A comparison between two four manifolds

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1 Introduction

Let $F \hookrightarrow E \to S^n$ be a fibration. Since the sphere is built out of two disks glued together and any bundle over a disk is trivializable, we may think of E as two copies of $D_n \times F$ identified along its boundary $S^{n-1} \times F$. In the case $F = G$ is a topological group and E is a principal bundle, the transitive action of G on the fibre leads us to observe that identifying two copies of $S^{n-1} \times G$ is equivalent to a map $\theta : S^{n-1} \to G$. Since two homotopic maps give the same bundle, G-bundles over S^n are classified by $\pi_{n-1}(G)$. For general fibre bundles with fibre F, we similarly have the equivalence between isomorphism classes of bundles and $\pi_{n-1}(\text{Aut}(F))$. In the discussion of fibre bundles in the smooth category, we would work with $\text{Diff}(F)$ while in the case of smooth oriented bundles it means $\text{Diff}^+(F)$. In what follows we shall assume $\text{Aut}(F)$ to be one of the above unless mentioned otherwise.

When considering G-bundles over S^1, we see that it is trivializable whenever G is connected. In general, any oriented bundle $F \hookrightarrow E \to S^1$ is trivial if the group $\text{Aut}(F)$ (of oriented preserving automorphisms of F) is connected. In particular, any smooth oriented S^3-bundle over the circle is trivial. Conversely, consider a smooth oriented S^1-bundle E over S^3. The isomorphism class is determined by $\pi_2(\text{Diff}^+(S^1))$. It is known that $\text{Diff}^+(S^1)$ deformation retracts to S^1, the group of left translations. Therefore, E is actually $S^1 \times S^3$.

Let us now consider S^2-bundles over S^2. This is classified by $\pi_1(\text{Diff}(S^2))$. It is well known by the work of S. Smale [2] that $\text{Diff}(S^2)$ deformation retracts to $O(3)$. Thus, if we are considering oriented S^2-bundles over S^2 then

$$\pi_1(\text{Diff}^+(S^2)) = \pi_1(SO(3)) = \mathbb{Z}_2,$$

whence there are two such bundles up to isomorphism. One of them, of course, is $S^2 \times S^2$. We will see that $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ is the other one. Our aim will be to compare these bundles as spaces by probing it using techniques from different fields.

2 Differences from various viewpoints

There are various ways in which one can compare the two manifolds mentioned above. We have divided the discussion into subtopics for the clarity of the exposition. We have also included brief descriptions of the necessary definitions to make this note as self-contained as possible.

2.1 As fibre bundles

Let us begin with a cellular description of $\mathbb{CP}^2$. Recall that $\mathbb{CP}^2$ can be thought of as built out of attaching a 4-ball along its boundary S^3 to a $\mathbb{CP}^1 = S^2$ via the Hopf map. Therefore, it has one 0-cell, one 2-cell and one 4-cell. The cohomology ring, therefore, is given by a single generator of degree 2. The manifold $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ can then be described via *mapping cylinders*. If we remove a disk from $\mathbb{CP}^2$ then this is just the space M_f constructed by taking $S^3 \times [0,1]$ and identifying $S^3 \times \{1\}$ to S^2 using the Hopf map $f : S^3 \to S^2$. Then take two copies of M_f, one where $[0,1]$ is oriented from left to right and the other where it is the opposite, and glue the two copies of $S^3 \times \{0\}$ by any oriented preserving diffeomorphism of S^3. Since $\text{Diff}^+(S^3)$ is connected we may choose the identity map for the identification. The resulting manifold $(M_f \coprod M_f)/\sim$ is $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$. In other words, one may think of this manifold as $S^3 \times [0,1]$ with its boundary $S^3 \times \{0\}$ and $S^3 \times \{1\}$ identified to S^2 via the Hopf map. Define a map

$$\pi : S^3 \times [0,1] \to S^2, \ \pi(x,t) = f(x), x \in S^3.$$

This induces a map $\widetilde{\pi} : \mathbb{CP}^2\#\overline{\mathbb{CP}}^2 \to S^2$. The fibre is $S^1 \times [0,1]$ with the end circles collapsed to a point, resulting in S^2.

One can do the same mapping cylinder construction with the map $p : T_1S^2 \to S^2$, where T_1S^2 is the unit tangent bundle of S^2. Let M_p denote the mapping cylinder of p. This space is precisely the unit disk bundle of TS^2. Take two copies of M_p, with orientation conventions as before, and glue T_1S^2 to itself with the identity map. Again, this results in a 4-manifold which fibres over S^2 with fibre S^2. In fact, $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ is almost a double cover of $S^2 \times S^2$. More precisely, take $S^2 \times S^2$ and use the branched double cover of $\pi : S^2 \to S^2$ with the north and south poles as the singular points to get another S^2-bundle over S^2. This resulting space is $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$. Both these manifolds are simply connected, as can be seen from the long exact sequence of homotopy groups arising from the fibrations or otherwise.

2.2 As complex manifolds

Let us recall the notion of a Kähler manifold. A Riemannian metric g on a complex manifold M is called *Hermitian* if it is compatible with the complex structure J of M, i.e., $g(JX,JY) = g(X,Y)$. The associated 2-form defined by $\omega(X,Y) := g(JX,Y)$ is called the *Kähler form*. The manifold will be called *Kähler* if $dw = 0$ or equivalently, $\nabla J = 0$ for the Levi-Civita connection ∇ on M. Complex submanifolds of Kähler manifolds provide examples of Kähler manifolds as do products. The prototype to keep in mind is $\mathbb{CP}^n$, the complex projective n-space, equipped with the Fubini-Study metric.

In complex algebraic geometry, $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ is commonly referred to as $\mathbb{CP}^2$ *blown up* at one point. The smooth as well as the complex structure doesn't depend on the point chosen. It can be shown via the Kodaira embedding theorem that a Kähler manifold blown up at one point is also Kähler. Since $\mathbb{CP}^2$ is a Kähler manifold, so is $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$. As an application of the Kodaira embedding theorem, yet again, this manifold embeds into some complex projective space.

The same holds for $S^2 \times S^2$ as well since a product of Kähler manifolds is again Kähler. Add embedding ???

For the Levi-Civita connection ∇ we define the *holonomy group* at $p \in M$ by

$$\text{Hol}_p(M) := \left\{ P_\gamma \in GL(T_pM) \,|\, \gamma : (S^1,1) \to (M,p) \right\},$$

where $P_\gamma : T_pM \to T_pM$ is the parallel transport map using ∇. Changing base points only changes the holonomy group by conjugation and we write $\text{Hol}(M)$ to mean the holonomy group up to conjugation. It is known that a connected Riemannian manifold M of dimension $2n$ is Kähler if and only if $\text{Hol}(M) \subseteq U(n)$. Moreover, it is Ricci-flat Kähler if and only if $\text{Hol}(M) \subseteq SU(n)$. For

$S^2 \times S^2$ we have

$$\mathrm{Hol}(S^2 \times S^2) = \mathrm{Hol}(S^2) \times \mathrm{Hol}(S^2) = S^1 \times S^1.$$

Show that $S^2 \times S^2$ is not Ricci-flat Kähler.
Find $\mathrm{Aut}_0(S^2 \times S^2, J)$ with the canonical complex structure J. (Guess $(GL_2(\mathbb{C})/\mathbb{C}^*) \times (GL_2(\mathbb{C})/\mathbb{C}^*)$)
Define $\mathrm{Aut}(M, J)$ and invoke the theorem comparing $\mathrm{Aut}_0(M \# \overline{\mathbb{CP}}^n)$ and $\mathrm{Aut}_0(M, p)$. This implies, in particular (work it out) that $\mathrm{Aut}(\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2)$ has the homotopy type of $U(2)$. Refer Griffiths-Harris 4.1 and Werner Ballmann *Lectures on Kähler manifolds* page 27.
A celebrated theorem of Bochner-Montogomery states that $\mathrm{Aut}(M, J)$ is a complex Lie group with respect to the compact-open topology if M is a closed complex manifold.

2.3 As symplectic manifolds

Both manifolds under consideration are symplectic. Let's recall that a *symplectic manifold* M is a smooth manifold equipped with closed non-degenerate 2-form ω, called the symplectic form. A diffeomorphism $f : (M_1, \omega_1) \to (M_2, \omega_2)$ which preserves the symplectic forms is called a *symplectomorphism.* The group of symplectomorphisms of (M, ω) will be denoted by $\mathrm{Symp}(M, \omega)$. There is special class of diffeomorphisms which arise as the integration of the flow of a vector field. Any differentiable function $H : M \to \mathbb{R}$ defines a unique vector field X_H satisfying the identity

$$\omega(X_H, Y) := dH(Y)$$

for any vector field Y on M. If M is compact then we can integrate this vector field to get a one parameter group of diffeomorphisms φ_t^H of M. It follows from $\omega(X_H, X_H) = 0$ that φ_t^H preserve the symplectic form. Any φ that arises as an element of a Hamiltonian flow φ_t^H is called a *Hamiltonian symplectomorphism.* The group consisting of such elements is denoted $\mathrm{Ham}(M, \omega)$.

Let $\gamma : [0, 1] \to \mathrm{Symp}(M, \omega)$ be a smooth curve joining the identity and an arbitrary element φ. The associated vector $\gamma'(0)$ is a vector field X on M. This corresponds via ω to a one form $\alpha \in \Omega^1(M)$. By definition the Lie derivative $\mathcal{L}_X \omega$ vanishes. Since $\mathcal{L}_X = d\iota_X + \iota_X d$ and $d\omega = 0$ we have $d\iota_X(\omega) = 0$ which implies

$$0 = d\iota_X(\omega)(V_1, V_2) = V_1(\omega(X, V_2)) - \omega(X, [V_1, V_2]) - V_2(\omega(X, V_1)) = -d\alpha(V_1, V_2).$$

Therefore, if $H^1(M; \mathbb{R}) = 0$ any closed 1-form is exact and $\alpha = dH$ for some differentiable function $H : M \to \mathbb{R}$. This implies that the connected component of the identity in $\mathrm{Symp}(M, \omega)$ is $\mathrm{Ham}(M, \omega)$.

Let $\sigma_0 \in \Omega^2(S^2)$ be the standard area form on S^2 with area one. Define

$$\sigma_0 \times \sigma_0 := \pi_1^*(\sigma_0) + \pi_2^*(\sigma_0)$$

to be the standard product symplectic form on $S^2 \times S^2$. It is known (see [1] for detailed references) that every symplectic form on $S^2 \times S^2$ is *standard.* This means it belongs to the following family

$$M_\lambda^0 = (S^2 \times S^2, \omega_\lambda^0),\ 1 \le \lambda \in \mathbb{R},$$

where $\omega_\lambda^0 = \pi_1^*(\lambda\sigma_0) + \pi_2^*(\sigma_0)$. Since $S^2 \times S^2$ is simply connected, the identity component in the group of symplectomorphisms of $S^2 \times S^2$ is the group of Hamiltonian symplectomorphisms.

Theorem 2.1. (Gromov) *The group of symplectomorphisms of M_1^0 has two components. Moreover, Ham$(S^2 \times S^2, \omega_1^0)$ is homotopy equivalent to $SO(3) \times SO(3)$.*

It is also known [1] that every symplectic form on $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ is *standard*, i.e., it belongs to the family

$$M^1_\lambda = (\mathbb{CP}^2\#\overline{\mathbb{CP}}^2, \omega^1_\lambda),\ -1 < \lambda \in \mathbb{R},$$

where ω^1_λ takes the value $2+\lambda$ on $S^2 \in \mathbb{CP}^2$ and $1+\lambda$ on $S^2 \in \overline{\mathbb{CP}}^2$. This is enough to specify a closed 2-form since $H^2(\mathbb{CP}^2\#\overline{\mathbb{CP}}^2;\mathbb{R}) = H^2(\mathbb{CP}^2;\mathbb{R}) \oplus H^2(\overline{\mathbb{CP}}^2;\mathbb{R})$ by Mayer-Vietoris.

Theorem 2.2. (Gromov) *The group of symplectomorphisms of M^1_λ deformation retracts onto the unitary group $U(2)$ when $-1 < \lambda \leq 0$.*

2.4 As spin manifolds

Let's consider the existence of spin structures. Recall that a manifold is *spin* if it is oriented and the second Stiefel-Whitney class w_2 vanishes. Since product of spin manifolds is spin, $S^2 \times S^2$ is spin. In fact, it has a unique spin structure since $H_1(S^2 \times S^2; \mathbb{Z}_2) = 0$. On the other hand, any compact, simply connected 4-manifold which has an odd intersection form cannot be spin. This implies that $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ is not spin. In fact, we'll see later that

$$w_2(\mathbb{CP}^2\#\overline{\mathbb{CP}}^2) = w_2(\mathbb{CP}^2) + w_2(\overline{\mathbb{CP}}^2) \tag{2.1}$$

under the identification $H^2(\mathbb{CP}^2\#\overline{\mathbb{CP}}^2;\mathbb{Z}_2) = H^2(\mathbb{CP}^2;\mathbb{Z}_2) \oplus H^2(\overline{\mathbb{CP}}^2;\mathbb{Z}_2)$. Recall that the Stiefel-Whitney classes of $\mathbb{CP}^2$ are given by $(1-c')^3$, where c' is the positive generator of the cohomology ring $H^*(\mathbb{CP}^2;\mathbb{Z})$. Therefore, $w_2(\mathbb{CP}^2) = -c' \neq 0$. This together with (2.1) implies that $w_2(\mathbb{CP}^2\#\overline{\mathbb{CP}}^2) \neq 0$.

However, both manifolds admit spin^c structures. Recall that an oriented manifold carries a *spin^c structure* if and only if w_2 is the mod 2 reduction of an integral cohomology class. The short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

induces a long exact sequence in cohomology

$$\cdots \longrightarrow H^i(X;\mathbb{Z}) \xrightarrow{2} H^i(X;\mathbb{Z}) \longrightarrow H^i(X;\mathbb{Z}_2) \xrightarrow{\beta} H^{i+1}(X;\mathbb{Z}) \longrightarrow \cdots.$$

The map β is called the *Bockstein homomorphism*. Therefore, an oriented manifold is called spin^c if $\beta(w_2) = 0$. This implies, in particular, that any spin manifold is also spin^c. Since $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ satisfies $H^1 = 0 = H^3$, w_2 is the reduction of an integral 2-cocycle, whence it is spin^c. In fact, for any spin^c manifold M, there are $2H^2(M;\mathbb{Z}) \oplus H^1(M;\mathbb{Z}_2)$ worth of spin^c structures on it. In our case, both manifolds have spin^c structures parametrized by $\mathbb{Z} \oplus \mathbb{Z}$.

2.5 Intersection form

For any compact, simply connected 4-manifold M Poincaré duality tells us that $H_2(M;\mathbb{Z}) \cong H^2(M;\mathbb{Z}) \cong \mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z})$ is a free abelian group. The *intersection form* on M is given by

$$Q_M : H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \to \mathbb{Z},\ Q_M(\alpha,\beta) := \alpha \cap \beta.$$

We say that two such intersection forms Q_{M_1} and Q_{M_2} are equivalent if there is an isomorphism $\varphi : H_2(M_1;\mathbb{Z}) \to H_2(M_2;\mathbb{Z})$ which commutes with the intersection forms. We have the classical result which says up to what extent Q_M classifies topological 4-manifolds.

Theorem 2.3. (Whitehead) *Two simply connected, closed topological* 4*-manifolds are homotopy equivalent if and only if the intersection forms are equivalent.*

As a consequence, the cohomology ring of M is determined by Q_M.

An intersection form is called *even* if $Q_M(\alpha, \alpha) \equiv 0 \bmod 2$ for any $\alpha \in H_2(M; \mathbb{Z})$. Clearly, we see that

$$H_2(S^2 \times S^2; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}, \;\; H_3(S^2 \times S^2; \mathbb{Z}) = 0$$

while using Mayer-Vietoris sequence for $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ we get exactly the same thing. However, two generators of $H_2(S^2 \times S^2; \mathbb{Z})$ can be taken to be the homology class represented by $\alpha = S^2 \times \{q_2\}$ and $\beta = \{q_1\} \times S^2$. It is clear that $\alpha \cap \beta = 1 = \beta \cap \alpha$, whence the intersection form is represented by the matrix

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The intersection form for $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ is just the product of the intersection forms of $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$ since the generators of second homology lives in the 2-skeleton and is unchanged by the connected sum. Therefore, the form is

$$A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Notice that these two matrices are not conjugate over the integers since 2 is not invertible in $\mathbb{Z}$. In fact, A_1 is an even form while A_2 is odd. However, these forms are conjugate over the rationals or the real numbers. Therefore, the cohomology rings of these two spaces, with coefficients in any ring R that admits an inverse of 2, are isomorphic. Moreover, for either manifold the signature $\sigma = 0$. It follows from M. Freedman's seminal work that there is a simply connected, closed non-smoothable 4-manifold M which has A_2 as the intersection form.

2.6 Rational homotopy groups

As observed earlier, both the manifolds under discussion are compact Kähler manifolds. By the famous result of P. Deligne, P. Griffiths, D. Mumford and D. Sullivan we know that any such manifold is *formal*. We would not delve into the technical meaning of the term but glean from its underpinnings. For our manifolds, the rational homotopy type is a formal consequence of the rational cohomology ring. In other words, the rational homotopy groups are determined by the ring in a functorial way. The cohomology ring is the homology of a differential graded algebra, viz., the space of cochains with the coboundary or the space of smooth forms with the de Rham differential. So, we have the algebra over $K = \mathbb{Q}, \mathbb{R}$

$$\mathcal{A} = \Lambda_K(x_1, x_2, x_3, x_4),\, dx_3 = x_1^2, dx_4 = x_2^2,\, |x_1| = 2 = |x_2|, |x_3| = 3 = |x_4|$$

whose homology gives us $H^*(S^2 \times S^2; K) \cong H^*(\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2; K)$. The algebra $\mathcal{A}$, by construction, is *minimal*. Then by D. Sullivan's rational homotopy theory it follows that the vector space generated by $\pi_n(S^2 \times S^2) \otimes \mathbb{Q}$ can be taken to be the underlying vector space of $\mathcal{A}$. Since the rational cohomology ring of $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ are isomorphic, so are its minimal models. Consequently, the two spaces have isomorphic rational homotopy groups. From the classical result of J. P. Serre we know that $\pi_n(S^2) \otimes \mathbb{Q}$ is non-zero and of rank 1 only when $n = 2, 3$. Therefore, we have

$$\pi_2(X) = \mathbb{Z} \oplus \mathbb{Z}, \;\; \pi_3(X) \otimes \mathbb{Q} = \mathbb{Q} \oplus \mathbb{Q}, \;\; X = S^2 \times S^2, \, \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2,$$

and the higher rational homotopy groups are zero. It is, thus, natural to ask if these two spaces are weakly homotopy equivalent or if there is a map $\psi : \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \to S^2 \times S^2$ which induces isomorphisms on $\pi_n \otimes \mathbb{Q}$. We'll take up this question in §3.

Add ??? stuff about localizing at prime p; fine except at the prime 2.

2.7 Whitehead products

Recall that the *Whitehead product* for any connected topological space is a map

$$[\,,\,] : \pi_i(X) \times \pi_j(X) \longrightarrow \pi_{i+j-1}(X), \;\; i, j \geq 1.$$

The boundary sphere of the $(i+j)$-cell in $S^i \times S^j$ maps onto $S^i \vee S^j$ via some natural map φ. One then defines the product by setting

$$S^{i+j-1} \xrightarrow{\varphi} S^i \vee S^j \longrightarrow X.$$

In particular, $[\alpha, \beta] = 0$ if and only if there exists a map $F : S^k \times S^l \to X$ such that $F \circ \iota_i : S^k \to X$ is homotopic to α and $F \circ \iota_2 : S^l \to X$ is homotopic to β. The Whitehead product is skew symmetric and satisfies the Jacobi identity.

Our basic example is $X = S^2$. Recall that the Hopf map $H : S^3 \to S^2$ has Hopf index 1 and this can be checked by measuring the linking number of two fibres $H^{-1}(p)$ and $H^{-1}(q)$. If $[S^3]$ denotes the positive generator of $\pi_3(S^2)$ then one can show via linking numbers that

$$\big[[S^2], [S^2]\big] = 2[S^3].$$

Let $[S^2]_i, i = 1, 2$ denote the homotopy class of the inclusion of S^2 into $S^2 \times S^2$ in the ith coordinate. Let $[S^3]_i$ denote the same for mappings of S^3 into $S^2 \times S^2$. It follows from the preceeding discussion that

$$\big[[S^2]_1, [S^2]_2\big] = 0 = \big[[S^2]_2, [S^2]_1\big]\,, \;\; \big[[S^2]_i, [S^2]_i\big] = 2[S^3]_i, \; i = 1, 2.$$

3 Homotopy groups

Based on what we claimed before, it's natural to ask if the homotopy groups are the same although Whitehead products distinguish between the two manifolds. For our purposes in this note, we consider the homotopy groups of $S^2 \times S^2$ to be known since one knows a great deal about $\pi_i(S^2)$. As a starting point, observe that both these manifolds have $\pi_2 = \mathbb{Z} \oplus \mathbb{Z}$. We'll analyze $\pi_i, i \geq 3$ in what follows.

Recall that the principal S^1-bundle $S^5 \xrightarrow{\pi} \mathbb{CP}^2$ corresponds to the first Chern class $c^{'} \in H^2(\mathbb{CP}^2; \mathbb{Z})$ of the tautological line bundle. Equivalently, if we want to kill the obstruction class to $\mathbb{CP}^2$ being spin then we construct a principal circle bundle via the correspondence of $H^2(X; \mathbb{Z})$ and $[X, \mathbb{CP}^\infty]$. Moreover, $w_2(\mathbb{CP}^2) = w_2(\overline{\mathbb{CP}}^2)$ since we're working over $\mathbb{Z}_2$. Therefore, we have a corresponding bundle $S^1 \hookrightarrow S^5 \to \overline{\mathbb{CP}}^2$ which kills $w_2(\overline{\mathbb{CP}}^2)$. Let D denote a disk of radius 1 around $[1 : 0 : 0] \in \mathbb{CP}^2$, viz.,

$$D := \{[1 : z_1 : z_2] \,|\, |z_1|^2 + |z_2|^2 \leq 1\}.$$

This implies that

$$\begin{aligned}
\pi^{-1}(D) &:= \{(\lambda\cos\theta, \lambda\sin\theta, x_3, \dots, x_6) \,|\, \theta \in [0, 2\pi),\ x_3^2 + x_4^2 + x_5^2 + x_6^2 = 1 - \lambda^2 \le \lambda^2 \neq 0\} \\
&= \{(\lambda\cos\theta, \lambda\sin\theta, x_3, \dots, x_6) \,|\, \theta \in [0, 2\pi),\ x_3^2 + x_4^2 + x_5^2 + x_6^2 = 1 - \lambda^2, \lambda \in [\tfrac{1}{\sqrt{2}}, 1]\} \\
&= \{(x_1, x_2, x_3 \dots, x_6) \,|\, x_3^2 + x_4^2 + x_5^2 + x_6^2 \le \tfrac{1}{2},\ x_1^2 + \dots + x_6^2 = 1\} \\
&\cong S^1 \times D_4.
\end{aligned}$$

Notice that the complement of $\pi^{-1}(D)$ is

$$\{(x_1, \dots, x_6) \,|\, x_1^2 + x_2^2 \le \tfrac{1}{2},\ x_1^2 + \dots + x_6^2 = 1\} \cong D_2 \times S^3.$$

Therefore, S^5 can be written as the union of $S^1 \times D_4$ and $D_2 \times S^3$.

Let $X := S^5 \setminus \pi^{-1}(D)$. Then we have a fibration

$$S^1 \hookrightarrow X \to \mathbb{CP}^2 \setminus D,\ \ S^1 \hookrightarrow X \to \overline{\mathbb{CP}}^2 \setminus D.$$

Let $\widetilde{X}$ be the space obtained by gluing two copies of X together by the identity along its boundary. It's clear that $\widetilde{X} = S^2 \times S^3$ and the two fibrations above blend into a fibration

$$S^1 \hookrightarrow S^2 \times S^3 \to \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2.$$

Therefore, for $i \ge 3$ the ith homotopy groups of $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ are equal to the homotopy groups of $S^2 \times S^3$ which is the same as that of $S^2 \times S^2$. We conclude that $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ have the same homotopy groups.

This may seem dubious at first based on what we have seen in §3. However, this is an instance of general phenomenon. Indeed, we have the rather surprising result, based heavily on a great result of S. Smale.

Proposition 3.1. *For a simply connected, closed, smooth 4-manifold M the homotopy groups are determined by its second Betti number.*

Before embarking on a proof, let's recall Smale's classification result [3] that we'll be using.

Theorem 3.2. (Smale) *Diffeomorphism classes of smooth, simply connected, closed 5-manifolds are in bijection with finitely generated abelian groups. The correspondence is given by $M \mapsto H_2(M;\mathbb{Z}) \oplus \frac{1}{2}\mathrm{Tor}(H_2(M;\mathbb{Z}))$.*

Remark Let's explain we mean by $T := \frac{1}{2}\mathrm{Tor}(H_2(M;\mathbb{Z}))$. There is a well defined bilinear pairing, called *linking duality*, denoted by $(\ ,\ ) : T \otimes T \to \mathbb{Q}/\mathbb{Z}$. Explicitly, for $\alpha \in T$ choose γ such that $\partial\gamma = n\alpha$ and define $(\alpha, \beta) := \frac{1}{n}(\gamma \cap \beta)$. This is non-degenerate by Poincaré duality and skew symmetric by definition. By the structure theorem for finite abelian groups we may write $T = \mathbb{Z}_{k_1} \oplus \dots \oplus \mathbb{Z}_{k_r}$ where $k_1 > 1$ and $k_i | k_{i+1}$. By relabelling we may rewrite this as $T = \mathbb{Z}_{k_1}^{l_1} \oplus \dots \oplus \mathbb{Z}_{k_s}^{l_s}$ where k_i strictly divides k_{i+1}. ??? Use Bockstein and Steenrod squares to show that l_i's are even.

We will also need the following.

Lemma 3.3. *Let M be a simply connected, closed 4-manifold with $H_2(M;\mathbb{Z}) = \mathbb{Z}^k$. Then there is a fibration $S^1 \hookrightarrow E \to M$ such that E is simply connected and spin with $H_2(E;\mathbb{Z}) = \mathbb{Z}^{k-1}$.*

Proof To construct E we need to deal with two cases - either M is spin or it's not. If M is spin, then choose any element α from a basis of $H^2(M;\mathbb{Z}) = [M, \mathbb{CP}^\infty]$. On the other hand, if M is not spin then w_2 is the reduction of some integral class α, which can be chosen such that it extends to a basis of $H^2(M;\mathbb{Z})$. In either case, since α corresponds to a homotopy class of maps, let $f : M \to \mathbb{CP}^\infty$ be a representative. In fact, one can assume that f is cellular and $f : M \to \mathbb{CP}^2$. Then $E := f^*S^5$ denote the pullback of the circle bundle over $\mathbb{CP}^2$. From the long exact sequence of homotopy groups of $S^1 \hookrightarrow E \xrightarrow{\pi} M$ we have

$$0 \longrightarrow \pi_2(E) \longrightarrow \pi_2(M) \longrightarrow \mathbb{Z} \longrightarrow \pi_1(E) \longrightarrow 0.$$

Notice that $\pi_2(M) = H_2(M;\mathbb{Z}) \cong H^2(M;\mathbb{Z})$ since $H_2(M;\mathbb{Z})$ is free and M is simply connected. Moreover, if $\overline{\alpha}$ denotes the dual of α then $\overline{\alpha} \in \pi_2(M)$ is mapped to $1 \in \pi_1(S^1)$ in the sequence above. This implies that $\pi_1(E) = 0$ and E is orientable. One also has the Gysin sequence

$$\cdots \longrightarrow H^i(M;\mathbb{Z}) \xrightarrow{\cup\alpha} H^{i+2}(M;\mathbb{Z}) \xrightarrow{\pi^*} H^{i+2}(E;\mathbb{Z}) \xrightarrow{\int} H^{i+1}(M;\mathbb{Z}) \longrightarrow \cdots .$$

Using the above with Poincaré duality we conclude that

$$H^2(E;\mathbb{Z}) = H^2(M;\mathbb{Z})/(\alpha),\ H^3(E;\mathbb{Z}) = H_3(E;\mathbb{Z}) = \mathbb{Z}^{k-1},\ H^4(E;\mathbb{Z}) = 0.$$

Since E is foliated by circles, which carry a nowhere vanishing vector field, we get a line bundle L on E. Notice that L is trivial since E is simply connected. Since E is the total space of fibration,

$$TE = \pi^*(TM) \oplus L = \pi^*(TM) \oplus \mathbb{R}.$$

Therefore,

$$w_2(E) = w_2(\pi^*(TM)) = \pi^*(w_2(M)) = \pi^*(\alpha) = 0$$

by construction and E is spin. □

Proof of the proposition Let M_1 and M_2 be two smooth, simply connected and closed 4-manifolds with the same second Betti number, say k. By the lemma, we have two simply connected, closed, spin 5-manifolds E_1 and E_2, which are also circle bundles over M_1 and M_2 respectively. By Smale's result E_1 and E_2 are diffeomorphic. Therefore, it follows from the long exact sequence of the circle fibrations that

$$\pi_i(M_1) = \pi_i(E_1) = \pi_i(E_2) = \pi_i(M_2),\ i \geq 3.$$

Since $\pi_2(M_1) = \pi_2(M_2) = \mathbb{Z}^k$ by Hurewicz, we conclude the result. □

One can be more precise since Smale's classification is quite explicit. There is a list of simply connected, closed spin 5-manifolds, one for each $q \geq 0$, denoted by M_q^5. For example, $M_0^5 = S^2 \times S^3$, $M_1^5 = S^5$ and $H_2(M_q^5;\mathbb{Z}) = \mathbb{Z}_q \oplus \mathbb{Z}_q$ for $q > 1$. Then given any simply connected closed 5-manifold M, write

$$H_2(M;\mathbb{Z}) = \mathbb{Z}^k \oplus \mathbb{Z}_{l_1} \oplus \cdots \oplus \mathbb{Z}_{l_n},$$

where $l_i | l_{i+1}$ and $l_i \geq 2$. If $M \neq S^5$ is spin then

$$M \cong \underbrace{M_0^5 \# \cdots \# M_0^5}_{k} \# M_{l_1}^5 \# \cdots \# M_{l_n}^5.$$

Corollary 3.4. *Let M be a simply connected, closed 4-manifold with $H_2(M;\mathbb{Z}) = \mathbb{Z}^k$. Then*

$$\pi_i(M) = \pi_i(\#^{k-1}S^2 \times S^3),\ i \geq 3.$$

We can see by Mayer-Vietoris that $H^2(\mathbb{CP}^2\#\overline{\mathbb{CP}}^2;\mathbb{Z}) = H^2(\mathbb{CP}^2;\mathbb{Z}) \oplus H^2(\overline{\mathbb{CP}}^2;\mathbb{Z})$. Recall that the first Chern class of $\mathbb{CP}^2$ is $-3c'$, where c' is the Chern class of the tautological line bundle over $\mathbb{CP}^2$. Similarly, $c_1(\overline{\mathbb{CP}}^2) = 3c''$ for the first Chern class c'' of the tautological line bundle over $\overline{\mathbb{CP}}^2$. Write $c_1(\mathbb{CP}^2\#\overline{\mathbb{CP}}^2) = mc' + nc''$ for integers $m, n \in \mathbb{Z}$. By the *Hirzebruch signature formula* $c_1^2 = 3\sigma + 2\chi$ we have

$$8 = c_1^2 = m^2(c')^2 + n^2(c'')^2 = m^2 - n^2,$$

whence $m = 3, n = 1$. Therefore, $w_2(\mathbb{CP}^2\#\overline{\mathbb{CP}}^2) = w_2(\mathbb{CP}^2) + w_2(\overline{\mathbb{CP}}^2)$ where we have used inappropriate notations at the expense of brevity. The spin manifold E that fibres over $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ (refer Lemma 4.3) is exactly what we had constructed before and identified as $S^2 \times S^3$.

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