Topics in Differential Topology

Blaine Lawson Jr.

Notes by Somnath Basu

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1 Theory of bundles

1.1 Vector Bundles

All vector spaces considered are assumed to be over $\mathbb{R}$ or $\mathbb{C}$ unless mentioned otherwise.

We shall briefly review the basic theory of vector bundles. Let X be a topological space.

Definition 1.1.1 *A continuous family of vector spaces over X is a topological space E with a continuous map $\pi : E \to X$ and has the structure of finite dimensional vector spaces on $E_x := \pi^{-1}(x)$, compatible with the topology induced from E.*
A **morphism** *from a family over X $(\pi : E \to X)$ to another $(\pi' : E' \to X)$ is a continuous map $\phi : E \to E'$ such that the following diagram commutes :*

$$\begin{tikzcd}
E \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & F \arrow[dl, "\rho"] \\
& X &
\end{tikzcd}$$

and $\phi_x := \phi|_{E_x} : E_x \to F_x$ is linear for all $x \in X$.
ϕ is called an **isomorphism** *if it is a homeomorphism.*

It is easily verified that ϕ is an isomorphism if and only if ϕ_x is for all x.

Definition 1.1.2 *A family $\pi : E \to X$ is* **trivial** *if it is isomorphic to $X \times \mathbb{R}^n \xrightarrow{\pi_1} X$ for for some n.*
A **vector bundle** *of rank n on X is a continuous family of vector spaces $\pi : E \to X$ which is locally trivial, i.e., there exists a covering of X by open sets $\{U_i\}_{i \in I}$ such that $\pi^{-1}(U_i)$ is homeomorphic (fibrewise) to $U_i \times \mathbb{R}^n$ (via continuous maps ϕ_i).*

If two such open sets intersect then let $x \in U_i \cap U_j$. We have

$$\phi_i^{-1} \circ \phi_j : (U_i \cap U_j) \times \mathbb{R}^n \to (U_i \cap U_j) \times \mathbb{R}^n$$

which preserves the fibre. Thus we have **transition** maps $g_{ij} : U_i \cap U_j \to GL_n(\mathbb{R})$ which satisfy the *cocycle* conditions :
(i) $g_{ij} g_{ji} = \mathrm{Id}$
(ii) $g_{ij} g_{jk} g_{ki} = \mathrm{Id}$.
This transition data is all one needs to reconstruct E from X. We shall denote such a transition data by $(\mathcal{U}, g)$.

Definition 1.1.3 *A vector bundle E over X (a C^k manifold) is of type C^k if E is a C^k manifold and $\pi : E \to X$ is C^k and local trivializations are C^k.*

In terms of the transition data it means that g_{ij} are C^k for all i, j.

Definition 1.1.4 *A* **cross section** *of a bundle $\pi : E \to X$ is a continuous map $s : X \to E$ such that $\pi \circ s = Id_X$.*

Denote the space of all sections by $\Gamma(E)$ and the space of all C^k sections by $\Gamma_k(E)$. Observe that both these constructs are vector spaces and $\Gamma(E)$ (resp. $\Gamma_k(E)$) is a module over $C(X)$ (resp. $C^k(X)$). For the trivial bundle $E = X \times \mathbb{R}^n$, $\Gamma(E) = C(X, \mathbb{R}^n)$. It can be shown that $\Gamma(E)$ is a free $C(X)$ module of rank n if and only if E is trivial of rank n. In fact, if X is compact then $\Gamma(E)$ is a f.g. projective $C(X)$ module and every f.g. projective $C(X)$ module is a vector bundle.

Exercise *Show that every cross section of the Möbius band to S^1 has at least one zero.*

Example (i) $\mathbb{CP}^n = \{\text{lines through the origin in } \mathbb{C}^{n+1}\}$ and its tautological line bundle T. The transition functions are $g_{ij} = z_i/z_j$ for the standard trivialization. This is an example of a holomorphic bundle over a complex manifold. It is known that any section of T must have a zero. Furthermore

Proposition 1.1.5 $\Gamma_{hol}(T) = \{0\}$

Proof If there was a section $\sigma : \mathbb{CP}^n \to T$ then composing with $p : T \to \mathbb{C}^{n+1}$ we have a holomorphic map $p \circ \sigma : \mathbb{CP}^n \to \mathbb{C}^{n+1}$ which by the maximum principle has to be a constant. Thus $p \circ \sigma(l) = v \in l \,\forall\, l$ whence $v = 0$. $\square$

Example (ii) Grassmanians - $G_k(V) = \{k \text{ dimensional subspaces passing through the origin in } V\}$ where V is f.d. vector space. In particular $G_1(\mathbb{C}^{n+1}) = \mathbb{CP}^n$. $G_k(V)$ and $G_{n-k}(V)$ can be identified with each other once we choose a metric on V. One can analogously study tautological bundles on these spaces. It is known that $G_k(\mathbb{R}^n)$ is a compact real analytic manifold of dimension $k(n-k)$ and is actually diffeomorhic to $O(n)/(O(k) \times O(n-k))$. Similar results hold for the complex cases.

Example (iii) Let $X_k := \{A \in M_n(\mathbb{R}) | \mathrm{rk} A = k\}$ be a subset of the $n \times n$ real matrices. One can associate natural bundles $E \to X_k$ and $Q \to X_k$ with $E_A = \ker A$ and $Q_A = \mathrm{Im} A$. We also have a short exact sequence of bundles :

$$0 \to E \to X_k \times \mathbb{R}^n \to Q \to 0.$$

Example (iv) $T \equiv \{A \in M_n(\mathbb{C}) | A^2 = A, \mathrm{rk} A = 1\}$ is an algebraic subvariety in $\mathbb{C}^{n^2}$. This effectively says that that the trivial bundle $\mathbb{C}^n = \ell \oplus \mathcal{K}$ where $A|_\ell = \mathrm{Id}|_\ell$ and $\mathcal{K} = \mathrm{Im} A$. There is the usual holomorphic map $\pi : T \to \mathbb{CP}^{n-1}$ sending A to its image, a line in $\mathbb{C}^n$. Note that $\pi^{-1}(\ell) = \{H | \text{hypersurfaces } H \text{ such that } H \cap \ell = \{0\}\} \cong \mathrm{Hom}(\ell^\perp, \ell)$. This is also called **torsor**.

If X is a manifold, i.e., a locally Euclidean space then one can define a linear space at each point of $x \in X$. This will be called the tangent space at x and can be defined in various ways. The manifold in question can be C^∞ or C^k depending on how the Euclidean pieces are glued together.

Definition 1.1.6 *Let X be a smooth manifold and $x \in X$. The germ of a (smooth) function at x is defined to be the equivalence pair (U, f) where U is a neighbourhood of x and $f : U \to \mathbb{R}$ is a smooth function under the equivalence relation $(U, f) \sim (V, g)$ if there exists a smaller neighbourhood W of x contained in $U \cap V$ such that $f|_W \equiv g|_W$. The set of all germs forms an $\mathbb{R}$-algebra and is denoted by $\mathcal{O}_{X,x}$.*
The (real) vector space of all derivations of $\mathcal{O}_{X,x}$ is called the **tangent space** *of X at x. It is denoted by $T_x X$ and the elements are called* **tangent vectors**.

There is a surjective $\mathbb{R}$-algebra homomorphism

$$\chi : C^\infty(X) \to \mathcal{O}_{X,x}, \;\; f \mapsto [f]$$

sending the function to its germ at x. There is also a natural evaluation map (a homomorphism of $\mathbb{R}$-algebras)

$$e : \mathcal{O}_{X,x} \to \mathbb{R}, \;\; [f] \mapsto f(x)$$

which is also surjective. The kernel is the unique maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$. Working locally we see that this tangent space can also be thought of as the "totality" of all directions in X at x. This turns out to be independent of the chart chosen. It can be shown that the $\mathbb{R}$ vector space T_xX of $\mathbb{R}$ derivations of $\mathcal{O}_{X,x}$ is isomorphic to the vector space $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{R})$ by mapping X to the linear functional $f \to X(f)$. The vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$ is called the **cotangent space** to X at x and denoted by T_x^*X. Taking the disjoint union of T_xX (resp. T_x^*X) and pulling back the topology from X we can make

$$TX := \coprod_{x \in X} T_xX \;(\mathrm{resp}\, T^*X := \coprod_{x \in X} T_x^*X)$$

into a smooth manifold of dimension $2n$ called the **tangent bundle** (resp. **cotangent bundle**). For any smooth map $f : X \to Y$ there is an induced map $f_* = Df : TX \to TY$ which obeys the chain rule.

Definition 1.1.7 *Let $f : X \to Y$ be a smooth map between manifolds (of dim $X = m$ and dim $Y = n$).*
(a) f is an **immersion** *if $f_x : T_xX \to T_{f(x)}Y$ is injective for all $x \in X$.*
(b) f is a **submersion** *if $f_x : T_xX \to T_{f(x)}Y$ is surjective for all $x \in X$.*

A local description of immersions and submersions can be given. One chooses a suitable chart around each point $x \in X$ and $f(x) \in Y$. Then the map f looks like inclusion of $\mathbb{R}^m$ into $\mathbb{R}^n$ via the first m coordinates if f is an immersion and looks like the projection onto the first n coordinates if f is a submersion. This follows from the implicit function theorem.

We can construct new vector bundles from given ones. A general guiding principle is that any natural operation of vector spaces carries over to vector bundles. Thus an inclusion of bundles $E \to X$ into $F \to X$ gives rise to the **quotient bundle** $F/E \to X$. Further given any two bundles E, F over X one can form the **direct sum bundle** $E \oplus F$, the **tensor product bundle** $E \otimes F$, the bundle $\mathrm{Hom}_{\mathbb{R}}(E, F)$, the **dual bundle of** E $E^* = \mathrm{Hom}_{\mathbb{R}}(E, X \times \mathbb{R})$.

Example $\bigwedge^p T^*X$ is called the bundle of exterior p forms. The direct sum

$$\bigwedge T^*X := \bigoplus_{p \geq 0} \bigwedge^p T^*X$$

is an algebra with a self map $d : \bigwedge^p T^*X \to \bigwedge^{p+1} T^*X$ such that $d^2 = 0$.

Replacing the fibre $\mathbb{R}^n$ in vector bundles with a topological space F would result in the notion of **fibre bundles** *which do not enjoy such liberties in construction.*

For any two bundles $h : E \to \tilde{E}$ over X choose a common chart for both bundles and denote the transition functions by g_{ij} and $\tilde{g}_{ij}$ respectively. It can be shown that E is isomorphic to $\tilde{E}$ if and only if there exists maps $h_i : U_i \to GL_n(\mathbb{R})$ such that

$$g_{ij}h_j = h_i\tilde{g}_{ij}.$$

Thus it provides a criteria for saying when a bundle is trivial, i.e., $g_{ij} = h_ih_j^{-1}$.

Definition 1.1.8 *Given continuous maps $f : X \to B$ and $g : Y \to B$ define $X \times_B Y = \{(x, y) \in X \times Y | f(x) = g(y)\}$.*

If $X \to B$ is a bundle then

$$\tilde{f} : X \times_B Y \to Y,\ (x, y) \mapsto y$$

is also a bundle with the same fibre as $X \to B$ and is called the **pullback** of $X \to B$ by g. It is easy to see that f is proper/finite/surjective/injective implies that $\tilde{f}$ is also so.

Definition 1.1.9 *Suppose X, Y, B are manifolds and $f : X \to B, g : Y \to B$ are smooth. Then f is transversal to g (write $f \pitchfork g$) if*

$$f_*T_xX + g_*T_yY = T_zB$$

for all $(x, y) \in X \times Y$ such that $f(x) = z = g(y)$.

Lemma 1.1.10 *For maps $f : X \to B, g : Y \to B$ such that $f \pitchfork g$, $X \times_B Y$ is a smooth submanifold of $X \times Y$ (of codimension = dim B).*

Proof Choose local coordinates $(x_i), (y_j), (z_k)$ on X, Y and Z respectively. Now $(x, y) \in X \times_B Y$ if and only if $F(x, y) := f(x) - g(y) = 0$. Then

$$F_* = f_* - g_* : T_xX \oplus T_yY \to T_zB$$

is surjective if and only if $f \pitchfork g$. A simple application of inverse function theorem then gives the result. ☐

This result has a number of corollaries :

Corollary 1.1.11 *If f is a submersion then $X \times_B Y$ is a submanifold and $\tilde{f} : X \times_B Y \to Y$ is also a submersion.*

Proof Since f is a submersion we have $f \pitchfork g$ and $X \times_B Y$ is a submanifold. Also

$$T_{(x,y)}X \times_B Y = \{(v, w) \in T_xX \oplus T_yY | f_*(v) = g_*(w)\}$$

and $\tilde{f}_*(v, w) = w$. Since f_* is surjective, given $w \in T_yY$, there exists $v \in TxX$ such that $f_*(v) = g_*(w)$ whence $\tilde{f}$ is also a submersion. ☐

Corollary 1.1.12 *If f is a smooth fibre bundle over B then $\tilde{f}$ is a smooth fibre bundle over Y.*

Corollary 1.1.13 *If $f \pitchfork g$ and f is an immersion then $\tilde{f}$ is an immersion.*

Proof If $\tilde{f}_*(v, w) = w = 0$ then $f_*(v) = g_*(w) = 0$ implies $v = 0$ since f_* is injective. ☐

Proposition 1.1.14 *Let $E \xrightarrow{\pi} B$ be a vector bundle of rank n and $g : Y \to B$ a continuous map. Then $\tilde{\pi} : E \times_B Y$ is vector bundle of rank n over Y and $\tilde{g}$ (refer figure below) is a morphism of bundles.*

$$\begin{tikzcd}
g^*(E) \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{\pi}"'] & E \arrow[d, "\pi"] \\
Y \arrow[r, "g"] & B
\end{tikzcd}$$

Here $g^*(E) = E \times_B Y$ is called the **pullback** *of E by g. Further, if π and g are smooth them $\tilde{\pi}$ is also smooth.*

Proof First notice that

$$\tilde{\pi}^{-1}(y) = \{(e,y) \in E \times Y | \pi(e) = g(y)\} = \pi^{-1}(g(y)) \cong E_{g(y)}$$

has the structure of an n dimensional vector space. If local trivializations of $\pi^{-1}(U)$ are given by cross sections $e_1, \cdots, e_n \in \Gamma(E|_U)$ then local trivializations of $\tilde{\pi}^{-1}(g^{-1}(U))$ are given by cross sections $e_1 \circ g, \cdots, e_n \circ g$ of $g^*(E)$. Further if g_{ij} are the transition functions for E then $g_{ij} \circ g$ are the transition functions for $g^*(E)$. $\square$

It is easily verified that

Exercise (i) $g^*(E \oplus F) = g^*E \oplus g^*F$
(ii) $g^*(E \otimes F) = g^*E \otimes g^*F$
(iii) $g^*(\bigwedge^k E) = \bigwedge^k g^*E$
(iv) $(g \circ f)^*E \cong f^*(g^*E)$.

Set $Vect_n(X) = \{$isomorphism classes of vector bundles of rank n on $X\}$. Any continuous map $g : X \to Y$ induces a map

$$g^* : Vect_n(Y) \to Vect_n(X).$$

We define

$$\nu(X) := \coprod_{n \geq 0} Vect_n(X)$$

and endowed with the operations $\oplus, \otimes$ this becomes a semi-ring. We define the group completion by setting

$$\mathcal{K}(X) = (\nu(X) \times \nu(X))/\sim$$

where $(E,F) \sim (E',F')$ if and only if $\exists G \in \nu(X)$ such that $G \oplus E' \oplus F \cong G \oplus E \oplus F'$. This turns $\mathcal{K}(X)$ into a ring and the induced map $g^* : \mathcal{K}(Y) \to \mathcal{K}(X)$ is a ring homomorphism. The group G acting on the fibre (for $\mathbb{R}^n$ it is usually $GL_n(\mathbb{R})$) of a bundle $E \to X$ is called the **structure group**. Recall that prescribing a bundle $E \to X$ is the same as giving cocycles with values in the structure group G. Let $G \subseteq GL_n(\mathbb{R})$ be Lie subgroup.

Definition 1.1.15 (Reduction of the structure group) *Let $E \to X$ be a vector bundle of rank n. Then a reduction of structure of E to $G \subseteq GL_n(\mathbb{R})$ is a cocycle $(\mathcal{U}, g)$ with $E \cong E(\mathcal{U}, g)$ and $g_ij : U_i \cap U_j \to G \subseteq GL_n(\mathbb{R})$.*

Suppose $T_0 \in (\mathbb{R}^n)^{\otimes n} \otimes (\mathbb{R}^n)* \otimes l$ such that $gT_0 = T_0$ for all $g \in G$. Then T_0 defines a global section

$$T \in \Gamma(E^{\otimes k} \otimes E^{*\otimes l})$$

given by $T(x) = T_0$ in each trivialization.
Conversely if $T \in \Gamma(E^{\otimes k} \otimes E^{*\otimes l})$ where $E = E(\mathcal{U}, g)$ then let T_i be the representation of T in the local trivialization over U_i, i.e.,

$$T_i : U_i \to \mathbb{R}^{\otimes k} \otimes \mathbb{R}^{*\otimes l}$$

??

Example (i) $G = O_n \subseteq GL_n(\mathbb{R})$ - a reduction to O_n determines a metric on E, i.e., $\langle , \rangle \in \Gamma(E^* \otimes E^*)$. Using a partition of unity it can be shown that every vector bundle over a paracompact space has a metric. In general the structure can always be reduced from $GL_n(\mathbb{R})$ to O_n since GL_n deformation retracts to O_n.

Example (iii) $GL_n^+(\mathbb{R}) \subseteq GL_n(\mathbb{R})$ - Amounts to choosing an orientation on E.

Example (iii) $GL_n(\mathbb{C}) \subseteq GL_{2n}(\mathbb{R})$ - Amounts to choosing $J : E \to E$ such that $J^2 = \mathrm{Id}$. In other words $J \in \Gamma(E^* \otimes E) = \Gamma(\mathrm{Hom}(E, E))$. This makes E_x into a complex vector space.

Example (iv) $SU_n \subseteq GL_{2n}(\mathbb{R})$ - Amounts to choosing (i) J as before, (ii) an inner product $\langle , \rangle$ such that $\langle Jv, Jw \rangle = \langle v, w \rangle$ and (iii) a global section $\phi \in \Gamma(\bigwedge_{\mathbb{C}}^n E)$.

Example (v) Octonions - Let $\ominus$ denote the octonions and $G_2 = \mathrm{Aut}(\ominus)$. We have $G_2 \subseteq SO(7) \subseteq GL(7) = GL(\mathrm{Im}\,\ominus)$. Reduction to G_2 gives a bundle **??**

1.2 G-Bundles

Let G be a topological group and P be a topological space.

Definition 1.2.1 *P is called a* **right G-space** *if there exists a continuous map $P \times G \to P$ such that*

$$(p \cdot g_1) \cdot g_2 = p \cdot (g_1 g_2) \, \forall \, p \in P, g_1, g_2 \in G.$$

P is a **free G space** *if there are no fixed points of the G action.*

Let $\pi : P \to P/G \equiv X$ be the orbit map. It is continuous if we put the quotient topology on X.

Definition 1.2.2 *A morphism of (right) G-spaces over X $(\pi : P \to X, \tilde{\pi} : \tilde{P} \to X)$ is a map $h : P \to \tilde{P}$ such that $\tilde{\pi} \circ h = \pi$ and $h(pg) = h(p)g$.*

The trivial right G space over X is $X \times G$ with right multiplication on G.

Definition 1.2.3 *A* **principal G bundle** *over a topological space X is a free right G space $\pi : P \to X$ which is locally trivial (with fibre G).*

Example (i) $H < G$ closed subgroup - $\pi : G \to G/H$ is a principal H bundle. For example $SO_n \to SO_n/SO_{n-1}$ corresponds to an oriented o.n. tangent frame bundle.

Example (ii) Universal cover - Let $\pi : \tilde{X} \to X$ be the universal cover of X. It is a principal $\pi_1(X)$ bundle.

Example (iii) Normal covers - Let $\pi : X_H \to X$ be a normal cover of X with $\pi_1(X_H) = H \triangleleft \pi_1(X)$. Then it is a principal $\pi_1(X)/H$ bundle.

Example (iv) Frame bundles - Let $E \to X$ be a vector bundle. One can construct the frame bundle $P_{GL}(E) \xrightarrow{\pi} X$ where $\pi^{-1}(x) =$ all basis of E_x. Observe that for any two frame B, B' of E_x there exists $g \in GL_n(R)$ such that $B = B'g$. This turns it into a principal $GL_n(\mathbb{R})$ bundle.
If we have a metric on E then we can define the bundle of o.n. frames (denoted by $P_O(E)$) which is a principal O_n bundle. Further, if E has an orientation then there is the $P_{SO}(E)$, a principal SO_n bundle consisting of oriented o.n. frames.

Example (v) Let $g \in SO_n$. Considering the columns of g as vectors in $\mathbb{R}^n$ we may think of g as a n-tuple of vectors, i.e., $g = (e_1 | \cdots | e_n)$. This allows us to define

$$\pi : SO_n \to S^{n-1}, \; g \mapsto e_1.$$

Observe that $\pi^{-1}(e_1) =$ all oriented o.n. bases of $e_1^{\perp} = T_{e_1} S^{n-1}$. This gives us a principal SO_{n-1} bundle.

Definition 1.2.4 *Let $E \to X$ be a vector bundle with a $G \subseteq GL_n(\mathbb{R})$ structure. Then E is given by a cocycle, i.e., $E = E(\mathcal{U}, g), g = \{g_{ij}\}_{i,j \in I}$ such that $g_{ij} : U_i \cap U_j \to G$. The* **associated principal G-bundle** *is defined as follows :*

For each $i \in I$ we take $U_i \times G$ with G acting on the right. A change of trivialization (or an equivalence relation $\sim$) would be given by

$$(U_i \cap U_j) \times G \to (U_i \cap U_j) \times G$$
$$(x, g) \mapsto (x, g_{ij}(x).g).$$

Set

$$P := \coprod_i (U_i \times G)/\sim$$

to be the required bundle over X.

Observe that $P_{GL_n}(E)$ is just the frame bundle and $P_{O_n}(E)$ is the o.n. frame bundle of the Riemannian vector bundle E. In general $P_G(E)$ is a subset of $P_{GL_n}(E)$. In other words we have

$$\begin{tikzcd}
P_G(E) \arrow[rr, hookrightarrow] \arrow[dr, "\pi_G"'] & & P_{GL}(E) \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}$$

and dividing the inclusion by G we have

$$\begin{tikzcd}
P_G(E)/G \arrow[rr] \arrow[dr, "\cong"'] & & P_{GL}(E)/G \arrow[dl, "\tilde{\pi}"'] \\
& X \arrow[ur, bend right, "s"'] &
\end{tikzcd}$$

Thus the following tells us when such reductions exist.

Lemma 1.2.5 *Let $P_G \to X$ be a principal G-bundle and $H \subset G$ be a closed subgroup. Then reductions $P_H \subset P_G$ are in one-to-one correspondence with sections s of the fibre bundle $P_G/H \to X$ with fibre G/H.*

Example (i) $H = \{1\}$ - The trivializations of X correspond bijectively to $\Gamma(P_G)$.
Example (ii) $H = O_n \subset GL_n = G$ for $P_{GL_n}(E) \to X$ - Since GL_n/O_n is just the positive definite inner products on $\mathbb{R}^n$,

$$P_{GL_n}(E)/O_n \cong \text{bundle of positive definite inner products on } E$$

Thus reductions to O_n are in bijective correspondence with $\Gamma(P_{GL_n}(E)/O_n$.

Using Čech cohomology we have another approach to principal G-bundles. Let $\rho : G \to GL_n$ be a representation of G (n arbitrary).

Definition 1.2.6 *Define the* **associated vector bundle** *for a principal G-bundle $P \to X$ and a given ρ to be*

$$E_\rho := P \times_G \mathbb{R}^n \equiv P \times \mathbb{R}^n/G$$

where G acts by

$$g(p, v) := (pg^{-1}, \rho(g)v).$$

The associated bundle construction will be shortened to ABC. If $\{g_{ij}\}$ are the transition functions for P then $\{\rho \circ g_{ij}\}$ are the transition functions for E_ρ. A special case is the inclusion $G \hookrightarrow GL_n$.
Example (i) Let $P = P_{GL_n}(E)$ and

$$\rho : GL_n \to GL(\underbrace{\mathbb{R}^n \oplus \cdots \oplus \mathbb{R}^n}_{m}).$$

Then $E_\rho = E \oplus \cdots \oplus E$.
Example (ii) Let $P = P_{GL_n}(E)$ and

$$\rho : GL_n \to GL(\underbrace{\mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n}_{m}).$$

Then $E_\rho = E \otimes \cdots \otimes E$.
Example (iii) Let $P \to X$ be a principal G-bundle and $\rho : G \to GL_n$. Then there are associated representations $\otimes^k \rho, \otimes^k \rho$ and $\wedge^k \rho$. Then

$$E_{\oplus^k \rho} = \oplus^k E_\rho,\ E_{\otimes^k \rho} = \otimes^k E_\rho,\ E_{\wedge^k \rho} = \bigwedge^k E_\rho.$$

For a fixed P, ABC sends representations of G into vector bundles (with G structure) on X.
Example (iv) Let $\tilde{X} \to X$ be the universal covering map. This is a principal $\pi_1(X)$-bundle. Let $\rho : \pi_1(X) \to GL_n$. Since $\pi_1(X)$ has the discrete topology, E_ρ is a vector bundle with *locally constant* transition functions.
Suppose $h : P \to \tilde{P}$ is an isomorphism. Then by

$$U_i \times G \overset{\phi_i}{\leftarrow} \pi^{-1}(U_i) \overset{h}{\to} \tilde{\pi}^{-1}(U_i) \overset{\tilde{\phi}_i}{\to}$$

$$(x, g) \mapsto (x, h_i(x)g)$$

we have maps $h_i : U_i \to G$. Using the commutative diagram below (corresponding to the a change of trivialization)

$$\begin{tikzcd}
(U_i \cap U_j) \times G \arrow[r, "h_i"] \arrow[d, "g_{ji}"'] & (U_i \cap U_j) \times G \arrow[d, "\tilde{g}_{ji}"] \\
(U_i \cap U_j) \times G \arrow[r, "h_j"] & (U_i \cap U_j) \times G
\end{tikzcd}$$

we have

$$\begin{tikzcd}
(x, g) \arrow[rr, "h_i"] \arrow[d, "g_{ji}"] & & (x, h_i(x)g) \arrow[d, "\tilde{g}_{ji}"] \\
(x, g_{ji}(x)g) \arrow[r, "h_j"] & (x, h_j(x)g_{ji}(x)g) = (x, \tilde{g}_{ji}(x)h_i(x)g) &
\end{tikzcd}.$$

As a consequence we get

$$\tilde{g}_{ij}(x) = h_j(x)g_{ji}(x)h_i^{-1}(x).$$

Using the Čech cohomology theory we see that

$$\mathrm{Prin}_G(X) \cong H^1(X, G).$$

Thus, for $G \subset GL_n$, a closed subgroup,

{isomorphism classes of rank n vector bundles with structure group G}

$$1 - 1 \updownarrow$$

{isomorphism classes of principal G-bundles}

since for a vector bundle E the associated principal G-bundle has the same transition functions. Conversely, given a principal G-bundle using the ABC we get a rank n vector bundle.

1.3 Classification of Bundles

We want to classify isomorphism classes of vector bundles of rank n over a compact, Hausdorff space X. For this we need to study the grassmanians. Recall that

$$G_n(\mathbb{R}^N) \equiv \{n - \text{dimensional linear subspaces of } \mathbb{R}^N\}$$

which is diffeomorphic to $O_N/(O_n \times O_{N-n})$. We have the tautological vector bundle

$$\begin{tikzcd} \mathbb{E}_n^N = \{(P, v) \in G_n(\mathbb{R}^N) \times \mathbb{R}^N | v \in P\} \arrow[d, "\pi"] \\ G_n(\mathbb{R}^N) \end{tikzcd}$$

The nested sequence of inclusions $\mathbb{R}^N \subset \mathbb{R}^{N+1} \subset \mathbb{R}^{N+2} \subset \cdots$ (via the first $N, N+1, \ldots$ coordinates resp.) we have the following :

$$\begin{tikzcd} \mathbb{E}_n^N \arrow[r, phantom, "\subset"] \arrow[d, "\pi"] & \mathbb{E}_n^{N+1} \arrow[r, phantom, "\subset"] \arrow[d, "\pi"] & \cdots \\ G_n(\mathbb{R}^N) \arrow[r, phantom, "\subset"] & G_n(\mathbb{R}^{N+1}) \arrow[r, phantom, "\subset"] & \cdots \end{tikzcd}$$

Definition 1.3.1 *Let $G_n(\mathbb{R}^\infty)$ be the union of $G_n(\mathbb{R}^N)$'s as N varies. We provide it with the* **direct limit topology** *coming from the compact sets*

$$K_1 \subset K_2 \subset K_3 \subset \cdots$$

where $K_k = G_n(\mathbb{R}^{n+k})$. A set $C \subseteq G_n(\mathbb{R}^\infty)$ is **closed** *if and only if $C \cap K_k$ is closed in K_k for all k.*

We may define a space $\mathbb{E}_n \to G_n(\mathbb{R}^\infty)$ by defining it to be the union of $\mathbb{E}_n^N$ and putting the direct limit topology. We shall need some facts from general topology to prove that this a vector bundle. We restate

Definition 1.3.2 *Let Y be a space with a filtration*

$$K_1 \subset K_2 \subset K_3 \subset \cdots$$

such that Y is the union of it and each K_j is a compact Hausdorff space. Further $K_j \subset K_{j+1}$ is an embedding. The **weak/direct limit/compactly generated** *topology is defined by saying :*

a subset C is closed if and only if $C \cap K_j$ is closed in K_j for all j.

Example (i) $G_n(\mathbb{R}^{n+1}) \subseteq G_n(\mathbb{R}^{n+2}) \subseteq \cdots$
Example (ii) $S^n \subset S^{n+1} \subset \cdots$
Example (iii) $\{K_i\}_{i \geq 1}, K_i = \{x \in \mathbb{R}^i \text{ s.t. } \|x\| \leq i\}$.

Lemma 1.3.3 *Let $Y = \bigcup_{i \geq 1} K_i$ be as above. Then a closed subset $C \subset Y$ is compact if and only if $C \subset K_n$ for some n.*

Proof The 'if' direction is trivial. Conversely, suppose on the contrary $C \not\subseteq K_n$ for all n. Then choose $x_n \in C \setminus K_n$. This sequence has no convergent subsequence, a contradiction. ☐

Definition 1.3.4 *Given topological spaces X, Y define $[X, Y]$ to be the homotopy classes of continuous maps from X to Y.*

It follows from the lemma that

Corollary 1.3.5 *If $Y = \bigcup_{i \geq 1} K_i$ has the weak topology and if X is compact then*

$$[X, Y] = \varinjlim_j [X, K_j].$$

Consequently we have :

$$\pi_n(Y) = [S^n, Y] = \varinjlim_j [X, K_j].$$

We state without proof the following :

Proposition 1.3.6 *Let $V_1 \subset V_2 \subset \cdots$ and $W_1 \subset W_2 \subset \cdots$ be locally compact Hausdorff spaces with weak topologies. Let there be filtrations*

$$K_1 \subset K_2 \subset \cdots \subset K_i' \subset V_i$$

$$L_1 \subset L_2 \subset \cdots \subset L_i' \subset W_i.$$

Then $V \times W$ is homeomorphic to the direct limit of the $K_j \times L_i$'s.

We are ready to prove that $\mathbb{E}_n \xrightarrow{\pi} G_n(\mathbb{R}^\infty)$ is a vector bundle. Given $P \in G_n(\mathbb{R}^\infty)$ (this means $P \in G_n(\mathbb{R}^N)$ for some N), set

$$\begin{aligned} U(P) &:= \big\{Q \in G_n(\mathbb{R}^\infty) | P^\perp \cap Q = \{0\}\big\} \\ &= \bigcup_{M \geq N} \big\{Q \in G_n(\mathbb{R}^M) | P^\perp \cap Q = \{0\} \, \mathrm{in} \, \mathbb{R}^M\big\}. \end{aligned}$$

This is an open set. Now pick a basis $v_1, \ldots, v_n$ of P. Define continuous sections

$$\sigma_k : U(P) \to \pi^{-1}(U(P)), \ Q \mapsto w_k \in Q \text{ s.t. } \mathrm{pr}^\perp(w_k) = v_k$$

where the map $\mathrm{pr}^\perp$ maps Q isomorphically to P via projection from $\mathbb{R}^M$ to P. Thus it is just the frame bundle of $G_n(\mathbb{R}^\infty)$. There are principal and o.n. frame bundles also.

Definition 1.3.7 *$St_n^\circ(\mathbb{R}^N)$ is the set of o.n. n-frames in $\mathbb{R}^N$. This is called the* **Stiefel manifold** *and is compact.*

Alternatively

$$St_n^\circ(\mathbb{R}^N) = \{(e_1, \cdots, e_n) \in \underbrace{\mathbb{R}^N \times \cdots \times \mathbb{R}^N}_{n} | e_i\text{'s are mutually o.n.}\}$$

and looks like the quotient O_N/O_{N-n}. There is a natural map

$$\rho : St_n^\circ(\mathbb{R}^N) \to S^{N-1}, \ (e_1, \cdots, e_n) \mapsto e_1.$$

This makes it into fibre bundle with $St^\circ_{n-1}(\mathbb{R}^{N-1})$ as its fibre. Similarly we have the fiber bundle

$$\begin{tikzcd}
St^\circ_{n-2}(\mathbb{R}^{N-2}) \arrow[r] & St^\circ_{n-1}(\mathbb{R}^{N-1}) \arrow[d] \\
 & S^{N-2}.
\end{tikzcd}$$

Proceeding recursively we get a fibre bundle

$$\begin{tikzcd}
S^{N-n} \arrow[r] & St^\circ_2(\mathbb{R}^{N-n+2}) \arrow[d] \\
 & S^{N-n+1}.
\end{tikzcd}$$

Using the long exact sequence for a fibration we see that $St^\circ_n(\mathbb{R}^N)$ is $(N-n-1)$ connected. Consequently

$$\pi_k(St^\circ_n) = \lim_{N\to\infty} \pi_k(St^\circ_n(\mathbb{R}^N)) = 0 \ \forall \ k.$$

Since St°_n has a CW complex structure, by Whitehead's theorem on homotopy equivalence of CW complexes we conclude :

Theorem 1.3.8 (Whitehead) *St°_n is contractible.*

Finally we state

Theorem 1.3.9 *Let X be a compact Hausdorff space. Then the induced bundle construction gives a bijection*

$$[X, G_n(\mathbb{R}^\infty)] \cong Vect_n(X),\ f \mapsto f^*\mathbb{E}_n.$$

Proof Given $E \to X$, a vector bundle of rank n, it suffices to find a continuous map $F : E \to \mathbb{R}^N$ lo large N which is linear and injective on every fibre $E_x, x \in X$. Then set $f(x) := [F(E_x)] \in G_n(\mathbb{R}^N)$. It is easily verified that $E \cong f^*\mathbb{E}_n(\mathbb{R}^N) = f^*\mathbb{E}_n$ producing the pullback :

$$\begin{tikzcd}
E \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & \mathbb{E}_n(\mathbb{R}^N) \subseteq \mathbb{E}_n \arrow[d] \\
X \arrow[r, "f"] & G_n(\mathbb{R}^N) \subseteq G_n(\mathbb{R}^\infty)
\end{tikzcd}$$

where $\tilde{f}(e) = (f(\pi(e), F(e))$.
Since the pullback by homotopic maps yield isomorphic bundles the map

$$[X, G_n(\mathbb{R}^\infty)] \overset{\Phi}{\to}\!\!\to Vect_n(X),\ f \mapsto f^*\mathbb{E}_n$$

is well defined and surjective. Thus every isomorphism class of vector bundle $E \to X$ gives a unique homotopy class in $[X, G_n(\mathbb{R}^\infty)]$. Using the fact that two bundles are isomorphic if and only if the maps from the base to $G_n(\mathbb{R}^\infty)$ are homotopic (Covering Homotopy Theorem) we get that Φ is a bijection.
For each $x \in X$ there are open sets $W \subseteq V \subseteq U$ containing x such that
(i) $\overline{W} \subset V, \overline{V} \subset U$

(ii) $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ is a local trivialization.
Cover X by finitely many of these $W_1, \ldots, W_l$. Choose $\rho_k : U_k \to [0,1]$ such that it is 1 on $\overline{W_k}$ and 0 on $U_k \setminus V_k$. Extend it to X by zero. Also let $\Phi_k := \mathrm{pr} \circ \phi_k : \pi^{-1}(U_k) \to \mathbb{R}^n$. Define

$$F : E \to \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{l},$$

$$F(e) = (\rho_1(\pi(e)\Phi_1(e), \ldots, \rho_l(e)\Phi_l(e)).$$

Then F is linear and injective. With a modification this construction works for X paracompact Hausdorff spaces and in particular for manifolds and metric spaces. □

The diagram below commutes upto homotopy

$$\begin{tikzcd}
\mathbb{R}^N \times \mathbb{R}^N \arrow[r, "\oplus"] \arrow[d] & \mathbb{R}^{2n} \arrow[d] \\
\mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \arrow[r, "\oplus"] & \mathbb{R}^{2N+2}
\end{tikzcd}$$

induces one between the grassmanians :

$$\begin{tikzcd}
G_n(\mathbb{R}^N) \times G_m(\mathbb{R}^N) \arrow[r] \arrow[d] & G_{n+m}(\mathbb{R}^{2N}) \arrow[d] \\
G_n(\mathbb{R}^{N+1}) \times G_m(\mathbb{R}^{N+1}) \arrow[r] & G_{n+m}(\mathbb{R}^{2N+2})
\end{tikzcd}$$

Passing to the limit gives a map $\sigma : G_n(\mathbb{R}^\infty) \times G_m(\mathbb{R}^\infty) \to G_{n+m}(\mathbb{R}^\infty)$ such that

$$\sigma^*(\mathbb{E}_{n+m}) = \mathbb{E}_n \oplus \mathbb{E}_m.$$

Thus if $f_E : X \to G_n(\mathbb{R}^\infty), f_F : X \to G_m(\mathbb{R}^\infty)$ classifies E, F respectively then $\sigma \circ (f_E, f_F) : X \to G_{n+m}(\mathbb{R}^\infty)$ classifies $E \oplus F$. Similarly we have tensor products

$$G_n(\mathbb{R}^\infty) \times G_m(\mathbb{R}^\infty) \xrightarrow{\tau} G_{nm}(\mathbb{R}^\infty)$$

sending (P, Q) to $P \otimes Q$. Also $\tau^*(\mathbb{E}_{mn}) = \mathbb{E}_n \otimes \mathbb{E}_m$.

??

1.4 Characteristic Classes

Recall that for a topological space X, $C_n(X)$ is just the free abelian groups generated by maps $f : \Delta^n \to X$. Equipped with the usual boundary map $\partial : C_n(X) \to C_{n-1}(X)$ such that $\partial^2 =$ this becomes a graded chain complex. The homology of this complex is the **simplicial homology** of X and denoted by $H_n(X, \mathbb{Z})$. If $f : X \to Y$ then there is an induced $f_* : C_*(X) \to C_*(Y)$ which descends to the homology. Now let Λ be an abelian group. Define

$$C^n(X, \Lambda) \equiv \mathrm{Hom}_{\mathbb{Z}}(C_n(X), \Lambda)$$

$$\delta : C^n(X, \Lambda) \to C^{n+1}(X, \Lambda),\ \ \delta\phi := \phi \circ \partial.$$

$\partial^2 = 0$ implies $\delta^2 = 0$. The homology of this complex will be the **simplicial cohomology** of X with coefficients with Λ and denoted $H^n(X, \Lambda)$. For f as before, there is an induced map $f^* : H^*(Y) \to H^*(X)$. Under the assumption that Λ is a ring, there is a product structure on the cohomology groups called the **cup product** :

$$H^l(X, \Lambda) \otimes H^m(X, \Lambda) \xrightarrow{\smile} H^{l+m}(X, \Lambda)$$

such that This turns $H^*(X, \Lambda)$ into a graded commutative ring. Finally, for $\alpha \in C^l(X, \Lambda), \beta \in C^m(X, \Lambda)$

$$\alpha \smile \beta(\langle v_0, \ldots, v_{l+m}\rangle) = \alpha(\langle v_0, \ldots, v_l\rangle)\beta(\langle v_l, \ldots, v_{l+m}\rangle).$$

For any $\mathcal{U} \in H^l(G_n(\mathbb{R}^\infty), \Lambda)$ (call it a $\mathcal{U}$-characteristic class) we set

$$\mathcal{U}(E) \equiv f_E^*(\mathcal{U}) \in H^l(X, \Lambda)$$

for any $f_E : X \to G_n(\mathbb{R}^\infty)$ classifying $E \in Vect_n(X)$.

Lemma 1.4.1 *If $f : Y \to X$ is a continuous map of vector spaces and $E \to X$ is a vector bundle over X then*

$$\mathcal{U}(f^*E) = f^*(\mathcal{U}(E).$$

Proof We have

$$Y \xrightarrow{f} X \xrightarrow{f_E} G_n(\mathbb{R}^\infty)\ .$$

Therefore $\mathcal{U}(f^*E) = (f_E \circ f)^*(\mathcal{U}) = f^*(f_E^*\mathcal{U}) = f^*(\mathcal{U}(E))$. □

So $E \cong F$ implies $\mathcal{U}(E) = \mathcal{U}(F)$ for any $\mathcal{U}$.

Example (i) $G_1(\mathbb{R}^\infty) = \mathbb{P}^\infty(\mathbb{R}) = S^\infty/\mathbb{Z}_2$ is also the direct limit of $\mathbb{P}^n(\mathbb{R})$'s. It is known that

$$H^*(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}_2) = \mathbb{Z}_2[x]/(x^{n+1})$$

and $H^*(\mathbb{P}^\infty(\mathbb{R}), \mathbb{Z}_2) = \mathbb{Z}_2[x]$. Let $w_1 = x \in H^1(\mathbb{P}^\infty, \mathbb{Z}_2) = \mathrm{Hom}(H_1(\mathbb{P}^\infty), \mathbb{Z}_2)$. Given a line bundle $\ell \to X$

$$w_1(L) = f_\ell^*(w_1) \in H^1(X, \mathbb{Z}_2) = \mathrm{Hom}(\pi_1(X), \mathbb{Z}_2).$$

$w_1(\ell)$ is the **orientation class**. For a loop $\gamma \subseteq X$, $\ell|_\gamma$ is trivial or the Möbius band if and only $w_1(\ell|_\gamma) = 0$ or 1 respectively. In fact, the following is an isomorphism

$$Vect_1^{\mathbb{R}}(X) \xrightarrow[w_1]{\cong} H^1(X, \mathbb{Z}_2).$$

To see this let $\ell \to X$ be a line bundle and $S(\ell) \to X$ be the unit sphere bundle which is also a principal $\mathbb{Z}_2$-bundle. Then $\ell = S(\ell) \times_{\mathbb{Z}_2} \mathbb{R}$ is the associate bundle. Thus $Vect_1^{\mathbb{R}}(X) \cong \mathrm{Prin}_{\mathbb{Z}_2}(X)$ is just the $\mathbb{Z}_2$ covering space of X. But the latter is just the group $\mathrm{Hom}(\pi_1(X), \mathbb{Z}_2) \cong H^1(X, \mathbb{Z}_2)$.

In the complex case $\mathbb{P}^\infty(\mathbb{C}) = G_1(\mathbb{C}^\infty)$ and $H^*(\mathbb{P}^\infty(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}[c_1]$ where c_1 generates $H^2(\mathbb{P}^\infty, \mathbb{Z}) = \mathbb{Z}$. Let $\lambda \to X$ be a $\mathbb{C}$-line bundle with a classifying map $f_\lambda : X \to \mathbb{P}^\infty$. As before

$$\lambda = P_{S^1}(\lambda) \times_{S^1} \mathbb{C}$$

where $P_{S^1}(\lambda)$ is the unit circle bundle of λ. Thus

$$Vect_1^{\mathbb{C}}(X) \cong \mathrm{Prin}_{S^1}(X) \cong H^1(X, S^1).$$

Lemma 1.4.2 *The map* $Vect_1^{\mathbb{C}}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z})$ *is an isomorphism.*

Proof The exact sequence of constant sheaves

$$0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 0$$

gives a long exact sequence in cohomology (via Čech cohomology) :

$$0 = H^1(X, \mathbb{R}) \to H^1(X, S^1) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R}) = 0.$$

The middle arrow is thus forced to be an isomorphism. To see that $H^i(X, \mathbb{R}) = 0, i = 1, 2$ we let $g_{ij} : U_i \cap U_j \to \mathbb{R}$, $g_{ij} + g_{jk} - g_{ki} \equiv 0$ on $U_i \cap U_j \cap U_k$ be a cocycle on $\mathcal{U} = \{U_i\}_{i \in I}$. Take a partition function of unity $\{\psi_i\}_{i \in I}$ define

$$h_i : U_i \to \mathbb{R},\ h_i(x) := \sum_j g_{ij}(x)\psi_j(x).$$

For $x \in U_i \cap U_j$

$$h_i(x) - h_j(x) = \sum_k g_{ik}(x)\psi_k(x) - g_{jk}(x)\psi_k(x) = \sum_k g_{ij}(x)\psi_k(x) = g_{ij}(x).$$

Hence $H^1(X, \mathbb{R}) = 0$. The case $H^2(X, \mathbb{R}) = 0$ is similar. This completes the proof. □

Let $j_{\mathbb{R}} : G_{n-1}(\mathbb{R}^\infty) \hookrightarrow G_n(\mathbb{R}^\infty), j_{\mathbb{C}} : G_{n-1}(\mathbb{C}^\infty) \hookrightarrow G_n(\mathbb{C}^\infty)$.

Proposition 1.4.3 (*i*) $j_{\mathbb{R}}$ *is an isomorphism on* $H^k(\cdot, \mathbb{Z}_2)$ *for* $k \le n - 1$.
(*ii*) $j_{\mathbb{C}}$ *is an isomorphism on* $H^k(\cdot, \mathbb{Z})$ *for* $k \le 2n - 1$.

Proof We have the fibration in the complex case :

$$U(n) \to St_n^{\mathbb{C}}(\mathbb{C}^\infty) \to G_n(\mathbb{C}^\infty)$$

which is the hermitian o.n. frame bundle of the fibre bundle $\mathbb{E}_n \to G_n(\mathbb{C}^\infty)$. Since $St_n^{\mathbb{C}}(\mathbb{C}^\infty)$ is contractible

$$\pi_k(G_n(\mathbb{C}^\infty) \cong \pi_k U(n)\ \forall\, k$$

Now the fibre bundle $U(n-1) \to U(n) \to S^{2n-1}$ yields

$$\cdots \to \pi_k S^{2n-1} \to \pi_{k-1} U_{n-1} \to \pi_{k-1} U_n \to \pi_{k-1} S^{2n-1} \to \cdots$$

and for $k-1 < 2n-1$ we get $\pi_{k-1} U(n-1) \cong \pi_k U(n)$. Since the diagram

$$\begin{tikzcd}
 & U(n) \arrow[rr] & & St_n^{\mathbb{C}} \arrow[dd] \\
U(n-1) \arrow[ur, "\pi_{k-1}", "\cong"'] \arrow[rr] & & St_{n-1}^{\mathbb{C}} \arrow[ur] \arrow[dd] & \\
 & & & G_n(\mathbb{C}^\infty) \\
 & & G_{n-1}(\mathbb{C}^\infty) \arrow[ur] &
\end{tikzcd}$$

commutes we get $\pi_k G_{n-1}(\mathbb{C}^\infty) \cong \pi_k G_n(\mathbb{C}^\infty)$. This implies that all the relative homology groups $H_i(G_n(\mathbb{C}^\infty), G_{n-1}(\mathbb{C}^\infty))$ are zero if $i \le 2n-1$. Consequently all relative cohomology groups are zero till $2n-1$ and hence the theorem follows. The real case is similar. $\square$

We state two main results which will be useful in various applications to follow :

Theorem 1.4.4 (Cohomology of grassmanians)
(i)$H^*(G_n(\mathbb{R}^\infty), \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \ldots, w_n]$ *where* $w_k \in H^k(G_n(\mathbb{R}^\infty), \mathbb{Z}_2)$. *Also, the map* $G_{n-1}(\mathbb{R}^\infty) \xrightarrow{g} G_n(\mathbb{R}^\infty)$ *induces*

$$g^* : H^*(G_n(\mathbb{R}^\infty), \mathbb{Z}_2) \to H^*(G_{n-1}(\mathbb{R}^\infty), \mathbb{Z}_2),\ w_i \mapsto w_i,\ i < n$$

and $ker\, g^* = (w_n)$.
(ii)$H^*(G_n(\mathbb{C}^\infty), \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_n]$ *where* $c_k \in H^{2k}(G_n(\mathbb{C}^\infty), \mathbb{Z})$ *and* $ker\, g^* = (c_n)$.

Theorem 1.4.5 *Let* $H^*(G_{n+m}), \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \ldots, w_{n+m}]$, $H^*(G_n, \mathbb{Z}_2) = \mathbb{Z}_2[\overline{w_1}, \ldots, \overline{w_n}]$, $H^*(G_m, \mathbb{Z}_2) = \mathbb{Z}_2[\tilde{w_1}, \ldots, \tilde{w_m}]$. *Then the characteristic classes satisfy :*

$$\textit{Real Case} - \sigma^*(1 + w_1 + \cdots + w_{n+m}) = (1 + \overline{w_1} + \cdots + \overline{w_n})(1 + \tilde{w_1} + \cdots + \tilde{w_m})$$

$$\textit{Complex case} - \sigma^*(1 + c_1 + \cdots + c_{n+m}) = (1 + \overline{c_1} + \cdots + \overline{c_n})(1 + \tilde{c_1} + \cdots + \tilde{c_m}).$$

Definition 1.4.6 *Let* $E \to X$ *be a vector bundle and* $f_E : X \to G_n(\mathbb{R}^\infty)$ *be a classifying map. Then* $w_k(E) = f_E^*(w_k)$ *is called the kth* **Stiefel-Whitney class** *of* E. *For the complex case,* $c_k(E) = f_E^*(c_k)$ *is called the kth* **Chern class** *of* E.

By the classifying theorem, there is a unique classifying map upto homotopy.

Definition 1.4.7 *Let* $E \to X$ *be a vector bundle.*
(i) (Real case) The **total Stiefel-Whitney class** *of* E *is* $w(E) = 1 + w_1(E) + \cdots + w_n(E)$.
(ii) (Complex Case) The **total Chern class** *of* E *is* $c(E) = 1 + c_1(E) + \cdots + c_n(E)$.

Let X, Y be manifolds with X compact. Suppose $f : X \to Y$ is a smooth immersion. Then $f^*(TY) = TX \oplus NX$ and

$$f^* w(TY) = w(f^* TY) = w(TX \oplus NX) = w(TX) w(NX).$$

Example (i) Let $f : X \to \mathbb{R}^n$ be a smooth immersion. Then $w(\mathbb{R}^n) = 1$ implies $w(TX)w(NX) = 1$.

Example (ii) Grassmanians - $T_P(G_n(\mathbb{R}^N)) \cong \mathrm{Hom}(P, P^\perp)$. At P we embed $\mathrm{Hom}(P, P^\perp)$ as a coordinate chart into $G_n(\mathbb{R}^N)$. For $n = 1$,

$$T\mathbb{P}^{N-1} = \mathrm{Hom}(\lambda, \lambda^\perp) = \lambda^* \otimes \lambda^\perp.$$

The exact sequence of bundles

$$0 \to (\lambda^* \otimes \lambda) \cong \mathbb{R} \to (\lambda^*)^N \to \lambda^* \otimes \lambda^\perp \to 0$$

imply $\mathbb{R} \oplus T\mathbb{P}^{N-1} = (\lambda^*)^N$. Thus

$$w(\mathbb{P}^{N-1}) := w(T\mathbb{P}^{N-1}) = w((\lambda^*)^N) = w(\lambda^*)^N = (1 + w_1(\lambda^*))^N = (1 + w_1)^N.$$

Example (iii) For the complex case we get

$$c(\mathbb{P}^{N-1}) = (1 + c_1(\lambda^*))^N = (1 - c_1(\lambda))^N.$$

Example (iv) Consider $\mathbb{P}^4(\mathbb{R})$. Then

$$w(\mathbb{P}^4) = (1 + w_1)^5 = 1 + w_1 + w_1^4.$$

If $f : \mathbb{P}^4 \to \mathbb{R}^k$ is an immersion then $w(\mathbb{P}^4)w(N\mathbb{P}^4) = 1$. If $w(N\mathbb{P}^4) = 1 + a_1 w_1 + \cdots + a_{k-4} w_1^{k-4}$ then solving for a_i's we get $a_1 = a_2 = a_3 = 1$ and $a_l = 0$ if $l \geq 4$. Thus $w(N\mathbb{P}^4) = 1 + w_1 + w_1^2 + w_1^3$. In particular, dim $\mathbb{N}\mathbb{P}^4 \geq 3$. Consequently

Theorem 1.4.8 *There is no immersion of* $\mathbb{P}^4$ *into* $\mathbb{R}^6$.

But we also have

Theorem 1.4.9 (Whitney) *There is an immersion of* $\mathbb{P}^4$ *into* $\mathbb{R}^7$*,*

It is a basic fact that a compact embedded submanifold $M \subseteq X$ of codimension q and oriented normal bundle defines an integral cohomology class $[M] \in H^q(X, \mathbb{Z})$. The idea is as follows:
Let $f : N \to X$ be a closed oriented manifold of dim q. By the transversality theorem make $f \pitchfork M$. Then $M \# N$ counted with proper signs gives an integer which is defined to be $[M](N)$. Let N_0, N_1 be two closed manifolds of dim q. If there is an oriented manifold W of dimension $q + 1$ such that $\partial W = N_0 = N_1$. Let $F : W \to X$ be a map. We may assume $F \pitchfork M$. Since

$$[M](\partial W) = [M](N_0) - [M](N_1)$$

on one hand and $\delta[M] = 0$ on the other

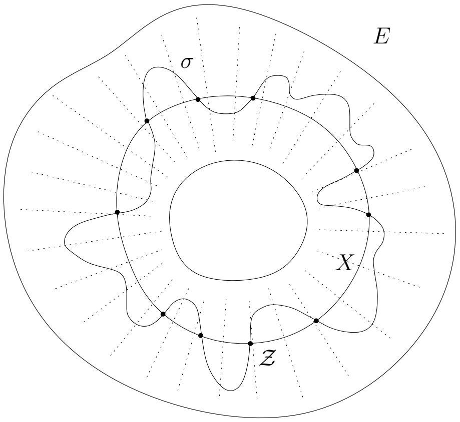
$$0 = \delta[M](W) = [M](N_0) - [M](N_1).$$

Yet another view is to treat $[M]$ as a closed differential form τ of deg q supported in $U_\varepsilon(M)$ such that

$$\int_{f(N)} \tau = \int_N f^*(\tau) = f(N) \# M, \ \int_{\text{normal disk}} \tau = 1.$$

Note that if X is oriented then $H_{n-q}(X, \mathbb{Z}) \cong H^q_{\mathrm{cpt}}(X, \mathbb{Z})$.

Let $E \xrightarrow{\pi} X^{\mathrm{cpt}}$ be a smooth complex vector bundle of rank n. Let $\mathcal{Z} \subseteq E$ be the **zero section**. It is a normally oriented submanifold. Let $\sigma : X \to E$ be a cross section s.t. $\sigma \pitchfork \mathcal{Z}$.



Then $\mathrm{zero}(\sigma) = \sigma^{-1}(\mathcal{Z})$ is a (complex codim n) normally oriented submanifold.

Exercise $\sigma_* : N(\mathrm{zero}(\sigma)) \xrightarrow{\cong} E|_{\mathrm{zero}(\sigma)}$.

Definition 1.4.10 *$c_n(E) = [zero(\sigma)]$.*

Let $\sigma_0, \sigma_1 \in \Gamma(E)$ be two sections transversal to $\mathcal{Z}$. Consider $\sigma : X \times [0,1] \to E$ defined by

$$\sigma(x,t) = (1-t)\sigma_0(x) + t\sigma_1(x).$$

$\sigma \pitchfork \mathcal{Z}$ in a neighbourhood of $\partial(X \times [0,1])$; so approximate σ by $\tilde{\sigma} \pitchfork \mathcal{Z}$ such that

$$\tilde{\sigma} = \begin{cases} = \sigma_0 & \text{near}\, X \times \{0\} \\ = \sigma_1 & \text{near}\, X \times \{1\}. \end{cases}$$

Therefore $\tilde{\sigma}^{-1}(\mathcal{Z})$ is a codim $2n$ normally oriented submanifold of $X \times [0,1]$ with $\sigma_i^{-1}(\mathcal{Z}), i = 0,1$ as boundary components. Thus the definition of $c_n(E)$ makes sense.

Remarks (i) Let $f : X \to Y$ be a smooth map and $E \to X$ a complex vector bundle of rank n.

$$\begin{tikzcd} f^*E \arrow[r, "\tilde{f}"] \arrow[d] & E \arrow[d, "\pi"'] \\ Y \arrow[r, "f"] & X \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

If $f \pitchfork \mathrm{zero}(\sigma)$ then $\sigma \circ f \pitchfork \mathcal{Z} \subseteq f^*E$ and $f^{-1}(\mathrm{zero}(\sigma)) = \mathrm{zero}(\sigma \circ f)$.

(ii) $c_n(E) = 0$ if and only if there exists $\sigma \in \Gamma(E)$ such that $\sigma(x) \neq 0$ for all $x \in X$.

Theorem 1.4.11 *Let $E \to X$ be a complex vector bundle of rank n over a compact manifold X. Suppose $f_E : X \to G_n(\mathbb{C}^\infty)$ is the classifying map. Then f_E is homotopic to $\tilde{f} : X \to G_{n-1}(\mathbb{C}^\infty) \subset G_n(\mathbb{C}^\infty)$ if and only if $c_n(E) = 0$.*

Proof Let $c_n(E) = 0$. Thus there is a non-vanishing section which implies $E \cong E_0 \oplus \mathbb{C}$. Consequently

$$f_E \cong f_{E_0 \oplus \ell} = \phi \circ \tilde{f}_{E_0}$$

where $\phi : G_{n-1}(\mathbb{C}^\infty) \subset G_n(\mathbb{C}^\infty)$ for $\mathbb{C}^N = \mathbb{C}^{N-1} \oplus \ell, N \geq n$. Conversely, if $\tilde{f}$ exists then $c_n(E) = \tilde{f}^*(c_n(\mathbb{E}_{n-1})) = 0$. ☐

??

Let N large and set ℓ_0 to be the first coordinate line in $\mathbb{C}^N$, i.e., $\mathbb{C}^N = \ell_0 \oplus \mathbb{C}^{N-1}$.

$$\Sigma_n = \{P \in G_n(\mathbb{C}^N) : P \subseteq \ell_0^\perp\} = G_n(\mathbb{C}^{N-1})$$

We have $j : G_{n-1}(\mathbb{C}^{N-1}) \to G_n(\mathbb{C}^\infty)$ sending $Q \mapsto \ell_0 \oplus Q$.

(1) $\operatorname{codim}_{\mathbb{C}}(\Sigma_n) = n(N-n) - n(N-n-1) = n$ and $\operatorname{codim}_{\mathbb{R}}(\Sigma_n) = 2n$.

(2) There is a section $u \in \Gamma(\mathbb{E}_n)$ given as follows : Fix a unit vector $u_0 \in \ell_0$ and set

$$u(P) = \pi_P(u_0)$$

where $\pi_P : \mathbb{C}^N \to P$ is the orthogonal projection on P. $\operatorname{zero}(u) = \{P | P \perp \ell_0\} = \Sigma_n$. Check that this vanishes non-degenerately and so $\Sigma_n = c_n(\mathbb{E}_n)$ defined as before.

(3) $G_{n-1}(\mathbb{C}^{N-1}) \hookrightarrow G_n(\mathbb{C}^N) \setminus \Sigma_n$ is a deformation retract. Define $\ell_{P,t} \equiv \mathbb{C}\{(1-t)u_0 + t\pi_P(u_0)\}$ and

$$\psi_t : G_n(\mathbb{C}^N) \setminus \Sigma_n \to G_n(\mathbb{C}^N) \setminus \Sigma_n, \; t \in [0,1]$$

$$\psi_t(P) = (P \cap \ell_0^\perp) \oplus \ell_{P,t}.$$

Thus $\psi_0(P) = \big((P \cap \ell_0^\perp) \oplus \ell_0\big) \in j(G_{n-1}(\mathbb{C}^{N-1}))$, $\psi_1(P) = P$ and ψ_t fixes $G_{n-1}(\mathbb{C}^{N-1})$ point wise.

(4) $\mathbb{E}_n|_{G_n \setminus \Sigma_n} \cong \mathbb{E}_{n-1} \oplus \mathbb{C}$. Recall that for a complex vector bundle $E \to X$, $c_n(E) = 0$ if and only if f_E is a homotopic to a map into $G_{n-1}(\mathbb{C}^{N-1})$.
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Let $E \to X$ be a rank n complex vector bundle. Then $c_n(E) = [\operatorname{zero}(\sigma)] \in H^{2n}(X, \mathbb{Z})$ for any section $\sigma \pitchfork \mathcal{Z}$. If E admits a nowhere vanishing section then $c_n(E) = 0$.

1.5 Connections

2 Transversality Theory

2.1 Transversality

We begin with a review of definitions :

Definition 2.1.1 *Let* $f : X \to Y$ *be a* C^1 *map between manifolds.* $y \in Y$ *is called a* **regular value** *if* $f_x : T_xX \to T_yY$ *is surjective for all* $x \in f^{-1}(y)$*.*

If $y \notin f(X)$ then it is also called a regular value. A value which is not a regular value is called a **critical value**. We shall use the following notations :
$R_f \subseteq Y$ - the set of regular values
$C_f \subseteq X$ - the set of critical points
$f(C_f) \subseteq Y$ - the set of critical values.

Definition 2.1.2 $S \subseteq X$ *is a* C^r *submanifold of codimension* k *if for all* $x \in S$ *there is an open set* U *containing* x *and a* C^r *chart*

$$\phi : U \overset{\cong}{\to} B \equiv \{x \in \mathbb{R}^n \; s.t. \; \|x\| < 1\}$$

such that $\phi(U \cap S) = B \cap \mathbb{R}^{n-k}$ *where* $\mathbb{R}^{n-k} \hookrightarrow \mathbb{R}^n$ *via the first* $n-k$ *coordinates.*

We know that if $f : X \to Y$ is a C^r map and $y \in Y$ is a regular value of f then $f^{-1}(y)$ is a C^r submanifold (of codimension = dim Y) in X. One can generalize this via transversality.

Definition 2.1.3 *Let* $f : X \to Y$ *be a* C^1 *map and let* $S \subseteq Y$ *be a submanifold. Then* f *is* **transversal** *to* S *(denoted* $f \pitchfork S$*) if* $f_x(T_xX) + T_{f(x)}S = T_{f(x)}Y$ *for all* $x \in f^{-1}(S)$*.*

If $f : X \to Y$ is a C^r map and $S \subseteq Y$ is a C^r submanifold of codimension k and $f \pitchfork S$ then $f^{-1}(S)$ is a submanifold (of codimension k) in X. Note that if dim X¡ codim S then $f \pitchfork S$ if and only if $f(X) \cap S = \phi$.

Definition 2.1.4 *A* C^1 *map* $f : X \to Y$ *is an* **embedding** *if it is an injective immersion. It will be called a* **proper embedding** *if it is proper and an embedding.*

Exercise *The image of a proper embedding is a closed set and a submanifold.*

We will also need

Theorem 2.1.5 (Sard's Theorem)
Let $f : X \to Y$ *be a* C^r *map where* $r > min\{0, dimX - dimY\}$*. Then* $f(C_f)$ *has measure zero and* R_f *is residue, i.e., contains a countable intersection of open dense sets.*

What follows is a discussion of embedding manifolds in $\mathbb{R}^n$.

Theorem 2.1.6 *Every compact* C^r *manifold* $(r \geq 1)$ *admits a proper embedding into* $\mathbb{R}^N$ *for some* N*.*

Proof There exists finitely many local coordinate charts $\phi_j : U_j \to 2B := B_2(0)$ and $X = \cup_{j=1}^{l} \phi_j^{-1}(B)$. Choose a smooth map $\rho : [0,2) \to [0,1]$ such that

$$\rho(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \geq 3/2. \end{cases}$$

Define $\rho_j(x) = \rho(\|\phi_j(x)\|)$ and extend by 0 on $X \setminus U_j$. Set

$$\Phi : X \to \mathbb{R}^{2l},\ x \mapsto (\rho_1\phi_1, \rho_1, \cdots, \rho_l\phi_l, \rho_l).$$

Check that Φ is an immersion. If $\Phi(x) = \Phi(y)$ then $\rho_j(x)\phi_j(x) = \rho_j(y) = \phi_j(y)$ and $\rho_j(x) = \rho_j(y)$ for all j. This implies that $x = y$. $\square$

Theorem 2.1.7 *Let X^n be a compact manifold of class $C^r, r \geq 2$. Then X admits a C^r embedding $X \hookrightarrow \mathbb{R}^{2n+1}$.*

Proof We may assume, using the previous theorem, that $X \subseteq \mathbb{R}^N$ for some N. Assume $N \geq 2n+2$. Fix a hyperplane $\mathbb{R}^{N-1} \subseteq \mathbb{R}^N$. For each $u \in S^{N-1} \backslash \mathbb{R}^{N-1}$ we have a linear projection $\pi_u : \mathbb{R}^N \to \mathbb{R}^{N-1}$ generated by

$$x \mapsto x,\ x \in \mathbb{R}^{N-1} \text{ and } u \mapsto 0.$$

We claim that for a residual set of such u's $\pi_u : X \to \mathbb{R}^{N-1}$ is an embedding. Applying induction with the claim then finishes the proof. So consider

$$F : X \times X \setminus \Delta \to S^{N-1},\ (x,y) \mapsto (x-y)/\|x-y\|.$$

Then $\pi_u(x) = \pi_u(y)$ if and only if $x - y = tu$ for some $t \in \mathbb{R}$ which is equivalent to $(x-y)/\|x-y\| = \pm u$. Since dim $(X \times X \setminus \Delta) = 2n < N-1$, by Sard's theorem $S^{N-1} \setminus \operatorname{Im} F$ is dense. So we can choose u such that $u \notin \operatorname{Im} F$. For such a choice of $u \in S^{N-1}$, π_u is one-to-one.
Now observe that $\pi_u|_X$ is an immersion is equivalent to $\pi_u|_{T x X}$ is injective which is equivalent to $u \notin T_x X$ for all $x \in X$. Thus it suffices to consider the unit tangent bundle $T_1 X$ - a compact manifold of class C^{r-1} and dimension $2n-1$. Sard's theorem applied to the (composed) map

$$T_1 X \subseteq \mathbb{R}^N \times S^{N-1} \overset{\pi_2}{\to} S^{N-1}$$

where dim $T_1 X = 2n - 1 < N - 1 =$ dim S^{N-1} we get that $S^{N-1} \setminus \pi_2(T_1 X)$ is open and dense. Thus

$$S^{N-1} \setminus (\pi_2(T_1 X) \cup \operatorname{Im} F) = (S^{N-1} \setminus \pi_2(T_1 X)) \cap (S^{N-1} \setminus \operatorname{Im} F)$$

is also dense. Consequently, π_u is an embedding for almost all $u \in S^{N-1}$. $\square$

Corollary 2.1.8 *If X^n is a compact C^r manifold $(r \geq 2)$ then it can be immersed in $\mathbb{R}^{2n}$.*

This follows from the proof above since the last part of the argument still goes through with one less dimension.

Theorem 2.1.9 *Let X^n be a compact C^r manifold with $r \geq 2$. Given any C^r map $f : X \to \mathbb{R}^N$ $(N \geq 2n+1)$ and $\varepsilon > 0$ there is an embedding $g : X \to \mathbb{R}^N$ such that $\max_{x \in X} \|f - g\| < \varepsilon$.*

??

Proposition 2.1.10 *Let U be a C^r manifold ($r \geq 2$) of dimension n. Let $\Phi : U \to \mathbb{R}^N$ be a C^r embedding. Suppose there exists a projection $\pi : \mathbb{R}^N \to \mathbb{R}^M \subseteq \mathbb{R}^N$ ($M \geq 2n+1$) to a subspace such that $\pi|_{\mathbb{R}^M} = Id|_{\mathbb{R}^M}$. Then given $\varepsilon > 0$ there exists a projection $\pi' : \mathbb{R}^N \to \mathbb{R}^M$ such that*

$$\|\pi(x) - \pi'(x)\| \leq \varepsilon \|x\| \; \forall \; x \in \mathbb{R}^N$$

and $\pi' \circ \Phi : U \to \mathbb{R}^M$ is a C^r embedding. Moreover, if Φ is an immersion and $M \geq 2n$ then $\pi' \circ \Phi$ is also a C^r immersion.

Proof Recall that in the proof 2.1.7 we fixed $\mathbb{R}^{N-1} \subseteq \mathbb{R}^N$ and fixed a unit vector $u \in S^{N-1} \setminus \mathbb{R}^{N-1}$. We considered $\pi_u : \mathbb{R}^n \to \mathbb{R}^{N-1}$ with $\pi_u(w + \lambda u) = w$ where $w \in \mathbb{R}^{N-1}$. Thus $\pi_u : \mathbb{R}^{N-1} \times \mathbb{R} \to \mathbb{R}^{N-1}$ looks like

$$\left(\begin{array}{ccc|c} 1 & & & v_1 \\ & \ddots & & \vdots \\ & & 1 & v_{N-1} \\ \hline 0 & \cdots & 0 & 0 \end{array}\right), u = \frac{1}{(1+|v|^2)^{\frac{1}{2}}} \begin{pmatrix} v_1 \\ \vdots \\ v_{N-1} \\ -1 \end{pmatrix}.$$

Write $x = (\tilde{x}, x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$. Then

$$\pi_u(x) = \tilde{x} + x_N v, \text{ where } v = (v_1, \cdots, v_{N-1}).$$

Now fix any $v \in \mathbb{R}^{N-1}$ and define

$$\pi_v : \mathbb{R}^{N-1} \times \mathbb{R} \to \mathbb{R}^{N-1}, \; x \mapsto \tilde{x} + x_N v.$$

Also

$$\|\pi_0(x) - \pi_v(x)\| = |x_N| \|v\| \leq \|v\| \|x\|$$

Choose v with sufficiently small norm. Going through the same arguments as in 2.1.7 we get the desired result. $\square$

Corollary 2.1.11 *Let $f : U \to \mathbb{R}^M$ be a C^r map ($r \geq 2$) and $\Phi : U \to \mathbb{R}^m$ be a C^r embedding (resp. immersion). Suppose $M \geq 2n+1$ (resp. $M \geq 2n$). Given $\varepsilon > 0$ there exists a linear map $L : \mathbb{R}^m \to \mathbb{R}^M$ such that*

(i) $f + L \circ \Phi : U \to \mathbb{R}^M$ is a C^r embedding (resp. immersion)

(ii) $\|L\| = \sup_{\|y\| \leq 1} \|Ly\| < \varepsilon$.

In particular if $\tilde{f} \equiv f + L \circ \Phi$ then $\|f(x) - \tilde{f}(x)\| \leq \varepsilon \|\Phi(x)\|, x \in U$.

Corollary 2.1.12 *Assume all the hypothesis of corollary 2.1.11 and let $U \subseteq \mathbb{R}^n$ be open. Then*

$$\|D^\alpha f(x) - D^\alpha \tilde{f}(x)\| \leq \varepsilon \|D^\alpha \Phi(x)\| \text{ for } \alpha = (\alpha_1, \cdots, \alpha_n), x \in U.$$

Exercise *Let X be a C^1 manifold. Then there exists a* **compact exhaustion**, *i.e., a nested sequence of compact sets $K_1 \subseteq K_2 \subseteq \cdots$ such that $X = \cup_i K_i$ and $K_i \subset {K_{i+1}}^\circ \, \forall \, i$.*

Assuming the exercise, set $A_i = K_i \setminus {K_{i-1}}^\circ$ and $B_i = {K_{i+1}}^\circ \setminus K_{i-2}$. A_i's are like annulus radiating outside and B_i's are open neighbourhoods of A_i's.

Theorem 2.1.13 *Every $C^r (r \geq 2)$ manifold of dimension n admits a proper embedding into $\mathbb{R}^{2n+1}$ and a proper C^r immersion into $\mathbb{R}^{2n}$.*

Proof Cover A_i with coordinate charts $\{(U_{i_\alpha}, \phi_{i_\alpha})\}_{\alpha=1}^{l_i}, \phi_{i_\alpha} \to 2B$ and $A \subset \cup_\alpha \phi_{i_\alpha}^{-1}(B)$ with $\overline{U_{i_\alpha}} \subseteq B_i$. Set

$$\Phi_i := (\rho_{i_1}\phi_{i_1}, \rho_{i_1}, \cdots, \rho_{i_{l_i}}\phi_{i_{l_i}}, \rho_{i_{l_i}}) : X \to \mathbb{R}^{2l_i}$$

where $\rho_{i_\alpha}(x) = \rho(\|\phi_{i_\alpha}(x)\|)$ and extended by 0 on $X \setminus U_{i_\alpha}$. The construction is similar to 2.1.6. Then Φ is a C^r embedding on a neighbourhood of A_i and is identically zero on $X \setminus B_i$. By choosing a projection $\pi_i : \mathbb{R}^{2l_i} \to \mathbb{R}^{2n+1}$ we get a map

$$\psi_i := \pi_i \circ \Phi_i : X \to \mathbb{R}^{2n+1}$$

which is an embedding on a neighbourhood of A_i and zero on $X \setminus B_i$. Since supp $\psi_i \subseteq A_{i-1} \cup A_i \cup A_{i+1}$,

$$\operatorname{supp}\psi_i \cap \operatorname{supp}\psi_j = \phi \text{ if } |i-j| > 3.$$

This prompts us to define

$$\tilde{\Psi} := \Big(\sum_{j\geq 1}\psi_{4j}, \sum_{j\geq 1}\psi_{4j-1}, \sum_{j\geq 1}\psi_{4j-2}, \sum_{j\geq 1}\psi_{4j-3}\Big) : X \to \mathbb{R}^{4(2n+1)}$$

which is a C^r embedding. We can successively project to get an embedding $\tilde{\tilde{\Psi}} : X \to \mathbb{R}^{2n+1}$. To complete the proof we shall need :

Lemma 2.1.14 *There is a C^r function $f : X \to [0, \infty)$ such that $f^{-1}[0, c]$ is compact for all $c \in \mathbb{R}$.*

Proof By Tietze's extension theorem there exist continuous maps $f_i : X \to [0, 1]$ such that

$$f_i(x) = \begin{cases} 1, & x \in A_i \\ 0, & x \in X \setminus B_i. \end{cases}$$

Define $f \equiv \sum_i i f_i$. Apply uniform approximation by a C^r function. □

Now consider $\Psi_1 := (\tilde{\tilde{\Psi}}, f) : X \to \mathbb{R}^{2n+2}$. For a compact subset $K \subseteq \mathbb{R}^{2n+2}$,

$$\Psi_1^{-1}(K) \subseteq f^{-1}(\pi(K))$$

where $\pi : \mathbb{R}^{2n+2} \to \mathbb{R}$ is the projection to the last coordinate. Thus Ψ_1 is proper. We project again and denote this new map by Ψ_1 again. Define

$$\Psi := \Psi_1 - vf,\ v \in \mathbb{R}^{2n+1}.$$

Given f this map is an embedding for almost all $v \in \mathbb{R}^{2n+1}$. Let f_0 be any proper function (as defined in the exercise) on X. Set $f := f_0 + e^{\|\Psi_1\|}$ and choose V such that $\|v\| \geq 1$. Then

$$\|\Psi_1 - v(f_0 + e^{\|\Psi_1\|})\| \geq \|v\|(f_0 + e^{\|\Psi_1\|}) - \|\Psi_1\| \geq f_0 + e^{\|\Psi_1\|} - \|\Psi_1\| > f_0.$$

Thus

$$\Psi^{-1}(\overline{B_R(0)}) = \{x | \|\Psi_1(x) - v(f_0(x) + e^{\|\Psi_1(x)\|})\| \leq R\} \subseteq f_0^{-1}[0, R]$$

is compact whence Ψ is proper. □

2.2 Function Spaces

We set $C^r(X,Y)=\{f|f:X\overset{C^r}{\to}Y\}$. Fix $f\in C^r(X,Y)$. Choose $U\subset X^n, K\,(\text{compact})\subseteq U, V\subset Y^m$ and C^r local coordinates $\phi:U\to\mathbb{R}^n,\psi:V\to\mathbb{R}^m$ such that $f(K)\subseteq V$. Set

$$\begin{aligned}\mathcal{U}_\varepsilon(f)&=\mathcal{U}^r(f,(U,\phi),(V,\psi),K,\varepsilon)\\ &:=\Big\{g\in C^r(X,Y)|g(K)\subseteq V,\sup_{\phi(K)}\sum_{|\alpha|\le r}\|D^\alpha(\psi\circ g\circ\phi^{-1})-D^\alpha(\psi\circ f\circ\phi^{-1})\|\le\varepsilon\Big\}.\end{aligned}$$

Definition 2.2.1 *The* **weak topology** *on $C^r(X,Y)$ is the topology generated by the weak basic neighbourhoods (a weak basic neighbourhood of f in $C^r(X,Y)$ is a finite intersection of sets as above).*

The proof of the following result is left as an exercise :

Theorem 2.2.2 *Suppose X is compact, of dimension n and of class $C^r, r\ge 2$. Then*
(i) C^r-embeddings are dense in $C^r(X,\mathbb{R}^N)$ if $N\ge 2n+1$ and
(ii) C^r-immersions are dense in $C^r(X,\mathbb{R}^N)$ if $N\ge 2n$.

For X compact, $C^r(X,\mathbb{R}^N)$ is a Banach space. Define

$$\|f-g\|=\sum_{i=1}^{l}\sup_{\phi_i(U_i)}\sum_{|\alpha|\le r}\|D^\alpha(f\circ\phi_i^{-1})-D^\alpha(g\circ\phi_i^{-1})\|$$

where $\{(U_i,\phi_i)\}_{i=1}^l$ is a finite (compact) cover of X. One can also define the strong topology on $C^r(X,Y)$ as follows. Fix $f\in C^r(X,Y)$; choose a locally finite set $\{(U_i,\phi_i)\}_{i\in I}$ of C^r-coordinates on X and a locally finite set $\{(V_i,\psi_i)\}_{i\in I}$ of C^r-coordinates on Y and $\{K_i^{\text{cpt}}\}_{i\in I}$ such that $f(K_i)\subseteq V_i$ and $K_i\subset U_i$. Given $\{\varepsilon_i\}_{i\in I},\varepsilon_i>0\,\forall\,i\in I$ set

$$\mathcal{U}:=\Big\{g\in C^r(X,Y)|g(K_i)\subseteq V_i\,\forall\,i\in I,\sup_{\phi_i(K_i)}\sum_{|\alpha|\le r}\|D^\alpha\tilde{g}_i-D^\alpha\tilde{f}_i\|\le\varepsilon_i\,\forall\,i\in I\Big\},$$

where

$$\tilde{g}_i=\psi_i\circ g\circ\phi_i^{-1},\ \tilde{f}_i=\psi_i\circ f\circ\phi_i^{-1}.$$

Definition 2.2.3 *We define the* **strong topology** *on $C^r(X,Y)$ using such $\mathcal{U}$ as basic neighbourhoods.*

Example $C^1(\mathbb{R},\mathbb{R})$ - Let $f\in C^1(\mathbb{R},\mathbb{R})$ and $\varepsilon:\mathbb{R}\to\mathbb{R}^{>0}$ be an arbitrary continuous function. Then

$$\mathcal{U}:=\{g\text{ s.t. }\|g(x)-f(x)\|_{C^1}<\varepsilon(x)\,\forall\,x\in\mathbb{R}\}$$

is strongly open.

Definition 2.2.4 *The* C^∞ **topology** *on* $C^\infty(X,Y)$ *is the union of all open sets from the injections*

$$C^\infty(X,Y) \subset C^r(X,Y)\ \forall\, r \geq 1$$

where $C^r(X,Y)$ *is equipped with the weak or strong topology.*

Note that the topology defined above doesn't depend on the ambient topology since we are taking union of all open sets. **??**

Notation $\text{Imm}^r(X,Y) = C^r$-immersions from X to Y (dim $Y \geq$ dim X).
$\text{Sub}^r(X,Y) = C^r$-submersions from X to Y (dim $X \geq$ dim Y).
$\text{Prop}^r(X,Y) =$proper C^r-maps from X to Y.
$\text{Emb}^r(X,Y) = C^r$-embeddings from X to Y (dim $Y \geq$ dim X).
$\text{Diff}^r(X) = C^r$-diffeomorphisms of X.

Proposition 2.2.5 *Imm*$^r(X,Y)$ *is open in the strong topology on* $C^r(X,Y)$.

Proof Let $f \in \text{Imm}^r(X,Y)$. Fix a locally finite coordinate covering $\{(U_i,\phi_i)\}_{i\in I}$ for X and choose compact subsets $K_i \subseteq U_i$ such that
(i) $X = \cup_{I\in I} K_i^\circ$
(ii) $f(K_i) \subseteq V_{\alpha(i)}$ where $\{(V_\alpha,\psi_\alpha)\}_\alpha$ is a coordinate covering on Y.
We define

$$T_i := \{L : \mathbb{R}^n \to \mathbb{R}^m | L = d(\psi_{\alpha(i)} \circ f \circ \phi_i^{-1})_x, x \in K_i\}.$$

Then T_i is compact and

$$T_i \hookrightarrow \text{Hom Inj}(\mathbb{R}^n,\mathbb{R}^m) \hookrightarrow \text{Hom}(\mathbb{R}^n,\mathbb{R}^m)$$

where the last inclusion is an open map. Therefore $\exists \varepsilon_i > 0$ such that

$$(T_i)_{\varepsilon_i} \subseteq \text{Hom Inj}(\mathbb{R}^n,\mathbb{R}^m).$$

With this choice of $\{\varepsilon_i\}_i$ we get a strong neighbourhood $\mathcal{U} \subseteq \text{Imm}^r(X,Y)$ of f. □

Similarly it can be shown

Proposition 2.2.6 *Sub*$^r(X,Y)$ *is open in the strong topology on* $C^r(X,Y)$.

Proposition 2.2.7 *Prop*$^r(X,Y)$ *is open in the strong topology on* $C^r(X,Y)$.

Proof Fix $f \in \text{Prop}^r(X,Y)$. Choose locally finite coordinate covering $\{(U_i,\phi_i)\}_{i\in I}$ of X, $K_i^{\text{cpt}} \subseteq U_i$ such that $X = \cup_i K_i$ and coordinates $\{(\tilde{V}_i,\psi_i)\}_i$ on Y such that $f(K_i) \subseteq V_i$. We shall need :

Lemma 2.2.8 *The* V_i*'s can be chosen to be locally finite on* Y.

Proof Choose a proper embedding $Y \subseteq \mathbb{R}^N$ for some N. If

$$\lim_{i\to\infty} d(f(K_i),0) \neq \infty$$

then there exists a subsequence $\{i_j\}_{j\geq 1}$ and $c > 0$ such that

$$\big(f(K_{i_j}) \cap \{x \text{ s.t. } \|x\| \leq c\}\big) \neq \phi\ \forall\, i_j.$$

Consequently

$$f^{-1}\big(\{x \text{ s.t. } \|x\| \leq c\}\big) \cap K_{i_j} \neq \phi\ \forall\, i_j.$$

Since K_i's are locally finite this is a contradiction. Now replace (if required) the original $\tilde{V_i}$'s by

$$V_i := \tilde{V_i} \cap \{y | d(y, f(K_i)) < 1\}.$$

This collection $\{V_i\}_i$ is locally finite. $\square$

Now choose $\tilde{\varepsilon}_i$ such that

$$[f(K_i)]_{\tilde{\varepsilon}_i} \equiv \{y \in Y | d(y, f(K_i)) < \tilde{\varepsilon}_i\} \subseteq V_i.$$

This gives a neighbourhood $\mathcal{U}$ of f in the strong topology on $C^r(X, Y)$. Choose ε_i such that

$$\sup_{\phi_i(K_i)} \|\psi_i \circ g \circ \phi_i^{-1} - \psi_i \circ f \circ \phi_i^{-1}\| < \varepsilon_i \Rightarrow d_{\mathbb{R}^n}(f, g) < \tilde{\varepsilon}_i \,\text{on}\, K_i.$$

We need to show that any $g \in \mathcal{U}$ is proper. Fix a compact set $C \subseteq Y$. C meets only finitely many of the V_i's, say $V_{i_1}, \ldots, V_{i_l}$. Since $f(K_i) \subseteq V_i$ holds for all i we get

$$g(K_i) \subseteq [f(K_i)]_{\tilde{\varepsilon}_i} \subseteq V_i.$$

Thus $g(K_i) \cap C \neq \phi$ for possibly $i = i_1, \ldots, i_l$. Therefore

$$g^{-1}(C) \subseteq K_{i_1} \cup \cdots \cup K_{i_l}.$$

Being a closed set of a compact set, it is also compact. Hence g is proper and $\mathcal{U} \subseteq \mathrm{Prop}^r(X, Y)$. $\square$

We may use this to prove :

Proposition 2.2.9 *$Emb^r(X, Y)$ is open in the strong topology on $C^r(X, Y)$.*

Proof The set of proper immersions are open since each of them are. Fix $f \in \mathrm{Emb}^r(X, Y)$ and choose a locally finite coordinate neighbourhoods $\{(U_i, \phi_i)\}_{i \geq 1}$ for X with compact subsets $K_i \subseteq L_i \subseteq U_i$ such that
(i) $K_i \subseteq L_i^\circ$
(ii) $X = \bigcup_{i \geq 1} K_i^\circ$.
Further, choose a locally finite family of coordinate charts $\{(V_i, \psi_i)\}_{i \geq 1}$ for Y such that $f(L_i) \subseteq V_i$. Also choose $\{\varepsilon_i\}_{i \geq 1}$ such that the neighbourhood $\mathcal{U}$ defined with these choices consists of proper immersions. We claim that by shrinking ε_i's sufficiently, we can make every $g \in \mathcal{U}$ and embedding on L_i and hence on X. We shall need :

Lemma 2.2.10 *Let $\mathcal{O}^{open} \subseteq \mathbb{R}^n$ and $C^{cpt} \subseteq \mathcal{O}$. Suppose $F : \mathcal{O} \to \mathbb{R}^m$ is a C^1 map which is an embedding on C. Then there exists $\varepsilon > 0$ such that if $G : \mathcal{O} \to \mathbb{R}^m$ is another C^1 map satisfying*

$$\sup_C \big\{\|G - F\| + \|DG - DF\|\big\} < \varepsilon \quad (*)$$

then $G|_C$ is an embedding.

Proof First observe that there exists $\epsilon > 0$ such that $(*)$ implies that G is an immersion on C. Now suppose the lemma fails. Then there exists $\{G_k\}_{k \geq 1} \subseteq C^1(\mathcal{O}, \mathbb{R}^m)$ such that

$$\|G_k - F\| + \|DG_k - DF\| \to 0 \;\text{as}\, k \to \infty.$$

But there are points $x_k \neq y_k$ in C such that $G_k(x_k) = G_k(y_k)$. Passing to a subsequence if necessary assume $x_k \to x$ and $y_k \to y$. This would imply $F(x) = F(y)$ whence $x = y$ since F is injective. Passing to a subsequence we may assume that

$$u_n := \frac{x_n - y_n}{\|x_n - y_n\|} \to u \in S^{n-1}.$$

By Taylor expansion we get

$$\frac{\|G_k(x_k) - G_k(y_k) - DG_k(y_k)(x_k - y_k)\|}{\|x_k - y_k\|} \xrightarrow{k\to\infty} 0.$$

But the the LHS of the above equals

$$\frac{\|DG_k(y_k)(x_k - y_k)\|}{\|x_k - y_k\|} = \|DG_k(y_k)u_k\| \to \|DF(y)u\| \neq 0$$

since F is an immersion. This completes the proof of the lemma. ☐

Now assume as before that $Y \subseteq \mathbb{R}^N$ for some N is a proper embedding. Set V_i's such that $V_i \subseteq \overline{[f(L_i)]_1}$. Set

$$A_i = f(K_i),\ B_i = f(X \setminus L_i),\ \eta_i = d(A_i, B_i).$$

$f(L_i)$ meets only finitely many V_j's, say $V_{j_1}, \ldots, V_{j_l}$. By shrinking ε_i's on each of the U_{j_k}'s we can arrange for

$$g(K_i) \cap g(K_{j_i} \cap (X \setminus L_i)) = \phi,\ g(K_i) \cap g(X \setminus L_i) = \phi.$$

On each U_j we change ε_j only finitely many times and hence it is permissible. This gives a strong neighbourhood $\mathcal{U}$ of f. Verify that $g \in \mathcal{U}$ is proper and an embedding. ☐

Exercise $f \in \mathrm{Diff}^r(X)$ if and only if $f : X \to X$ is a proper embedding.

Corollary 2.2.11 *$\mathrm{Diff}^r(X)$ is open in the strong topology.*

The remaining section will deal with various approximation results.

Theorem 2.2.12 *If $\dim Y \geq 2\dim X$ and $r \geq 2$ then $\mathrm{Imm}^r(X, Y)$ is strongly dense in $C^r(X, Y)$.*

Proof Fix $f \in C^r(X, Y)$ and a strong neighbourhood $\mathcal{U}$ of f as before. We may assume $X \subseteq K_i^\circ$. We shall construct a sequence of functions $f_k : X \to Y$ such that (i) $f_k \in \mathcal{U}$ (ii) $f_k|_{\cup_{j=1}^k K_j}$ is an immersion and (iii) f_k differs from f_{k-1} only on U_k. Then $f_k\tilde{f}$ which is an immersion by the local finiteness of U_i's. So suppose inductively that f_{k-1} is given. Consider

$$\begin{tikzcd}
K_k \subseteq U_k \arrow[r, "f_{k-1}"] \arrow[d, "\cong"', "\phi_k"] & V_k \arrow[d, "\cong"', "\psi_k"] \\
\phi_k(K_k) \subseteq \phi_k(U_k) \arrow[r, "\tilde{f}_{k-1}"] & \psi_k(V_k) \subseteq \mathbb{R}^m
\end{tikzcd}$$

where $\tilde{f}_{k-1} = \psi_k \circ f_{k-1} \circ \phi_k^{-1}$. Choose $g_k : \phi_k(U_k) \to \mathbb{R}^M$ such that

(i) g_k is an immersion on a neighbourhood of $\phi_k(U_k)$ and

(ii) $g_k \equiv 0$ outside a bigger compact set in U_k.

Let $\pi : \mathbb{R}^M \to \mathbb{R}^m$ be such that (here to use the cor 2.1.11 we need $m \geq 2n$)

(i) $\|\pi_*(v)\| \leq \varepsilon\|v\|$,

(ii) $\tilde{f}_{k-1} - \pi g_k$ is an immersion on a neighbourhood of $\phi_k(U_k)$.
Choosing ε small enough we can make

$$\sup_{\phi_k(U_k)} \sum_{|\alpha||leqr} \|D^\alpha \tilde{f}_{k-1} - D^\alpha(\tilde{f}_{k-1} + \pi g_k)\| = \sup_{\phi_k(U_k)} \sum_{|\alpha|\leq r} \|D^\alpha(\pi g_k)\|$$

as small as we line; in particular less than ε_k. Define $f_k \equiv \psi_k^{-1}(\tilde{f}_{k-1} + \pi g_k)\phi_k$. It will be an immersion on all of $K_1 \cup \cdots \cup K_k$ for ε sufficiently small. □

Lemma 2.2.13 (Basic Approximation Lemma)
Fix $U^{open} \subseteq \mathbb{R}$. *Let* $F : U \to \mathbb{R}^m$ *be of class* $C^s, 0 \leq s < \infty$. *Given* $\varepsilon > 0, r > s$ *and* $K^{cpt} \subseteq L^\circ \subseteq L^{cpt} \subseteq U$ *there exists* $G : U \to \mathbb{R}^m$ *of class* C^s *such that*
(i) $G \equiv F$ *in* $U \setminus L$
(ii) G *is of class* C^r *on a neighbourhood of* K
(iii) G *is of class* C^r *on an open subset where* F *is of class* C^r *and*
(iv) $\sup_U \sum_{|\alpha|\leq s} \|D^\alpha f - D^\alpha g\| < \varepsilon$.

Proof For a suitable choice of $\xi \in C_c^\infty(\mathbb{R})$ with $\xi(t) = \xi(-t)$ we set $\phi_\varepsilon(x) = \xi(\|x\|/varepsilon)/\varepsilon^n$ such that its integral over $\mathbb{R}^n$ is 1. Define

$$F_\varepsilon(x) := \int_{\mathbb{R}^n} \phi_\varepsilon(y-x)F(y)dy.$$

This is well defined and smooth. We also have

$$\sup_L \sum_{|\alpha|\leq s} \|D^\alpha F_\varepsilon - D^\alpha F\| < e(\varepsilon$$

where $e(\varepsilon) \to 0$ as $\varepsilon \to 0$. Choose $\lambda \in C_0^\infty(U)$ such that $\lambda \equiv 1$ on a neighbourhood of K and $\lambda \equiv 0$ on a neighbourhood of $U \setminus L$. Consider $G = \lambda F_\varepsilon + (1-\lambda)F$. Then

$$D^\alpha G - D^\alpha F = D^\alpha(\lambda(F_\varepsilon - F)) = \sum_\beta c_{\beta(\alpha-\beta)} D^\beta(\lambda) D^{\alpha-\beta}(F_\varepsilon - F)$$

and can be made as small as we want. Finally observe that G is smooth on a neighbourhood of K.
□

Theorem 2.2.14 *Let* X, Y *be* C^r *manifolds* $(r > s \geq 0)$. *Then* $C^r(X,Y)$ *is strongly dense in* $C^s(X,Y)$.

Proof Use the lemma and the argument of the immersion case. □

The proofs of the following two results are left as exercises.

Theorem 2.2.15 *Let* X, Y *be smooth manifolds and* $s \geq 0$. *Then* $C^\infty(X,Y)$ *is strongly dense in* $C^s(X,Y)$.

Lemma 2.2.16 *Let* $U^{open} \subseteq X$ *be a* C^r *manifold and* $f : X \to Y^{open} \subseteq \mathbb{R}^m$ *be a* C^r *map. Let* $f(U) \subseteq V^{open} \subseteq Y$. *Then there exists an open neighbourhood* $\mathcal{U}$ *of* $f|_U$ *in* $C^r_{str}(U,V)$ *such that the map* $: \mathcal{U} \to C^r(X,Y)$ *defined by setting*

$$g \mapsto \begin{cases} g(x), & x \in U \\ f(x), & x \notin U \end{cases}$$

is well defined and continuous.

As a consequence we get

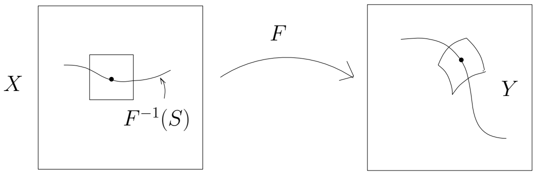
Theorem 2.2.17 *Every C^r-manifold ($r \geq 1$) has a compatible C^s-structure for $\infty \geq s > r$.*

2.3 Transversality Theorem

In this section we shall discuss the main theorem and a few applications.

Theorem 2.3.1 (Transversality Theorem)
Let U, X, Y be manifolds and $S \subseteq Y$ a submanifold such that $F : U \times X \to Y$ is map of class $C^r, r > \max\{0, dim\, X - codim\, S\}$. Then $F \pitchfork S$ implies that $F_u(\cdot) \equiv F(u, \cdot)$ is $\pitchfork S$ for a.e. $u \in U$.

Proof Localize and reduce to $\dim S = 0$.



Fix $y_0 \in S, (u_0, x_0) \in U \times X$ and $F(u_0, x_0) = y_0$. Choose coordinate neighbourhoods V of y_0 in Y, i.e.,

$$(\eta, \xi) : V \to \mathbb{R}^s \times \mathbb{R}^m,\ y_0 \mapsto (0, 0)$$

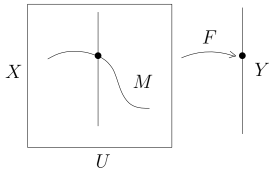
and $V \cong \{(\eta, \xi)\,\text{s.t.}\, \|\eta\| \leq 1, \|\xi\| \leq 1\}$ such that $V \cap S \cong \{(\eta, 0)\,\text{s.t.}\, \|\eta\| \leq 1\}$. Choose a product neighbourhood $U_0 \times X_0$ of (u_0, x_0) such that $F(U_0 \times X_0) \subseteq V$. On this neighbourhood

$$F \pitchfork S \Leftrightarrow 0\,\text{is a regular value of}\, \eta \circ F.$$

In the reduced case S is a point in Y, which is a regular value of F. Set $M := F^{-1}(S)$ a C^r-submanifold of $U \times X$ of codim $= \dim Y$. The following lemma will complete the proof :

Lemma 2.3.2 *$y \in Y$ is a regular value of $F_u : X \times Y$ if and only if $u \in U$ is a regular value of $\pi : M \to U$ where $\pi = pr_1|_M$.*

Proof Fix (u, x) with $F(u, x) = y$. Let $\dim Y = m$ and $\dim X = n$. The local picture look like



We have the following grid :

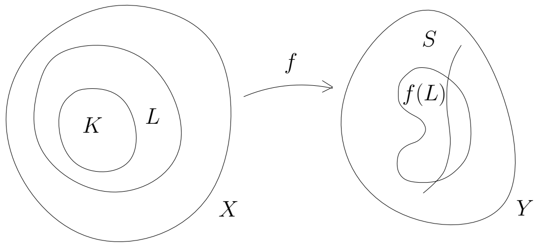
$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & T_xX \arrow[d] \arrow[dr, "(F_u)_*"] & & \\
0 \arrow[r] & T_{(u,x)}M \arrow[r] \arrow[dr, "\pi_*"'] & T_uU \times T_xX \arrow[r] \arrow[d] & T_yY \arrow[r] & 0 \\
 & & T_uU \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

$(F_u)_*$ is surjective $\Leftrightarrow$ dim ker $(F_u)_* = m - n \Leftrightarrow$ dim $(T_xX \cap T_{(u,x)}M) = m - n \Leftrightarrow$ dim (ker π_*) $=$ $m - n \Leftrightarrow \pi_*$ is surjective (dim $U = k$, dim $M = k + n - m$). Also y is a regular value of $F_u \Leftrightarrow (F_u)_*$ is surjective for all x such that $F(u, x) = y \Leftrightarrow \pi_*$ is surjective $\forall\, x \in F^{-1}(y) \cap \pi^{-1}(u) = M \cap \pi^{-1}(u) \Leftrightarrow u$ is a regular value of π. $\square$

Theorem 2.3.3 *Let $f : X \to Y$ be a C^r-map and $S \subseteq Y$ be a C^r-submanifold with $r > \max\{0, dim\, X - codim\, S\}$. Then given a strong neighbourhood $\mathcal{U}$ of f, there exists $g \in \mathcal{U}$ such that $g \pitchfork S$. Furthermore, if $f \pitchfork S$ on some closed set $C \subseteq X$ then we can assume $f \equiv g$ on C.*

Proof It suffices to consider the local case

$$K^{\text{cpt}} \subseteq L^\circ \subseteq L^{\text{cpt}} \subseteq X^{\text{open}}(\subseteq \mathbb{R}^n) \xrightarrow{f} Y^{\text{open}}(\supseteq S) \subseteq \mathbb{R}^m.$$



It suffices to show that for any $\varepsilon > 0$ there is a C^r-map $g : X \to Y$ satisfying :

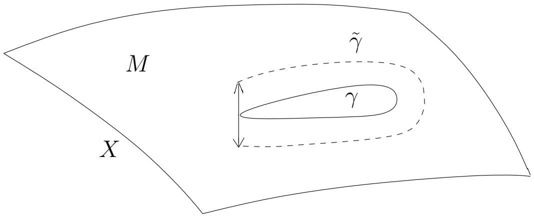
(i) $g \equiv f$ on $X \setminus L$
(ii) $g \pitchfork S$ on a neighbourhood of S
(iii) $g \pitchfork S$ on a neighbourhood of C
(iv) $\sup_X \sum_{|\alpha| \leq r} \|D^\alpha g - D^\alpha f\| < \varepsilon$.

Choose $\lambda \in C_0^\infty(X)$ such that $\lambda \equiv 1$ on a neighbourhood of K and $\lambda \equiv 0$ on $X \setminus L$. **??**

Applications

Proposition 2.3.4 *Let $M^{n-1} \subseteq X^n$ be a closed, smooth hypersurface (proper). Assume that X is connected and simply connected. Then M is orientable and $X \setminus M$ has 2 components.*

Proof The given hypothesis on X implies that it is orientable. If M isn't orientable then there is a loop γ (based at $p \in M$) reversing orientation on TM and NM. Choose a unit normal vector along γ and let the new loop $\tilde{\gamma}$ be as in the figure.



Since $\pi_1(X) = \{0\}$ there is a map $F : D^2 \to X$ such that $F|_{\partial D^2} = \tilde{\gamma}$. We may assume w.l.o.g that $F \pitchfork M$ in a neighbourhood of $p \in \partial D^2$. Approximate F by $\tilde{F}$ such that $\tilde{F} \pitchfork M$ and $\tilde{F} \equiv F$ near p. Then $\tilde{F}^{-1}(M)$ is a compact 1-dim submanifold of D with only 1 boundary point p, a contradiction. □

Proposition 2.3.5 *Let $E \to X$ be a smooth vector bundle with rank $E >$ dim X. Then there exists a section which nowhere zero. If rank $E =$ dim X then there is a section which has isolated non-degenerate zeroes.*

Proof Exercise. □

If $\sigma : X \to E$ has a zero, say $\sigma(x) = (x, 0)$ then the composite map

$$d\sigma_x : T_xX \to T_(x,0)E \overset{\text{pr}}{\to} E_x$$

is an isomorphism. Consider $\sigma_0 \equiv 0$ and apply transversality theorem to get $S =$ zero section $\subseteq E$. In general, there exists σ with $\sigma \pitchfork S$ and $\sigma^{-1}(S)$ being a submanifold of codim m. Also

$$[\sigma^{-1}(S)] = w_m(E) \in H^m(X, \mathbb{Z}_2).$$

If E is oriented then $\sigma^{-1}(S)$ is normally oriented and $[\sigma^{-1}(S)] \in H^m(X, \mathbb{Z})$ is the **Euler class**.

Theorem 2.3.6 *Let X be a compact manifold with boundary $\partial X \neq \phi$. Then there are no smooth maps $f : X \to \partial X$ such that $f|_{\partial X} = Id|_{\partial X}$.*

Proof Given such an f, assume smoothness of the boundary (Collar Neighbourhood Theorem). Fix any $p \in \partial X$ such that p is regular value of f restricted to the collar. One can approximate f by $\tilde{f} \pitchfork p$ such that $\tilde{f} = f$ on a neighbourhood of ∂X. Then $\tilde{f}^{-1}(p)$ is a compact 1-dim manifold with one boundary point, a contradiction. □

Using this we can prove the smooth version of

Theorem 2.3.7 (Brouwer Fixed Point Theorem)
Any continuous map $F : D^n \to D^n$ has a fixed point.

Proof If such a map exists then this produces a map $\tilde{f} : D^n \to S^{n-1}$ which is identity on S^{n-1}. This is a contradiction. □

We shall use transversality to define the mod 2 degree of a map $f : X \to Y$. Note that the arguments in the theorem hold for non-compact spaces if the maps are proper and the homotopies are proper.

Theorem 2.3.8 *Let X, Y be compact n-manifolds without boundary.*
- *(i) Given a smooth map $f : X \to Y$ we have $\#\{f^{-1}(p)\} \equiv \#\{f^{-1}(q)\}\, mod 2$ for regular values $p, q \in Y$.*
- *(ii) If f is homotopic to g then $deg_2(f) = deg_2(g)$.*

As a consequence of this theorem

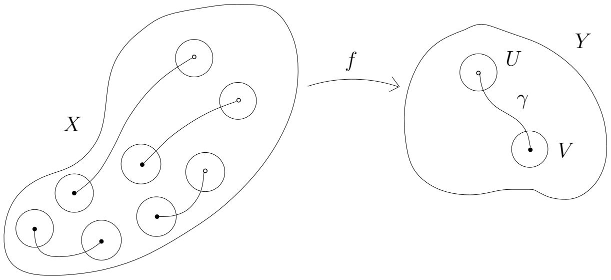
Definition 2.3.9 *Define* $deg_2(f) = \#\{f^{-1}(p)\}\, mod\, 2$.

This is also defined for continuous maps by choosing a smooth map in its homotopy class.

Proof For suitable neighbourhoods U, V of p, q respectively

$$f^{-1}(p) \subseteq f^{-1}(U) = \coprod_{i=1}^{r} U_i,\ f^{-1}(q) \subseteq f^{-1}(V) = \coprod_{j=1}^{s} V_j.$$

Let γ be an embedded curve joining p and q.



Then $f \pitchfork \gamma$ on $f^{-1}(U) \cup f^{-1}(V)$. Now make $f \pitchfork \gamma$ everywhere and also call this new function f. $f^{-1}(\gamma)$ is a compact 1-dim manifold with $\{p_1, \ldots, p_r, q_1, \ldots, q_s\}$ as boundary points, whence $r + s$ is even. Thus $r \equiv s \bmod 2$.

For the case where $F : X \times I \to Y$ a homotopy between $f = F_0$ and $g = F_1$ and the proof in general, refer to the beautiful book (*J. W. Milnor - Topology from the Differentiable Viewpoint*). ☐

3 Cobordism Theory

3.1 Cobordism

Definition 3.1.1 *Let X_0, X_1 be smooth compact n-manifolds without boundary. We say X_0 and X_1 is* **cobordant** *if there is a compact $(n+1)$-manifold Y and a diffeomorphism $\partial Y \sim X_0 \coprod X_1$.*

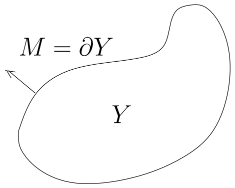
By the **Collar Neighbourhood Theorem** this is an equivalence relation. Let Ω_n denote the equivalence classes of n-manifolds. It is a group under $\coprod$. Also, let $\Omega_* = \oplus_{n \geq 0} \Omega_n$, which is a ring under $\times$. Note that every element is a 2-torsion. Check that $\Omega_0 = \mathbb{Z}_2, \Omega_1 = 0, \Omega_2 = \mathbb{Z}_2$. We have proved before

Theorem 3.1.2 *Let $f : X^m \to Y^n$ be a C^∞-map between compact manifolds $(n \leq m)$.*
(i) $[f^{-1}(p)] \in \Omega_{m-n}$ is independent of the regular value p.
(ii) $[f^{-1}(p)]$ depends only on the homotopy class of f.

Let $k = m - n$ and let $w = P(w_1, \ldots, w_{m-n})$ be a polynomial in Stiefel-Whitney classes. Then $w([f^{-1}(p)]) \in \mathbb{Z}_2$ is well defined. This gives rise to many mod 2 degrees. We have :

Theorem 3.1.3 (Thom)
Let $\alpha \in \Omega_k$. Then $\alpha = 0$ if and only if $w(\alpha) = 0$ for all w.

Let $M = \partial Y$. Then $TM \oplus \mathbb{R} = TY|_M$ and $M = 0$ in $H^k(Y)$.



Thus we have

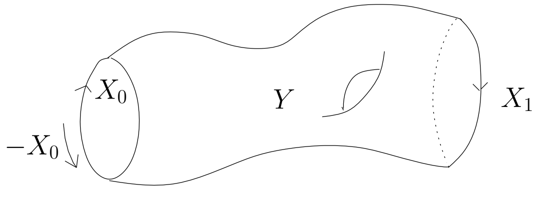
$$0 = w(TY)[\partial Y] = w(TM \oplus \mathbb{R})[M] = w(TM)[M].$$

Definition 3.1.4 *A compact, oriented n-manifold is* **oriented cobordant to zero** *if there is a compact, oriented $(n+1)$-manifold Y and an orientation preserving diffeomorphism $\partial Y \overset{\cong}{\to} X$.*

Given a manifold X with an orientation, let $-X$ denote the same manifold with the opposite orientation. Then

$$\partial(X \times [0,1]) = X \coprod (-X) \overset{\text{o. cob}}{\sim} 0.$$

We say X_0 is oriented cobordant to X if $X_1 \coprod (-X_0) \overset{\text{o. cob}}{\sim} 0$.



Define Ω_n^{SO} to be equivalence class of n-manifolds. This is an abelian group under $\coprod$ and $-[X] = [-X]$. Also $\Omega_*^{SO} = \oplus_{n\geq 0}\Omega_n^{SO}$ is a ring under $\times$. Similar arguments (as before) show that

Theorem 3.1.5 *Let $f : X \to Y$ be a smooth map between compact oriented manifolds (or a smooth proper map). Then the oriented cobordism class $[f^{-1}(p)] \in \Omega_{m-n}^{SO}$ is independent of the regular value and depends only on the proper homotopy class of f.*

Remarks (i) Note that $f^{-1}(p) \equiv M$ has an oriented normal bundle and $df_x : N_xM \overset{\cong}{\to} T_{f(x)}Y$ for all $x \in M$. Orientations of NM and X determine an orientation for M.

(ii) If dim $X =$ dim Y then $[f^{-1}(p)] \in \Omega_0^{SO} = \mathbb{Z}$ is just the degree of f (also equals the number of algebraic preimages of a regular value). Otherwise, suppose $\omega \in H^n(Y, \mathbb{R})$ is a smooth n-form such that $\int_Y \omega = 1$. Define deg $f = \int_X f^*\omega$. For a regular value p of f, let ω_ε be a n-form (compactly supported around p) with unit volume such that $\omega_\varepsilon \to 0$. This implies

$$\int_X f^*\omega_\varepsilon \to \sum n_j\delta_{x_j}$$

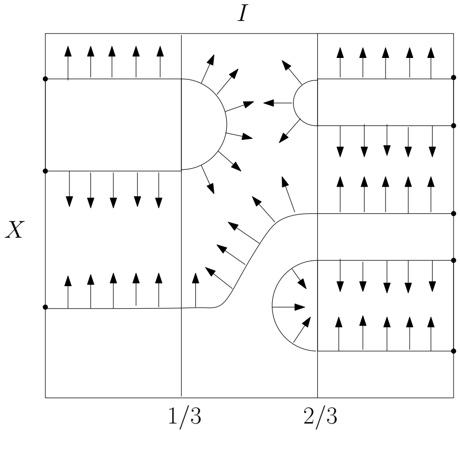
where $n_j = \pm 1$ and x_j's are the preimages of p.

(iii) Let $f : X \to Y$ with dim $X >$ dim Y. $\lim_{t\to 0} f^*\omega_t =$ current of integration over oriented submanifold $f^{-1}(p)$, i.e.,

$$\lim_{t\to 0}\int_X f^*\omega_t \wedge \alpha = \int_M \alpha,\ \forall\, \alpha \in \mathcal{E}^{m-n}(X).$$

Given any polynomial $P = F(p_1, \ldots, p_l), k = m - n = 4l, P([f^{-1}(p)]) \in \mathbb{Z}$ is an invariant. We get $[f^{-1}(p)] \in \Omega_{4l}^{SO}$ and depends only on the homotopy class of f.

Definition 3.1.6 *A* **framed submanifold** *of a manifold X is a compact submanifold $M \subseteq X$ together with a trivialization of the normal bundle.*

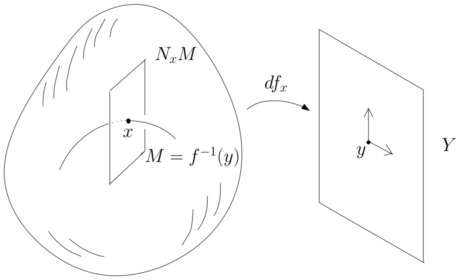


Definition 3.1.7 *Two framed submanifolds $(M, \nu), (M', \nu')$ of X are framed cobordant in X if there is a framed submanifold $(L, \tilde{\nu})$ in $X \times [0, 1]$ such that (refer to the figure above)*

$$L = (M \times [0, 1/3], \nu)\, in\, X \times [0, 1/3]$$

$$L = (M \times [2/3, 1], \nu')\, in\, X \times [2/3, 1].$$

Let $f : X^m \to Y^n$ be a smooth proper map with $m \geq n$ and Y oriented. Let $y \in Y$ be a regular value of f and let $v_1, \ldots, v_n$ be a basis of T_yY with positive orientation. The map $df_x : N_xM \to T_yY$ is an isomorphism. Let $\underline{\nu}$ be the pullback orientation on NM.



Theorem 3.1.8 (*i*) *The framed cobordism class of* $(f^{-1}(y), \underline{\nu})$ *is independent of the choice of regular value and the choice of an oriented basis.*
(*ii*) *The framed cobordism class of* $(f^{-1}(y), \underline{\nu})$ *depends only on the proper homotopy class of* f.

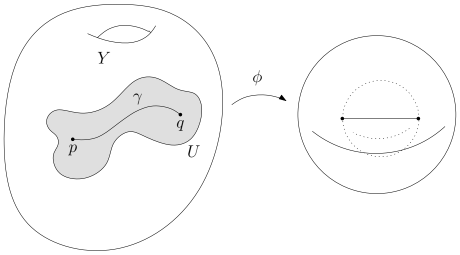
Proof We break up the proof into various steps. In the proof M denotes $f^{-1}(y)$.
Step 1 : We proceed to show independence of the choice of oriented basis $v_1, \ldots, v_m$ of T_yY. Let $v'_1, \ldots, v'_m$ be another. The two bases can be joined by a smooth family of bases $\nu(t) = (v_1(t), \ldots, v_n(t))$. Construct the framed bordism $(M \times [0,1], \tilde{\nu})$ where

$$\tilde{\nu}(t) = \begin{cases} \underline{\nu}, & 0 \leq t \leq 1/3 \\ \nu(3t-1), & 1/3 \leq t \leq 2/3 \\ \underline{\nu}', & 2/3 \leq t \leq 1. \end{cases}$$

Step 2 : If $f \sim g$ via the homotopy H and p is a regular value of both, then the corresponding framed submanifolds are framed cobordant. Choose ε suitably and define $F : X \times [0,1] \to Y$ such that

$$F(x,t) = \begin{cases} f(x), & 0 \leq t < 1/3 + \varepsilon \\ H(x, \frac{t-1/3-\varepsilon}{1/3-2\varepsilon}), & 1/3 + \varepsilon \leq t \leq 2/3 - \varepsilon \\ g(x), & 2/3 - \varepsilon < t \leq 1. \end{cases}$$

By transversality theorem, we can make $F \pitchfork p$, keeping it fixed on $X \times ([0,1/3] \cup [2/3,1])$. Now choose an ordered basis $v_1, \ldots, v_m$ of T_yY. Set $L = F^{-1}(p)$ with the pullback framing $\underline{\nu}$ coming from $v_1, \ldots, v_m$.
Step 3 : Suppose p, q are regular values of f. Join p to q by a smooth embedded arc γ. There exists a neighbourhood U of γ and a diffeomorphism $\phi : U \to B_2(0)$ satisfying $\phi(\gamma(t)) = \{(t, 0, \ldots, 0) | -1 \leq t \leq t\}$.



Hence there is a 1-parameter family of diffeomorphisms $\psi_t : Y \to Y, t \in [0,1]$ such that (i) ψ_t is compactly supported in U for all t (supp $\psi_t = \{x | \psi_t(x) \neq x\}$).

(ii) $\psi_0 = \mathrm{Id}$.

(iii) $\psi_1(p) = q$.

(iv) ψ_t is constant if $t \in [0, 1/3] \cup [2/3, 1]$.

Let $F(x,t) := \psi_t(f(x))$; q is a regular value of both $f = F(x,0)$ and $\psi_1 \circ f = F(x,1)$. By step 2 the corresponding framed submanifolds are framed cobordant. But the framed cobordism for $\psi_1 \circ f$ (for q) is the same as the framed cobordism for f (for p). ☐

As a consequence, for Y oriented with dim $Y \leq$ dim X, we get

$$[X,Y] \simeq \pi_0(\mathrm{Map}(X,Y)) \hookrightarrow \text{framed cobordism classes of framed submanifolds (of codim = dim } Y\text{) of } X.$$

Consider $Y = S^m$ with a fixed orientation, i.e., S^m is the compactification of $\mathbb{R}^m$ with an oriented basis $v_1, \dots, v_m$ of $T_0 S^m$. The following is a partial converse of the result proved before :

Theorem 3.1.9 *Given $(M, \underline{\nu})$, a framed submanifold of codim m in a compact manifold X, there exists $f : X \to S^m$ with 0 as a regular value and $(M, \underline{\nu})$ as its associated framed submanifold.*

Proof We use the

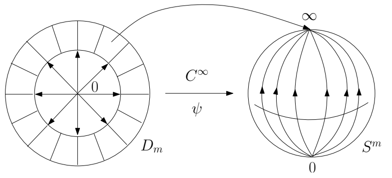
Product Neighbourhood Theorem *Given a framed submanifold $(M, \underline{\nu})$ of codim m, there is an open neighbourhood U of M and a diffeomorphism $\phi : M \times D^m \to \overline{U}$ such that $\phi_*(e_j) = v_j$ along $M \times \{0\}$, where e_j's are the standard vector fields on $D^m \subseteq \mathbb{R}^m$.*

For a quick proof of this, set

$$\Phi(x, t_1, \dots, t_m) = \exp_x\Big((\sum_{i=1}^{m} t_i v_i(x) \Big)$$

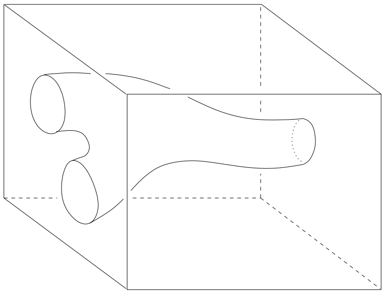
for some Riemannian metric on Y. Now apply the inverse function theorem to get a diffeomorphism in a neighbourhood.
Define $f : X \to S^m$ by

$$f(x) = \begin{cases} U \overset{\Phi^{-1}}{\to} M \times D^m \overset{\mathrm{pr}_2}{\to} D^m \overset{\psi}{\to} S^m, & x \in U \\ \infty, & x \in X \setminus U. \end{cases}$$



$S^m \equiv D^m/(D^m \setminus \frac{1}{2} D^m)$

This construction also applies to framed cobordism. ☐



As an upshot we have :

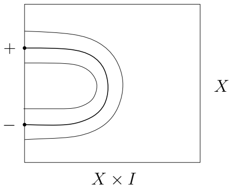
Theorem 3.1.10 *For X compact, $[X, S^m] \overset{1-1}{\longleftrightarrow}$ framed cobordism classes of X (of codim m).*

Corollary 3.1.11 *Suppose X is compact and oriented of dim m. Then $[X, S^m] \cong \mathbb{Z}$ and two maps $f, g : X \to S^m$ are homotopic if and only if deg $f =$ deg g.*

Proof We are interested in framed cobordism classes of 0-dimensional submanifolds. A framed submanifold is a finite set of points $x_1, \ldots, x_l \in X$ with a choice of basis of each $T_{x_i}X$. Moreover, a framed cobordism depends only on the orientation of each frame (or only on $\sum_{i=1}^{l} n_i$ where

$$n_i = \begin{cases} +1 & \text{if orientation agrees with } X \\ -1 & \text{otherwise.} \end{cases}$$

As the picture suggests



there is a cancellation of opposite pairs. □

Corollary 3.1.12 *Suppose X is compact, non-orientable of dim m. Then $f, g : X \to S^m$ are homotopic if and only if $deg_2 f = deg_2 g$.*

Specializing to $X = S^{m+k}$ we get that $\pi_{m+k}(S^m)$ is the framed cobordism classes of k-dimensional framed submanifolds of S^{m+k}. For $M^k \subseteq S^{m+k}$, add the normal to S^{m+k} (equator of S^{m+k+1} in S^{m+k+1}) to get a framing of $M^k \subseteq S^{m+k+1}$. This gives map

$$\pi_{m+k}(S^m) \to \pi_{m+k+1}(S^{m+1}).$$

Exercise (i) Prove that this is induced by the suspension map.
Exercise (ii) Prove a special case of the **Freudenthal Suspension Theorem** :

$$\pi_{m+k}(S^m) \xrightarrow{\Sigma} \pi_{m+k+1}(S^{m+1}) \text{ is an isomorphism if } m > k+1.$$

This implies that

$$\pi_{m+1}(S^m) \cong \mathbb{Z}_2, m > 2.$$

3.2 Thom Construction

Suppose $X^n \subseteq \mathbb{R}^{n+m}$ is a compact submanifold with $\partial X \neq \phi$. Let N be the normal bundle of X. We need :

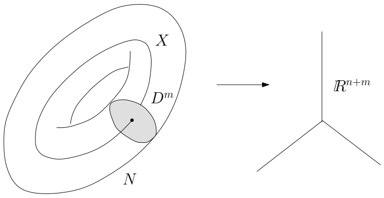
Theorem 3.2.1 Tubular Neighbourhood Theorem
There exists $\varepsilon > 0$ and a diffeomorphism

$$\Phi : \{v \in N \, s.t. \, \|v\| < \varepsilon\} \equiv N_\varepsilon \to U_\varepsilon \equiv \{x \in \mathbb{R}^{n+m} | d(x, X) < \varepsilon\}$$

sending the zero section to X.

Proof Define a map $e : N \to \mathbb{R}^{n+m}$ which sends $v \in N_x$ to $x + v$. $de = \mathrm{Id}$ along points of X. Apply the inverse function and compactness of X to get Φ and ε. ☐

So we get $X \subseteq U \cong N$ and there is a classifying map $f : X \to G_m(\mathbb{R}^{n+m})$ for $N \to X$.



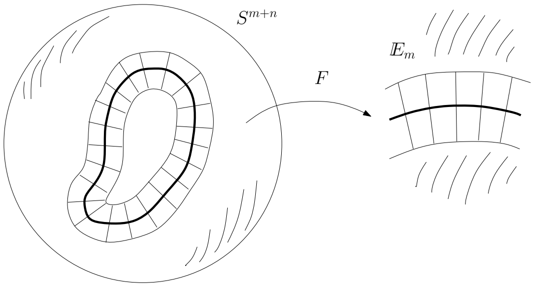
Definition 3.2.2 *Given a vector bundle $E \to X$ with a metric, the* **Thom space** *of E is the quotient*

$$\tau(E) \equiv E/(E - D^\circ(E)) \equiv D(E)/\partial D(E) \equiv E \cup \{\infty\}$$

where $D(E) = \{v \in E \, s.t. \, \|v\| \leq 1\}$.

The compactification of $U \to N \to \mathbb{E}_m$ yields

$$\begin{tikzcd}
\mathbb{R}^{n+m}/(\mathbb{R}^{n+m} \setminus U) \arrow[r] & \tau(N) \arrow[r] & \tau(\mathbb{E}_m) \ni \infty \\
S^{n+m}/(S^{n+m} \setminus U) \arrow[u, equal] & & \\
\infty \in S^{n+m} \arrow[u] \arrow[uurr, "F"] & & , F(\infty) = \infty.
\end{tikzcd}$$



Thus, associated to X is a base point map $F_X : S^{n+m} \to \tau(\mathbb{E}_m)$.

Proposition 3.2.3 *The corresponding element* $[F_X] \in \pi_{n+m}(\tau(\mathbb{E}_m))$ *is independent of choices (identification of* U, N *etc.) and independent of the choice of classifying maps* $X \to G_m(\mathbb{R}^{m+n'})$ *for* $n' \geq n+2$.

Proof The first part is straightforward. For the last part, the map $G_m(\mathbb{R}^{m+k}) \to G_m(\mathbb{R}^{m+k+1})$ is k-connected. Since any two maps f and g

$$\begin{tikzcd}
 & G_m(\mathbb{R}^{m+k}) \arrow[dr, "\subseteq"] & \\
X \arrow[ur, "f"] \arrow[dr, "g"'] & & G_m(\mathbb{R}^{m+k'}) \\
 & G_m(\mathbb{R}^{m+k'}) \arrow[ur, "="] &
\end{tikzcd}$$

classifying the same bundle are homotopic if $k' \geq k$, we are done. □