

4.3. LINK BETWEEN SYMBOLIC DYNAMICS AND EXPANDED COMPACT SUBSETS 31

Remark: Actually the map π is injective on (an intersection of dense open subset and so) a dense subset of Σ_A .

Let us illustrate this theorem by an example.

Example 47. Let K be the circle (\mathbb{R}/\mathbb{Z}) . Let f be the map $x \in K \mapsto 10 \cdot x \in K$. Let π be the following map from the symbolic space $\Sigma_{10} = \{1, \dots, 10\}^{\mathbb{N}}$ which sends $(x_i)_i$ to $\sum_{i \geq 0} x_i / 10^i$. The map π is continuous and well defined. Also for any $x \in S^1$ if $x = 0, x_1 x_2 \dots x_i \dots$ we have $\pi((x_i)_i)$ equals to x . Thus π surjective. Finally, we notice that

$$f \circ \pi((x_i)_i) = 10 \cdot 0, x_1 x_2 \dots x_i \dots [1] = x_1, x_2 \dots x_i \dots [1] = 0, x_2 \dots x_i \dots [1] = \pi \circ \sigma((x_i)_i)$$

The map π is “mostly injective” but is not injective: the point $(1, 9, 9, 9, 9, 9, \dots)$ and $(2, 0, 0, 0, \dots)$ are sends to the same point.

Example 48. example de l’endo du tore avec une partition infy (fractal de Rauzy?).

The proof of this theorem follows from the existence of a Markov Partition.

Definition 49. A family $R = \{R_1, \dots, R_N\}$ of disjoint subsets of \mathbb{R}^n is a Markov partition of K if:

1. $\cup_{i=1}^N R_i$ contains K ,
2. for every i, j we have:

$$f(R_i) \cap R_j \neq \emptyset \Rightarrow f(R_i) \supset R_j.$$

Let us show that the following proposition (showed further) is sufficient the prove theorem 46.

Proposition 50. For any $\epsilon > 0$, there exists a Markov partition for any expanded compact subset, whose elements have diameter less than ϵ .

Proof of theorem 46. Let $\epsilon > 0$ be less than the expansiveness constant of K and the constant of property 44. Let $(R_i)_{i=1}^N$ be a Markov partition of K whose elements have diameter less than ϵ . Let $A \in \mathcal{M}_n(\{0, 1\})$ such that with $A := [a_{ij}]$, $a_{ij} = 1$ iff $f(R_j) \supset R_i$. Let $(x_i)_i \in \Sigma_A$.

We notice that $\cap_{n \geq 1} f^{-n}(cl(R_{x_n}))$ is a decreasing intersection of non empty compact subset. By expansiveness, this intersection consists of a unique point $\pi((x_i)_i)$. We notice that we have well:

$$f \circ \pi = \pi \circ \sigma.$$

On the other hand, for $y \in K$, let $x_n \in \{1, \dots, N\}$ s.t. $f^n(y)$ is in R_{x_n} . We notice that $x = (x_n)_n$ is in Σ_A and that $\pi(x)$ is equal to y , by uniqueness.

It remains to prove that π is continuous onto K . By property 44, for ϵ small enough the diameter of $\cap_{n=1}^k f^{-n}(cl(R_{x_n}))$ is less than $\lambda^k \cdot \epsilon$, for any $(x_i)_i \in \Sigma_A$. When $(y_i)_i$ is close to $(x_i)_i$, the coordinate x_i and y_i are equal for every i less than a large k , thus the distance between $\pi((x_i)_i)$ and $\pi((y_i)_i)$ is less than $\lambda^k \cdot \epsilon$ and so is small (since $|\lambda| < 1$). Consequently the map π is continuous. □

Proof of proposition 50. We can suppose ϵ less than the constant of property 44. We suppose f $\lambda < 1$ -expansive on K .

As K is bounded there exists a finite family of open subsets $(U_i)_{i=1}^N$ with diameter less than $\epsilon(1 - \lambda)/2$, such that K is contained in $\cup_{i=1}^N U_i$. For any $j, k \geq 0$ and $x \in f^{-k}(U_j) \cap K$, let:

$$U_{xj}^k := f_{|B(x, \epsilon)}^{-1} \left(f_{|B(f(x), \epsilon)}^{-1} \left(\dots \left(f_{|B(f^k(x), \epsilon)}^{-1}(U_j) \right) \dots \right) \right) \cap K.$$

We notice that the diameter of U_{xj}^k is less than $\lambda^k \cdot \epsilon(1 - \lambda)/2$.

For $i \in \{1, \dots, N\}$, let us construct $U_i^{(n)}$ by induction.

Let $U_i^{(0)}$ be equal to U_i . For $k \geq 0$, let $U_i^{(k+1)}$ be the union of $U_i^{(k)}$ with the subsets U_{xj}^{k+1} intersecting $U_i^{(k)}$, for $x \in K$ and $j \in \{1, \dots, N\}$.

We notice that by induction the diameter of $U_i^{(k)}$ is less than $\sum_{p=0}^k \lambda^p \cdot \epsilon(1 - \lambda)$ and that $(U_i^{(k)})_k$ is an increasing sequence of open subsets of K . Thus the open subset $U_i^{(\infty)}$ of K equal to $\cup_{k \geq 0} U_i^{(k)}$ has diameter less than $\epsilon = \sum_{p=0}^{\infty} \lambda^p \cdot \epsilon(1 - \lambda)$.

Let $(R_k)_k$ be the nonempty subsets of the form $\cap_{i \in I} U_i^{(\infty)} \setminus \cup_{i \in I^c} U_i^{(\infty)}$ for $I \subset \{1, \dots, N\}$.

We notice that $(R_k)_k$ is a partition. Let k, k' such that $f(R_k)$ intersects $R_{k'}$, with $R_k := \cap_{i \in I} U_i^{(\infty)} \setminus \cup_{i \in I^c} U_i^{(\infty)}$ and $R_{k'} := \cap_{i \in I'} U_i^{(\infty)} \setminus \cup_{i \in I'^c} U_i^{(\infty)}$.

Let us show that $(R_k)_k$ is a markov partition: for every j, k , if $f(R_k)$ intersects R_j then $f(R_k)$ contains R_j .

For we first prove the following lemma:

Lemma 51. *For every l , the $\phi(U_i^{(\infty)})$ such that $\phi(U_i)$ intersects U_l cover $U_l^{(\infty)}$, where ϕ is a section of f (that is of this form $f_{|B(x,\epsilon)}^{-1}$) whose domain contains $U_i^{(\infty)}$.*

Proof. Let $x \in U_l^{(\infty)}$, then there exists a path $(U_{i_k})_{k=0}^j$ such that i_0 is l and $f^j(x)$ belongs to U_{i_j} and $f(U_{i_{k+1}})$ intersects U_{i_k} for every k . Then we notice that $i = i_1$ is convenient. \square

This lemma implies that each times that some $\phi(R_i)$ intersects some $U_l^{(\infty)}$, then $\phi(R_i)$ is contained in $U_l^{(\infty)}$. To prove this implication we see that if $\phi(R_i)$ intersects some $U_l^{(\infty)}$, then there exists U_j such that $\phi(U_j)$ intersects U_l and such that $\phi(U_j^{(\infty)})$ intersects $\phi(R_i)$. The last statement implies that $\phi(R_i)$ is contained in $\phi(U_j^{(\infty)})$. Also as $\phi(U_j)$ intersects U_l , the set $\phi(U_j^{(\infty)})$ is contained in $\phi(U_l^{(\infty)})$. Thus $\phi(R_i)$ is well contained in $U_l^{(\infty)}$ whenever $\phi(R_i)$ intersects $U_l^{(\infty)}$. Consequently $\phi(R_i)$ is contained in R_l whenever $\phi(R_i)$ intersects R_l .

This implies the markov partition property since, for any m, i if $f(R_m)$ intersects R_i then R_m intersects $\phi(R_i)$ (for some good section ϕ) and so $\phi(R_i)$ is contained in R_m , and R_i is well contained in R_m . \square

Lien avec les fractals?

Chapter 5

Structural stability in chaotic dynamics

5.1 Structural stability o north-south dynamics of the circle

5.2 structural stability of the doubling angle map of the circle

Chapter 6

Quasi-periodic phenomena : homeomorphisms of the circle

Let f be a homeomorphism of the circle \mathbb{S}^1 which preserves the orientation. We wonder about the existence of a semi-conjugation between f and a rotation of the circle. That is the existence of a continuous map h from \mathbb{S}^1 onto \mathbb{S}^1 such that:

$$h \circ f = R_\alpha \circ h,$$

for some rotation R_α of the circle with angle α . For in this section we explain the rotation number $\alpha_f \in \mathbb{S}^1$ of f , which satisfies the following properties:

Theorem 52 (Poincaré). *For all homeomorphisms f and f' , the rotation numbers satisfy the following properties:*

1. *if f and f' are semi-conjugated then $\alpha_f = \alpha_{f'}$,*
2. *if α_f is irrational, then f is semi-conjugated to the rotation R_{α_f} ,*
3. *if α_f is rational then f has a periodic point.*

To define the rotation number, let us admit some properties about the liftings. For let us denote by $\pi : \mathbb{R} \rightarrow \mathbb{S}^1$ the canonical map.

Proposition 53. *For every homeomorphism f of the circle, there exists a homeomorphism F of the real line \mathbb{R} such that:*

1. *the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{F} & \mathbb{R} \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{S}^1 & \xrightarrow{f} & \mathbb{S}^1 \end{array},$$

2. $F(x + n) = F(x) + n$, for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$,
3. if F' is another map satisfying the first above property, then $F - F'$ is a constant map equal to an integer,
4. if f preserves the orientation, then F is increasing.

The map F is called a *lifting* of f .

Let us fix such a lifting F of f . let $u_n : x \mapsto F^n(x) - x$, where the map F^n is the n^{th} composition of F . The rotation number is defined as:

$$\alpha_f := \pi \left(\lim_{n \rightarrow \infty} \frac{u_n(x)}{n} \right).$$

We notice that $\alpha_f = \pi \left(\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} F^{k+1}(x) - F^k(x) \right)$.

The existence of this limit and these non-dependences in F nor in x are to be proved. Let us begin by showing this non-dependence. The non-dependence in F follows obviously from 53.3. For we notice that, by proposition 53.2, for every $x, y \in \mathbb{R}$ we have:

$$|u_n(x) - u_n(y)| \leq 2 \tag{6.0.1}$$

This prove the independence of α_f in the point x . Also this shows that:

$$u_{n+m}(x) \leq u_n(x) + u_m(x), \tag{6.0.2}$$

by taking $y := F^m(x)$. Now the existence of the limit follows only from this lemma:

Lemma 54. *For every $\epsilon > 0$, there exists $N \geq 0$ and such that for every $n \geq N$, for m sufficiently large we have:*

$$\frac{u_m(x)}{m} \leq \frac{u_n(x)}{n} + \epsilon.$$

Proof. Let $m = k \cdot n + r$, with $0 \leq r \leq n$. Then we have:

$$\frac{u_m(x)}{m} = \frac{u_{k \cdot n + r}(x)}{m} \leq \frac{u_{k \cdot n}(x) + u_r(x) + 2}{m} \leq \frac{k u_n(x) + 2k}{kn} + \frac{u_r(x) + 2}{m} \leq \frac{u_n(x)}{n} + \frac{2}{n} + \frac{u_r(x) + 2}{m}$$

For N sufficiently large, $2/n$ is less than $\epsilon/2$ and for m sufficiently large $\frac{u_r(x)+2}{m}$ is less than $\epsilon/2$. \square

As moreover the symmetric inequality is done by the same way, we have that $(u_m(x)/m)_{m \geq 0}$ converges well to a real α_f .

Let us show now that if f and f' are conjugated, then their rotation number is the same: Let h be the conjugacy such that:

$$h \circ f = f' \circ h.$$

Then for two liftings F and H of respectively f and h , we notice that the following map is a lifting of f' :

$$F' = H \circ F \circ H^{-1}.$$

Also we have the existence of $M \geq 0$ s.t.

$$x - M \leq H(x) \leq x + M$$

Thus $|F'^n \circ H(x) - F^n(x)| \leq M$. Thus α_f and $\alpha_{f'}$ are equal.

Proof of Poincaré theorem. Let us show first that f has a periodic point if and only if α_f is a rational number. If f has a periodic point p of period l , then there exist $\hat{p} \in \pi^{-1}(\{p\})$, $k \in \mathbb{Z}$ and a lifting F of f such that

$$F^l(\hat{p}) = \hat{p} + k$$

From this we get that:

$$\lim_{n \geq \infty} F^n(x)/n = k/l$$

Thus the angle α_f is rational. Suppose now that α_f is rational. We may suppose that $\alpha_f = 0$ (by considering f^n instead of f which satisfies $\alpha_{f^n} = n\alpha_f$). We suppose for the sake of contradiction

that f has no fixed point. Then any lifting F of f cannot have a fixed point. Let us assume for instance that:

$$F(x) > x, \quad \forall x \in \mathbb{R}$$

Then either $F^n(0) < 1$ for every $n \geq 0$, either there exists $k \geq 0$ such that $F^k(0) \geq 1$. In the later case, then $F^{mk}(0) \geq m$. So

$$\lim_{n \rightarrow \infty} \frac{F^n(0)}{n} \geq \frac{1}{k}$$

This is a contradiction. In the first case, $(F^n(0))_n$ is an increasing sequence bounded by 1. Such a sequence must converges to a fixed point of F . Contradiction.

Let us now prove the second statement of the Poincaré theorem. We suppose that α_f is irrational. We want to show that f is semi conjugated to the rotation of angle α_f . For let us fix a lifting F of f and put $\alpha_F = \lim F^n(0)/n$. The proof of this theorem consists of constructing a map between

$$G' := \{F^n(0) + m; n, m \in \mathbb{Z}\}$$

and

$$G := \{n \cdot \alpha_F + m; n, m \in \mathbb{Z}\}$$

Since α_F is irrational, the subgroup G of \mathbb{R} is dense. We will show at the end the

Lemma 55. *For every $n, m, n', m' \in \mathbb{Z}$, if $F^n(0) + m < F^{n'}(0) + m'$ then $n \cdot \alpha_F + m < n' \cdot \alpha_F + m'$.*

Then we can define:

$$H : x \in \mathbb{R} \mapsto \sup_{\{n, m \in \mathbb{Z} : F^n(0) + m \leq x\}} (n\alpha_F + m).$$

The map H is well define, continuous and onto \mathbb{R} by the above lemma and the fact that G is dense in \mathbb{R} . Also by definition of H , we have for every $x \in \mathbb{R}$ and $m \in \mathbb{Z}$:

$$H(x + m) = H(x) + m \tag{6.0.3}$$

$$H \circ F(x) = H(x) + \alpha_f \tag{6.0.4}$$

By (6.0.3), there exists a continuous map h of the circle such that:

$$f \circ \pi = \pi \circ H$$

As H is surjective, h must be surjective. Also by (6.0.4), we have:

$$h \circ f = R_{\alpha_f} \circ h$$

This implies the second statement of the Poincaré Theorem. \square

Proof of lemma 55. For let us consider the function:

$$\phi : x \mapsto F^n(x) + m - F^{n'}(x) - m'.$$

We notice that ϕ is positive at zero, continuous, and cannot vanish (since else f as a periodic point). So ϕ is positive everywhere. In other words, for every $x \in \mathbb{R}$, we have:

$$F^n(x) + m < F^{n'}(x) - m'.$$

Also the maps $f_1 : x \mapsto F^n(x) + m$ and $f_2 : x \mapsto F^{n'}(x) - m'$ are increasing since f preserves the orientation. Let us show by induction on $k > 0$ that:

$$f_1^k(0) < f_2^k(0)$$

For $k = 1$ this is the hypothesis of the lemma. Let $k > 0$ and suppose that $f_1^k(0) < f_2^k(0)$. Since f_1 is increasing, we have:

$$f_1^{k+1}(0) < f_1 \circ f_2^k(0).$$

Since the map ϕ is positive, we have

$$f_1 \circ f_2^k(0) < f_2^{k+1}(0).$$

Thus, we have well $f_1^{k+1}(0) < f_2^{k+1}(0)$. This induction implies that

$$n\alpha_F + m = \lim_{k \rightarrow \infty} \frac{f_2^k}{k} \leq \lim_{k \rightarrow \infty} \frac{f_1^k}{k} = n'\alpha_F + m'.$$

\square

Exercise 56. *P. 103-114 Devaney § 1.14.*

Chapter 7

Completeness of some spaces

7.1 Definitions and basic properties

Throughout this section, we denote by (E, d) a metric space.

A sequence $(x_n)_{n \geq 0} \in E^{\mathbb{N}}$ is *Cauchy* if for every $\epsilon > 0$, there exists $N > 0$ such that for $p, q \geq N$, we have:

$$d(x_q, x_p) \leq \epsilon.$$

The space (E, d) is *complete* if every Cauchy sequence of (E, d) converges.

Property 57. *Every Cauchy sequence is bounded.*

Proof. Let $(x_n)_n \in E^{\mathbb{N}}$ be a Cauchy sequence. Then there exists $N \geq 0$ s.t. for every $n \geq 0$, $d(x_n, x_N) \leq 1$. Thus the diameter of $\{x_n\}_n$ is less than:

$$\max_{k \leq N} 2 \cdot d(x_k, x_N) + 2 < \infty$$

□

Example 58. 1. *The subset $(0, 1]$ endowed with the euclidean distance is not complete, since the sequence $(\frac{1}{n+1})_{n \geq 0}$ is Cauchy but does not converge in $(0, 1]$.*

2. *The set of rational numbers \mathbb{Q} endowed with the euclidean distance is not complete, since the sequence $(x_n)_n$ defined by:*

$$x_n := \frac{[10^n \cdot \sqrt{2}]}{10^n}, \text{ with } [\cdot] : \mathbb{R} \rightarrow \mathbb{N} \text{ equal to the entire part,}$$

is a Cauchy sequence that does not converge in \mathbb{Q} .

3. (non-trivial) The space of real numbers \mathbb{R} endowed with the euclidean norm is complete.
4. (non-trivial) The space of continuous bounded functions of \mathbb{R} endowed with the following norm is complete:

$$\|f\|_{C^0} = \sup_{x \in \mathbb{R}} |f(x)|.$$

5. (non-trivial) The space of C^k -functions of $[0, 1]$ endowed with the following norm is complete:

$$\|f\|_{C^k} = \sum_{j=0}^k \sup_{x \in \mathbb{R}} |f^{(j)}(x)|.$$

6. (non-trivial) The space of non empty compact subsets of \mathbb{R}^n , endowed with the Hausdorff distance, is complete.

All the interest of complete spaces follows from the very useful following theorem:

Theorem 59. *Every contracting map of a complete space has an unique fixed point.*

Proof. Let (E, d) be a metric space and let $f : E \rightarrow E$ be a contracting map. To say that f is contracting means that there exists $\lambda < 1$ such that

$$\forall x, y \in E, \quad d(f(x), f(y)) \leq \lambda \cdot d(x, y).$$

Let $x \in E$, and let $(x_n)_n$ be the sequence defined by:

$$x_n := f^n(x), \quad \forall n \geq 0.$$

Existence

Let $\epsilon > 0$. There exists $N \geq 0$ such that

$$\frac{\lambda^N}{1 - \lambda} \cdot d(x_1, x_0) \leq \epsilon$$

Then, for every $p \geq q \geq N$, we have:

$$d(x_q, x_p) \leq d(x_q, x_{q+1}) + d(x_{q+1}, x_{q+2}) + \cdots + d(x_{p-1}, x_p).$$

By induction we can show that for any $n \geq 0$:

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1).$$

Thus we have :

$$\begin{aligned} d(x_q, x_p) &\leq (\lambda^q + \lambda^{q+1} + \dots + \lambda^{p-1})d(x_0, x_1) = \lambda^q \left(\sum_{k=0}^{p-q} \lambda^k \right) \cdot d(x_0, x_1). \\ &\leq \lambda^N \sum_{k=0}^{\infty} \lambda^k \cdot d(x_0, x_1) = \frac{\lambda^N}{1-\lambda} \cdot d(x_1, x_0) \leq \epsilon. \end{aligned}$$

Consequently, $(x_n)_n$ is a Cauchy sequence and so converges to some x . As $f(x_n) = x_{n+1}$, when n approaches infinity, by continuity of f , we have $f(x) = x$.

Uniqueness

Let x, x' be two fixed points of f . Then we have:

$$d(x, x') = d(f(x), f(x')) \leq \lambda d(x, x').$$

Thus x and x' are equal. □

7.2 Proof of the complete spaces of some spaces

7.2.1 Proof of the completeness of \mathbb{R}

There are several ways to construct \mathbb{R} . In my opinion, the most canonical way is to see the set of real numbers is as the set of number with possibly infinite decimal expression:

$$\mathbb{R} := \left\{ x = x^0, x^1 x^2 \dots x^n \dots ; x^0 \in \mathbb{Z}, x^n \in \{0, 1, \dots, 9\} \forall n \geq 1 \text{ s.t. } \forall N \geq 0 \exists k \geq N x^k \neq 9 \right\}$$

The last condition " $\forall N \geq 0 \exists k \geq N x^k \neq 9$ " is to avoid the fact that $0, 999999 \dots$ is equal to 1.

From the expression of \mathbb{R} , one can easily define the addition, the subtraction, the absolute value and order relation \leq . So we can define the following Euclidean distance:

$$d(x, x') \in \mathbb{R} \mapsto |x - x'| \in \mathbb{R}^+.$$

The following injection of \mathbb{Q} respect \leq :

$$q \in \mathbb{Q} \mapsto (q^n)_{n \geq 0}, \quad \text{with } q^0 := [q] \text{ and } q^{n+1} = [10^{n+1} \cdot q] - 10 \cdot q^n$$

Theorem 60. (\mathbb{R}, d) is a complete metric space.

Proof. Let $(x_n)_n$ be a Cauchy sequence.

Let us construct by induction on N a family $(J_n)_{n=0}^N$ of subsets of \mathbb{N} satisfying:

1. $(J_n)_n$ is decreasing: $J_0 \supset J_1 \supset \cdots \supset J_N$,
2. the cardinality of J_N is infinite (and so is the cardinality of J_k , for $k \leq N$),
3. for all $k \leq N$, there exists u_k , s.t. for every $n \in J_k$, x_n^k is equal to u_k .

Step $N = 0$

As $(x_n)_n$ is a Cauchy sequence, it is bounded, so there exists $M \geq 0$ such that $(x_n)_{n \geq 0} \subset (-M, M)^\mathbb{N}$. So this infinite family is contained in the following finite union of boxes:

$$(-M, -M + 1] \cup \cdots \cup (-1, 0] \cup [0, 1) \cup \cdots \cup [M - 1, M).$$

So there exists an infinite subset J_0 of \mathbb{N} s.t. for all $k, k' \in J_0$, we have $x_k^0 = x_{k'}^0$.

Step $N \rightarrow N + 1$

Let $(J_n)_{n=0}^N$ be the family constructed by induction. The infinite family $(x_n^{N+1})_{n \in J_N}$ can have the following finite possible values:

$$\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}$$

So this family takes infinity many times one of the above values, say u_{N+1} .

$$\text{Let } J_{N+1} := \{k \in J_N : x_n^{N+1} = u_{N+1}\}$$

This concludes the proof of the induction.

If for every $N \geq 0$ there exists $k \geq N$ such that u_k is not equal to 9, Let $x := u^0, u^1 u^2 \cdots \in \mathbb{R}$.

Other wise, let $N \geq 0$ be minimal such that, for every $k > 0$, we have $u^k = 9$.

$$\text{Let } x^k := \begin{cases} u^k & \text{if } k < N \\ u^k + 1 & \text{if } k = N \\ 0 & \text{else} \end{cases}$$

Let $x := x^0, x^1 x^2 \cdots \in \mathbb{R}$. Let us prove that in all the cases, the sequence $(x_n)_n$ converges to x .

We define the increasing sequence of integers $(n_k)_k$, by induction: n_0 is the least integer of J_0 , and n_{k+1} is the least integer of J_{k+1} which is greater than n_k .

One can easily show that $(x_{n_k})_k$ converges to x .

To conclude, we can proof in our particular case that Cauchy sequences which have a converging subsequences, converge.

Let $\epsilon > 0$ and let $N > 0$ such that:

- for any $n_k \geq N$, $d(x_{n_k}, x) < \epsilon/2$,
- for any $p, q \geq N$, $d(x_p, x_q) < \epsilon/2$.

From this we conclude that for every $n \geq N$,

$$d(x, x_n) \leq d(x, x_{n_k}) + d(x_n, x_{n_k}) \leq \epsilon.$$

□

7.2.2 Completeness of the space of continuous, bounded, real functions of \mathbb{R}^n

Let us denote by $C_b^0(\mathbb{R}^n)$ the space of continuous, bounded, real functions of \mathbb{R}^n . We endow the vector space $C_b^0(\mathbb{R}^n)$ with the following norm:

$$\|f\| = \sup_{x \in \mathbb{R}^n} |f(x)|.$$

Theorem 61. *The normed vector space $(C_b^0(\mathbb{R}^n), \|\cdot\|)$ is complete.*

Proof. Let $(f_n)_n$ be a Cauchy sequence of $C_b^0(\mathbb{R}^n)$. We notice that for every $x \in \mathbb{R}^n$, the real sequence $(f_n(x))_n$ is Cauchy, and so converges to some $f(x)$.

f is bounded.

As $(f_n)_n$ is Cauchy, it is Bounded. So there exists $M \geq 0$ such that $\|f_n\| \leq M$, for every $n \geq 0$. This means that for all $n \geq 0$, for all $x \in \mathbb{R}^n$, $|f_n(x)|$ is less than M . When n approaches the infinity, we have that $|f(x)| \leq M$, and so that f is bounded.

f is continuous.

Let $x \in \mathbb{R}^n$ and $\epsilon > 0$.

As $(f_n)_n$ is Cauchy, there exists $N \geq 0$ s.t. for every $p \geq N$, we have:

$$\|f_p - f_N\| \leq \epsilon/4.$$

As f_N is continuous, there exists $\delta > 0$, such that for every $y \in B(x, \delta)$, we have:

$$\|f_N(y) - f_N(x)\| \leq \epsilon/2.$$

From the two above inequalities, we have:

$$\|f_p(y) - f_p(x)\| \leq \|f_p(y) - f_N(y)\| + \|f_N(y) - f_N(x)\| + \|f_N(x) - f_p(x)\| \leq \epsilon$$

Thus when p approaches infinity, we have:

$$\|f(y) - f(x)\| \leq \epsilon$$

$(f_n)_n$ converges to f in $(C_b^0(\mathbb{R}^n), \|\cdot\|)$

For every $\epsilon > 0$, since $(f_n)_n$ is Cauchy, there exists $N \geq 0$ s.t. for every $p \geq q \geq N$, we have:

$$\|f_p - f_q\| \leq \epsilon.$$

This means that, for every $x \in \mathbb{R}^n$, we have

$$|f_p(x) - f_q(x)| \leq \epsilon.$$

When, p approaches infinity, we get for every $x \in \mathbb{R}^n$:

$$|f(x) - f_q(x)| \leq \epsilon.$$

And so:

$$\|f - f_q\| \leq \epsilon.$$

□

7.2.3 Completeness of the space of compact subsets

Theorem 62. *The space \mathcal{K} of nonempty compact subsets of \mathbb{R}^n , endowed with the Hausdorff distance d_{HD} is complete.*

Proof. Let $(K_n)_n$ be a Cauchy sequence of non empty compact subsets of \mathbb{R}^n .

For every $n \geq 0$, let $H_n := cl(\cup_{k \geq n} K_k)$. Let $K := \cap_{n \geq 0} H_n$.

Let us prove that K is a non empty compact subset and that:

$$d_{HD}(K_n, K) \rightarrow 0$$

First of all K is closed since it is an intersection of closed subsets. Since $(K_n)_n$ is a Cauchy sequence, it is d_{HD} -bounded. This implies that $\cup_n K_n$ is bounded in \mathbb{R}^n , and that H_0 is bounded and so compact. As K is a subset of H_0 it is bounded. Consequently, K is compact.

Let us show that K is not empty. We remind that H_0 is compact. For every $n \geq 0$, let $x_n \in K_n$. By compactness of H_0 , we can extract from $(x_n)_n$ a subsequence $(x_{n_i})_n$ that converges to some x . As for every $n_i \geq n$, x_{n_i} belongs to the closed subset H_n , the limit point x belongs to H_n . So x belongs to the intersection K of $(H_n)_n$. Thus K is not empty.

Let us show that $(K_n)_n$ converges to K for the Hausdorff distance. Let $\epsilon \geq 0$. As $(K_n)_n$ is a Cauchy sequence, there exists $N \geq 0$ st, for every $n \geq N$,

$$d_{HD}(K_N, K_n) < \epsilon/2 \tag{7.2.1}$$

This implies that K_n is included in $B(K_N, \epsilon/2) \subset \mathbb{R}^n$.

On the other hand, equation 7.2.1 implies that for every $n \geq N$, the subset K_N is included in $B(K_n, \epsilon/2)$. Thus for every $x \in K_N$, this implies the existence of $k_n \in K_n$ such that:

$$d(x, k_n) \leq \epsilon/2$$

we may extract from $(k_n)_n$ a subsequence $(k_{n_i})_i$ that converges to some k . We have that:

$$d(x, k) \leq \epsilon/2$$

As before, the point x belongs to K .

In other words k belongs to K and $d(x, k) \leq \epsilon/2$. Thus K_N is contained in $cl(B(K, \epsilon/2))$. By equation 7.2.1, for every $n \geq N$, K_n is contained in $B(K, \epsilon)$.

From the two last paragraphs, we conclude that for every $n \geq N$, $d_{HD}(K, K_n) \leq \epsilon$. \square

Chapter 8

Midterm at home

8.1 Hausdorff dimension

Let K be a compact subset of \mathbb{R}^n .

For $s \geq 0$, the *Hausdorff outer measure of dimension s* , denoted H^s , is defined by:

$$H_\delta^s(K) = \inf \left\{ \sum_{i=1}^{\infty} \text{diam}(A_i)^s \right\},$$

where the infim

$$H^s(K) = \limsup_{\delta \rightarrow 0} H_\delta^s(K).$$

Question 8.1.1. *Show that the function $s \mapsto H^s(K)$ is non-increasing as function into $[0, \infty]$.*

Question 8.1.2. *Show that for all values of s , except possibly one, $H^s(K)$ is either 0 or ∞ .*

Question 8.1.3. *Show that $K \subset \mathbb{R}^n$ has finite Hausdorff dimension: that is there is a real number $0 \leq d < \infty$ such that if $s < d$ then $H^s(K) = \infty$ and if $s > d$, then $H^s(K) = 0$.*

We define the Hausdorff dimension of K as:

$$\dim_H(K) = \inf\{s : H^s(K) = 0\}.$$

Question 8.1.4. Shows that if K and K' are two compact subsets of \mathbb{R}^n :

$$H^s(K \cup K') \leq H^s(K) + H^s(K')$$

$$\dim_H(K \cup K') = \max\left(\dim_H(K), \dim_H(K')\right)$$

Question 8.1.5. Compute the Hausdorff dimension of all the fractals that you have already seen in your homework.

8.2 Hyperbolic Julia set

Giving $c \in \mathbb{C}$, let $P_c : z \mapsto z^2 + c$.

The polynomial function P_c is *Hyperbolic* if the sequence $(P_c^n(0))_n$ approaches infinity as n approaches infinity.

Question 8.2.1. Show that this is the case for $c \in \mathbb{R} \setminus [-2, 1/4]$.

Let $f := P_c$ be a hyperbolic polynomial function. Let $S := \mathbb{C} \setminus \{f^n(0) : n \geq 0\}$ and $\mathbb{D} := \{z \in \mathbb{C}; \|z\| \leq 1\}$. Let us admit that there exists an analytic map π from \mathbb{D} onto S without critical points.

Question 8.2.2. Show that there exists a map g from \mathbb{D} into \mathbb{D} such that $f \circ \pi \circ g = \pi$.

Question 8.2.3. Prove that g is without critical point.

Question 8.2.4. Prove that g cannot be surjective, but is injective**.

Question 8.2.5. Prove that:

- i. for any n , f has periodic a point p of period n , that lies in S . P
- ii. we can chose π s.t. it sends 0 to p (We can consider a composition of π with a Moebius function that preserves the unit circle $z \mapsto \frac{az+b}{cz+d}$).
- iii. Prove by using the Schwarz lemma, that $\|(g^n)'(0)\|$ is less than 1 and so that p is a repulsive periodic orbit of f .

Question 8.2.6. *Let J be the closure of the set of periodic points of f . Show that J is a compact set of \mathbb{C} .*

Question 8.2.7. **** Show that there exists $C > 0$ and $\lambda > 1$ such that for every $n \geq 0$ and any $x \in J$,*

$$|(f^n)'(x)| \geq C \cdot \lambda^n$$

Question 8.2.8. ****** Show that J is a fractal of \mathbb{C} .*

8.3 Structural stability of hyperbolic compact subset

Definition 63. *Let f be a C^1 map of an euclidean space or of a torus M . Let K be a compact subset of this space that preserves K :*

$$f(K) \subset K$$

The compact K is said to be structurally stable if for every f' C^1 -close to f there exists a homeomorphism $h : K \rightarrow M$, close to the canonical inclusion of $K \subset M$, such that f' preserves $h(K)$, and for every $x \in K$:

$$f' \circ h(x) = h \circ f(x)$$

For instance, when K is a simple point $\{x\}$, to say that K is preserved by f means that x is a fixed point. To say that K is structurally stable means that every f' C^1 -close to f has a fixed point x' close to x . The aim of this problem is to find sufficient conditions implying the structural stability of compact subsets. We will use the implicit function theorem for some Banach spaces.

Question 8.3.1. *Prove that the vector space C_b^1 of maps f from \mathbb{R}^2 into \mathbb{R}^2 such that:*

$$\|f\|_1 = \sup_{x \in \mathbb{R}^2} (\|f(x)\| + \|d_x f\|) < \infty$$

is a complete vector space endowed with the norm $\|f\|_1$.

Let us consider the following map:

$$\begin{aligned} \phi &: C_b^1 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \\ (f', y) &\longmapsto (f'(y) - y) \end{aligned}$$

Question 8.3.2. Show that, for every $(f_0, x_0) \in C_b^1 \times \mathbb{R}^2$, there exists a linear function L from $C_b^1 \times \mathbb{R}^2$ into \mathbb{R}^2 , such that

$$\phi(f, x) = \phi(f_0, x_0) + L(f - f_0, x - x_0) + o(f - f_0, x - x_0)$$

for every (f, x) . Give the expression of L .

The map L is called the differential of ϕ at (f_0, x_0) and is denoted by $T_{(f_0, x_0)}\phi$.

We denote by \mathcal{L} the space of continuous linear map from $(C_b^1 \times \mathbb{R}^2, \|\cdot\|_1 + \|\cdot\|)$ into $(\mathbb{R}^2, \|\cdot\|)$.

Question 8.3.3. Show that a linear map l from $C_b^1 \times \mathbb{R}^2$ into \mathbb{R}^2 belongs to \mathcal{L} if and only if

$$\|l\|_{\mathcal{L}} = \sup\{\|l(f, x)\| : x \in \mathbb{R}^2, f \in B \text{ s.t. } \|x\| + \|f\|_1 \leq 1\}$$

is a finite number. Show that the differential of ϕ at any point (f_0, x_0) is continuous.

Question 8.3.4. Prove that $(\mathcal{L}, \|\cdot\|_{\mathcal{L}})$ is a Banach space (that is a complete normed vector space).

Question 8.3.5. Prove that the differential of $T_{f,x}\phi$ depends continuously on (x, f) .

This means that ϕ is a C^1 -maps of the Banach space $C_b^1 \times \mathbb{R}^2$ into \mathbb{R}^2 .

Theorem 64 (Implicit functions theorem for Banach spaces). Let B, B' be two Banach spaces. Let $\psi : B \times B' \rightarrow B'$ be a C^1 -map such that:

- $\psi(0, 0) = 0$,
- the restriction of $T_{(0,0)}\psi$ to $\{0\} \times B'$ is a bijection onto B' .

Then there exists a neighborhood U of $0 \in B$ and a neighborhood V' of 0 in B' and a C^1 -map ρ from V into V' such that

$$\rho(0, 0) = 0$$

$$\forall (x, y) \in V \times V', \quad \psi(x, y) = 0 \iff y = \rho(x)$$

Question 8.3.6. What is the restriction of $T_{(f,x)}\phi$ to \mathbb{R}^2 ?

Question 8.3.7. Prove that if $d_x f$ has no eigenvalue of modulus 1, then the restriction of $T_{(f,x)}\phi$ to \mathbb{R}^2 is invertible.

Question 8.3.8. If $f(x) = x$ and $d_x f$ has no eigenvalue of modulus 1, by using the implicit function theorem, prove that there exists a C^1 -map ρ from a neighborhood U of f and a C^1 map from U into \mathbb{R}^2 such that for every $f' \in U$ the point $\rho(f')$ is fixed by f' .

Question 8.3.9. Generalize the above result for a periodic point x .

Question 8.3.10. What if the map f is a diffeomorphism of the plan, and so is not C^1 -bounded? Generalize the above results to \mathbb{R}^n .

Hyperbolicity and compact

Question 8.3.11. Let $A \in \mathcal{M}_2(\mathbb{R})$, with no eigenvalue of modulus 1. Show that there exist two subspaces E^s and E^u of \mathbb{R}^2 , $\lambda < 1$ and $C > 0$ such that for every $n \geq 0$, we have:

$$E^u \oplus E^s = \mathbb{R}^2$$

$$\forall u \in E^s, \|A^n(u)\| \leq C \cdot \lambda^n \|u\|.$$

$$\forall u \in E^u, \|A^n(u)\| \geq \frac{\lambda^{-n}}{C} \|u\|.$$

What if $A \in \mathcal{M}_n(\mathbb{R})$, for $n \geq 0$?

Definition 65. Let f be a C^1 -map of \mathbb{R}^2 . A compact subset K of M such that $f(K) = K$ is hyperbolic if

- the restriction of f to K is a bijection onto K ,
- there exists a continuous map:

$$x \in K \mapsto (e_s(x), e_u(x)) \in \mathbb{R}^2$$

such that the vectors $(e_s(x), e_u(x))$ span \mathbb{R}^2 and there exist $\lambda < 1$ and $C > 0$ such that for every $n \geq 0$, we have:

$$\forall u \in E^s, \|d_x f^n(e_s(x))\| \leq C \cdot \lambda^n.$$

$$\forall u \in E^u, \|d_x f^{-n}(e_u(x))\| \geq \frac{\lambda^{-n}}{C}.$$

For $\epsilon \geq 0$, let $h_\epsilon : (x, y) \mapsto (x^2 - 2 + y, \epsilon \cdot x)$. This map belongs to the so called Hénon family. And its chaotic properties showed that the simplest model of meteorology are chaotic.

Question 8.3.12. *Prove that for $\epsilon > 0$, h_ϵ is a diffeomorphism. Prove that for $\epsilon = 0$, h_ϵ has the same dynamic as $x \mapsto x^2 - 2$. **By using the homework question about this map, show that for $\epsilon > 0$ small enough, h_ϵ preserves and is hyperbolic to a non trivial compact set K .*

We now fix $\epsilon > 0$ small enough s.t. h_ϵ preserves and is hyperbolic to a non trivial compact set K .

Structural stability Let B' be the Banach space of continuous map from K into \mathbb{R}^2 endowed with the C^0 norm.

We denote by h^* the inverse of the restriction of h_ϵ to K .

Question 8.3.13. *Prove that the following map is well defined and of class C^1 :ined and of class C^1 :*

$$\begin{aligned} \phi : C_b^1 \times B' &\rightarrow B' \\ (f, \sigma) &\longrightarrow [x \in K \mapsto (h_\epsilon + f) \circ \sigma \circ h^*(x) - \sigma] \end{aligned}$$

Question 8.3.14. *Show, by using the implicit function theorem, that there exists a neighborhood V of $0 \in C_b^1$ and a C^1 -map $\rho : C_b^1 \rightarrow B'$ such that:*

- $\rho(f_0) = 0$,
- for every $f \in V$, for every $x \in K$, $(h_\epsilon + f') \circ \sigma(f') = \sigma(f') \circ h_\epsilon(x)$.

Question 8.3.15. *Show that $\sigma(f')$ is an injection for every $f' \in V$ sufficiently close to 0. Show that this implies that $\sigma(f')$ is a homeomorphism onto its image.*

Chapter 9

Exam at home

The purpose of this exam is to show that the repulsive compact subsets of a manifold M (that we are allowed to take equal to \mathbb{R}^n or to the torus \mathbb{T}^n) are C^1 -structurally stable. This is the so-called Shub theorem (1980's).

We remind that giving a C^1 -map of M , a compact subset K is repulsive if the following conditions are satisfied:

- $f(K) \subset K$,
- there exists $\lambda < 1$, such that for every $x \in K$, the differential $T_x f$ of f at x is invertible with inverse of norm less than λ . In other words, $T_x f$ has inverse λ -contracting.

The C^1 -structural stability of K means that for f' C^1 -close to f , there exists a homeomorphism h_f of K onto its image in K such that with $K' := h_f(K)$, we have:

- $f'(K) \subset K'$,
- for every $x \in K$, $f' \circ h_f(x) = h_f \circ f(x)$.

By C^1 -close, we mean that for any compact subset C of M , the following semi distance may be asked small:

$$d_C(f, f') := \max_{x \in C} \{d(f(x), f'(x)) + \|T_x f - T_x f'\|\}$$

9.1 Examples

Question 9.1.1. *Prove that the following compact subset are repulsive for the associated dynamics:*

1. $M_1 := \mathbb{C}$, $K_1 := \{z \in \mathbb{C} : |z| = 1\}$, $f_1 := z \mapsto z^2$.
2. $M_2 := \mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$, $K_2 := M$, $f_2 := (x, y) \in M_2 \mapsto (2x + y, 3y)$.
3. $M_3 := M_1 \times M_2$, $K_3 := K_1 \times K_2$, $f_3 := (p, p') \in M_1 \times M_2 \mapsto (f_1(p), f_2(p'))$.

9.2 Some preliminary implementations

Question 9.2.1. *Show the existence $\epsilon > 0$ and $\lambda' < 1$, such that for every $x \in K$ the restriction of f to the ϵ -ball centered at x is λ'^{-1} -expanding. By λ'^{-1} -expanding, we mean that this restriction is a diffeomorphism onto its image, whose inverse is λ' -contracting.*

Question 9.2.2. *By taking an appropriate compact subset C , show the existence of $\epsilon > 0$, $\lambda' < 1$ and $\delta > 0$, such that if $d_C(f, f')$ is less than δ , for every $x \in K$, the restriction of f' to the ϵ -centered at x is λ'^{-1} -expanding.*

9.3 Expansiveness of f on K

Question 9.3.1. *Show that for $(x, x') \in K^2$, if for every $n \geq 0$, the points $f^n(x)$ and $f^n(x')$ are ϵ -distant, then x and x' are equal.*

9.4 Completeness of a space

Let $H := C^0(K, M)$ be the set of continuous maps h from K into M .

Question 9.4.1. *Show that the map:*

$$d_H : (h, h') \in H^2 \mapsto \max_{x \in K} d(h(x), h(x')) \in \mathbb{R},$$

defines a distance on H

Question 9.4.2. *Show that H endowed with the distance d_H is a complete metric space.*

9.5 Definition of $h_{f'}$ as a fixed point

Question 9.5.1. Prove the existence of $\delta' < \delta$ such that the following map is well defined and λ' -contracting for every $f' \in C^1(M, M)$ such that $d_C(f, f')$ is less than δ' :

$$\phi_{f'} : V_H \rightarrow V_H$$

$$h \mapsto \left[x \in K \mapsto f'_{|B(x, \epsilon)}^{-1} \circ h \circ f(x) \right],$$

with $V_H := \{h \in H : d(h(x), x) \leq \epsilon/4\}$.

Question 9.5.2. Show that $\phi_{f'}$ has a unique fixed point $h_{f'}$.

Question 9.5.3. Show that $h'_{f'}$ satisfies:

$$f' \circ h_{f'}(x) = h_{f'} \circ f(x), \quad \forall x \in K.$$

Question 9.5.4. Show that $h_{f'}$ is injective.

Question 9.5.5. Show carefully that $h_{f'}$ is a homeomorphism onto its image.

Question 9.5.6. Prove that in example 2, the homeomorphism $h_{f'}$ is onto M_2 .