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MAT 541 - Intermediate Algebraic Topology II

Final Examination

Problem 1 Let R denote the group ring of the fundamental group of a finite connected regular cell complex X. (Recall *regular* means the closure of each cell is an embedded ball.)

(a) Describe the boundary operator of degree minus one of a chain complex over R freely generated by the cells of X, each decorated by a homotopy class paths to the base point. Say why the boundary squared is zero.

Solution Let $\mathcal{C}$ be a regular cell decomposition of X with a chosen base point x_0. For each cell in (C) independently choose an orientation and choose labels given by homotopy classes of path from x_0 to a chosen point of the cell. Define the ith chain group to be consisting of integral linear combination of such i cells with group elements in $\pi_1(X, x_0)$. For $r = n[\gamma] \in R = \mathbb{Z}[\pi_1(X)]$ and $\sigma = m[\gamma'] \cdot (\eta, p) \in C_i(X, R)$ the R module structure is given as follows :

$$r \cdot \sigma = mn[\gamma' * \gamma] \cdot (\eta, p * \gamma)$$

and extended linearly. Thus $C_*(X, R) = \oplus_{i \geq 0} C_i(X, R)$ where each $C_i(X, R)$ is a finitely generated free R module. Observe that we have the usual

$$\partial \eta = \sum_{\text{faces of } \eta} [\eta_i, \eta] \eta_i$$

where the number $[\eta_i, \eta]$ equals ± 1 depending on whether the original orientation (of η_i) and the induced orientation (of η_i from η) agree or disagree. We define the boundary map of degree -1 :

$$\partial : C_*(X, R) \to C_*(X, R)$$

$$\partial r \cdot (\eta, p) = \sum_{\eta_i \in \partial \eta} [\eta_i, \eta] r \cdot (\eta_i, p_{\eta_i \eta} * p), \; r \in R$$

where $p_{\eta_i \eta}$ is just a path (in η) joining the the chosen point of η to the chosen point of η_i. Hence $p_{\eta_i \eta} * p$ is just a path from x_0 to the point of η_i. The boundary squared is zero because of the'diamond' trick, i.e., cancel in pairs. However we have the path labels to worry about. We observe that if $\eta \subseteq \eta_1, \eta_2 \subseteq \nu$ then any path in ν joining any two points in it has a unique homotopy class since ν is contractible. Thus $p_{\eta\eta_1} * p_{\eta_1 \nu} \sim p_{\eta\eta_2} * p_{\eta_2 \nu}$ and thus the boundary squared is zero.

(b) Use tensor products over R to describe a chain complex for computing the "twisted" homology with coefficients in any module M over R.

Solution For any R module M we form the new complex $C_*(X, R) \otimes_R M$ and extend ∂ by identity on M, i.e.

$$\partial(\sigma \otimes m) = \partial \sigma \otimes m, \; \sigma \in C_*(X, R), m \in M.$$

This gives a chain complex $(C_*(X, R) \otimes_R M, \partial)$ and its homology is called the "twisted" homology with coefficients in M.

(c) Define a R module associated to any connected covering space Y over X and use (b) to define a chain complex for computing the homology of Y.

Solution We first deal with the cases of connected normal covers, i.e., covering spaces Y of X such that $\pi_1(Y)$ is a normal subgroup of $\pi_1(X)$. Then the group of Deck transformations of Y (over X) is exactly $\pi_1(X)/\pi_1(Y)$. The universal cover $\tilde{X}$ has $\pi_1(X)$ as it transformation group. Also $\mathbb{Z}[\pi_1(X)/\pi_1(Y)]$ is a R module by the conjugation action. Since X is finite, we can subdivide X so that inverse of each cell under any covering map $\pi : Y \to X$ is disjoint homeomorphic copies of the base cell. One way to reconstruct Y is to take these base cells and use the transformation groups to move to another homeomorphic copy sitting on top of it. Tensoring with $\mathbb{Z}[\pi_1(X)/\pi_1(Y)]$ is basically reducing the coefficient group from R (the total transformation group) to the transformation group of Y. Thus elements of

$$C_i(X,R) \otimes_R \mathbb{Z}[\pi_1(X)/\pi_1(Y)]$$

can be thought of actual integral linear combinations of i cells is Y. Thus the homology of this complex should give us the usual Z homology of Y.

Examples (i) $(C_*(X,R) \otimes_R R, \partial) = (C_*(X,R), \partial)$ gives the homology of $\tilde{X}$.

(ii) $(C_*(X,R) \otimes_R \mathbb{Z}, \partial)$ gives the homology of X (R is given the trivial action on $\mathbb{Z}$).

However this doesn't go through if the covering is not normal since in that case the covering transformations of Y if $N_{\pi_1(X)}(\pi_1(Y)/\pi_1(Y)$. This is no longer a R module under the conjugation action. However instead of taking the complex $C_*(X,R)$ we could take $C_*(X,R')$ where $R' = \mathbb{Z}[N_{\pi_1(X)}(\pi_1(Y)]$. So if we define our new complex associated with Y to be

$$C_*(X,R') \otimes_{R'} \mathbb{Z}[N_{\pi_1(X)}(\pi_1(Y)/\pi_1(Y)]$$

then the term to the right of the tensor is a R' module. This is a chain complex (with the usual ∂ map) over R' but it computes the homology of Y with $\mathbb{Z}$ coefficients.

Problem 2 **(a)** Define the cells of the pair subdivision of a finite regular cell complex X, its grading and boundary operator over the group ring.

Solution The cells of a pair subdivision of X are just $(\eta, \nu), \eta \subseteq \nu$ for cells η, ν in X. The cells of X are independently given orientations. Fix a base point x_0 of X and for each cell σ choose a point σ_0 on it, say the barycenter if σ is a regular simplex. To define the chain complex over the group ring R we consider 'labelled' cells (η, ν, p) where p is a homotopy class of paths connecting x_0 and ν_0. The labelled cells (η, ν, p) have grading $|\nu| - |\eta|$. For each degree i take integral linear combinations of labelled i-cells with group elements of $\pi_1(X)$ (call it $P_i(X, R)$), i.e., we have $P_i(X, R)$ is a free R-module. This is finitely generated since X is compact. The boundary and the coboundary operator is defined first for actual cells in X as follows :

$$\partial \eta = \sum_{\eta_i \subseteq \eta} [\eta_i, \eta] \eta_i$$

where η_i's are the faces of η and the number $[\eta_i, \eta]$ equals ± 1 depending on whether the original orientation (of η_i) and the induced orientation (of η_i from η) agree or disagree. In general

$$[\eta_i, \eta] = \begin{cases} 1 & \text{if } \eta_i \text{ is a face of } \eta \text{ and the two orientations agree} \\ -1 & \text{if } \eta_i \text{ is a face of } \eta \text{ and the two orientations disagree} \\ 0 & \text{otherwise.} \end{cases}$$

$$\delta \eta = (-1)^{|\eta|-1} \sum_{\eta \subseteq \eta'} [\eta, \eta'] \eta'$$

where the sum runs over simplices containing η as a face. Now we define ∂ for labelled cells :

$$\partial(\eta, \nu, p) := (\delta \eta, \nu, p) + (-1)^{|\eta|} (\eta, \partial \nu, p_{\partial \nu}).$$

For each face ν' in $\partial \nu$, $p_{\nu'}$ is just the homotopy class of the concatenation of a path representing p (from x_0 to ν_0) with the natural path joining ν_0 to ν'_0. Notice that this doesn't change the homotopy class since ν is contractible. Finally we need to check that $\partial^2 = 0$ on labelled cells to make $(P_*(X, R), \partial)$ into a chain complex.

$$\begin{aligned} \partial^2(\eta, \nu, p) &= \partial((\delta \eta, \nu, p) + (-1)^{|\eta|} (\eta, \partial \nu, p_{\partial \nu})) \\ &= (-1)^{|\eta|+1} (\delta \eta, \partial \nu, p_{\partial \nu}) + (-1)^{|\eta|} (\delta \eta, \partial \nu, p_{\partial \nu}), \text{ since } \partial^2 = 0 = \delta^2 \text{ on cells} \\ &= 0. \end{aligned}$$

In fact the same calculation without the labels shows that the usual boundary operator $\partial : P_*(X) \to P_*(X)$ defined on cells of pair subdivisions also satisfies $\partial^2 = 0$.

Remark The R module structure of $P_*(X, R)$ is given by : $n[\gamma].(\eta, \nu, p) = n(\eta, \nu, p * \gamma)$ on the generators of $\pi_1(X, x_0)$ and then extending linearly.

(b) Define a coalgebra structure on the pair complex of X and show it is a chain mapping and that it is coassociative : first ignoring the group ring and then including the group ring.

Solution (i) *Pair complex with* $\mathbb{Z}$ *coefficients -* $P_*(X)$

We define maps (with the grading of cells as defined in $(2a)$)

$$\Delta : P_*(X) \to P_*(X) \otimes P_*(X),\ (\eta, \nu) \mapsto \sum_{\eta \subseteq \eta' \subseteq \nu} (\eta, \eta') \otimes (\eta', \nu)$$

$$\partial : P_*(X) \to P_*(X),\ (\eta, \nu) \mapsto (\delta\eta, \nu) + (-1)^{|\eta|}(\eta, \partial\nu),\ \text{degree}\, \partial = -1$$
$$\delta : P_*(X) \to P_*(X),\ (\eta, \nu) \mapsto (\partial\eta, \nu) + (-1)^{|\eta|}(\eta, \delta\nu),\ \text{degree}\, \delta = 1$$

and extend all of them by linearity. It is clear that $\partial^2 = 0$ whence $P_*(X)$ is a chain complex. By Leibnitz rule one can extend ∂ to

$$\partial_\otimes : P_*(X) \otimes P_*(X) \to P_*(X) \otimes P_*(X)$$

to make it a chain complex. We need to show that Δ is a chain map and $(\Delta \times \mathrm{I})\Delta = (\mathrm{I} \times \Delta)\Delta$, i.e., it is coassociative. As observed in $(2a)$

$$\begin{aligned}
\partial(\eta, \nu) &= (\delta\eta, \nu) + (-1)^{|\eta|}(\eta, \partial\nu) \\
&= (-1)^{|\eta|-1} \sum_{\eta \subseteq \eta' \subseteq \nu} [\eta, \eta'](\eta', \nu) + (-1)^{|\eta|} \sum_{\eta \subseteq \nu_i \subseteq \nu} [\nu_i, \nu](\eta, \nu_i). \\
\Delta\partial(\eta, \nu) &= (-1)^{|\eta|-1} \sum_{\eta \subseteq \eta' \subseteq \eta'' \subseteq \nu} [\eta, \eta'](\eta', \eta'') \otimes (\eta'', \nu) \\
&\quad + (-1)^{|\eta|} \sum_{\eta \subset \nu_i' \subseteq \nu_i \subseteq \nu} [\nu_i, \nu](\eta, \nu_i') \otimes (\nu_i', \nu_i).
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\partial_\otimes\Delta(\eta, \nu) &= \sum_{\eta \subseteq \tilde\eta \subseteq \nu} \partial(\eta, \tilde\eta) \otimes (\tilde\eta, \nu) + \sum_{\eta \subseteq \tilde\eta \subseteq \nu} (-1)^{|\eta|-|\tilde\eta|}(\eta, \tilde\eta) \otimes \partial(\tilde\eta, \nu) \\
&= \sum_{\eta \subseteq \tilde\eta \subseteq \nu} \{(\delta\eta, \tilde\eta) + (-1)^{|\eta|}(\eta, \partial\tilde\eta)\} \otimes (\tilde\eta, \nu) \\
&\quad + \sum_{\eta \subseteq \tilde\eta \subseteq \nu} (-1)^{|\eta|-|\tilde\eta|}(\eta, \tilde\eta) \otimes \{(\delta\tilde\eta, \nu) + (-1)^{|\tilde\eta|}(\tilde\eta, \partial\nu)\} \\
&= \underbrace{\sum_{\eta \subseteq \eta' \subseteq \tilde\eta \subseteq \nu} (-1)^{|\eta|-1}[\eta, \eta'](\eta', \tilde\eta) \otimes (\tilde\eta, \nu)}_{I_1} + \sum_{\eta \subseteq \tilde\eta \subseteq \nu} (-1)^{|\eta|}(\eta, \partial\tilde\eta) \otimes (\tilde\eta, \nu) \\
&\quad + \underbrace{\sum_{\eta \subseteq \tilde\eta \subseteq \nu_i \subseteq \nu} (-1)^{|\eta|}[\nu_i, \nu](\eta, \tilde\eta) \otimes (\tilde\eta, \nu_i)}_{I_2} + \sum_{\eta \subseteq \tilde\eta \subseteq \nu} (-1)^{|\eta|-|\tilde\eta|}(\eta, \tilde\eta) \otimes (\delta\tilde\eta, \nu) \\
&= \Delta\partial(\eta, \nu) + \sum_{\eta \subseteq \tilde\eta \subseteq \nu} (-1)^{|\eta|}(\eta, \partial\tilde\eta) \otimes (\tilde\eta, \nu) + \sum_{\eta \subseteq \tilde\eta \subseteq \nu} (-1)^{|\eta|-|\tilde\eta|}(\eta, \tilde\eta) \otimes (\delta\tilde\eta, \nu)
\end{aligned}$$

since $I_1 + I_2 = \Delta\partial(\eta, \nu)$. Thus it suffices to show

$$I = \sum_{\eta\subseteq\tilde{\eta}\subseteq\nu} (-1)^{|\eta|}(\eta, \partial\tilde{\eta}) \otimes (\tilde{\eta}, \nu) + \sum_{\eta\subseteq\tilde{\eta}\subseteq\nu} (-1)^{|\eta|-|\tilde{\eta}|}(\eta, \tilde{\eta}) \otimes (\delta\tilde{\eta}, \nu) = 0.$$

A simple calculation shows that

$$\begin{aligned} I &= \sum_{\eta\subseteq\tilde{\eta}_i\subseteq\tilde{\eta}\subseteq\nu} (-1)^{|\eta|}[\tilde{\eta}_i, \tilde{\eta}](\eta, \tilde{\eta}_i) \otimes (\tilde{\eta}, \nu) \\ &\quad + \sum_{\eta\subseteq\tilde{\eta}\subseteq\tilde{\eta}'\subseteq\nu} (-1)^{|\eta|-|\tilde{\eta}|}[\tilde{\eta}, \tilde{\eta}'](-1)^{|\tilde{\eta}|-1}(\eta, \tilde{\eta}) \otimes (\tilde{\eta}', \nu) \\ &= 0. \end{aligned}$$

Hence Δ is a chain map.
To verify coassociativity we check that

$$(\Delta \times \mathrm{I})\Delta(\eta, \nu) = \textstyle\sum_{\eta\subseteq\eta'\subseteq\nu} \Delta(\eta, \eta') \otimes (\eta', \nu) = \sum_{\eta\subseteq\tilde{\eta}\subseteq\eta'\subseteq\nu}(\eta, \tilde{\eta}) \otimes (\tilde{\eta}, \eta') \otimes (\eta', \nu)$$

which is exactly

$$(\mathrm{I} \times \Delta)\Delta(\eta, \nu) = \textstyle\sum_{\eta\subseteq\tilde{\eta}\subseteq\nu}(\eta, \tilde{\eta}) \otimes \Delta(\tilde{\eta}, \nu) = \sum_{\eta\subseteq\tilde{\eta}\subseteq\eta'\subseteq\nu}(\eta, \tilde{\eta}) \otimes (\tilde{\eta}, \eta') \otimes (\eta', \nu).$$

Thus $P_*(X)$ is a coassociative coalgebra.

(ii) *Pair complex with R coefficients - $P_*(X, R)$*

To simplify notation somewhat we shall denote by $p_{\eta\nu}$ the canonical path joining the barycenter of ν to that of η, where $\eta \subseteq \nu$ is an inclusion of regular cells (thought of here as simplices). Then define

$$\Delta : P_*(X, R) \to P_*(X, R) \otimes P_*(X, R),$$
$$(\eta, \nu, p) \mapsto \sum_{\eta\subseteq\eta'\subseteq\nu} (\eta, \eta', p_{\eta'\nu} * p) \otimes (\eta', \nu, p).$$

By the very definition of $p_{\eta\nu}$, if $\eta \subseteq \nu' \subseteq \nu$ then $p_{\eta\nu'} * p_{\nu'\nu} \sim p_{\eta\nu}$. We shall be implicitly using this in various equalities. We verify coassociativity first :

$$\begin{aligned} (\Delta \times \mathrm{I})\Delta(\eta, \nu, p) &= \sum_{\eta\subseteq\eta'\subseteq\nu} \Delta(\eta, \eta', p_{\eta'\nu} * p) \otimes (\eta', \nu, p) \\ &= \sum_{\eta\subseteq\tilde{\eta}\subseteq\eta'\subseteq\nu} (\eta, \tilde{\eta}, p_{\tilde{\eta}\eta'} * p_{\eta'\nu} * p) \otimes (\tilde{\eta}, \eta', p_{\eta'\nu} * p) \otimes (\eta', \nu, p) \\ &= \sum_{\eta\subseteq\tilde{\eta}\subseteq\eta'\subseteq\nu} (\eta, \tilde{\eta}, p_{\tilde{\eta}\nu} * p) \otimes (\tilde{\eta}, \eta', p_{\eta'\nu} * p) \otimes (\eta', \nu, p) \end{aligned}$$

which is exactly

$$\begin{aligned} (\mathrm{I} \times \Delta)\Delta(\eta, \nu, p) &= \sum_{\eta\subseteq\tilde{\eta}\subseteq\nu} (\eta, \tilde{\eta}, p_{\tilde{\eta}\nu} * p) \otimes \Delta(\tilde{\eta}, \nu, p) \\ &= \sum_{\eta\subseteq\tilde{\eta}\subseteq\eta'\subseteq\nu} (\eta, \tilde{\eta}, p_{\tilde{\eta}\nu} * p) \otimes (\tilde{\eta}, \eta', p_{\eta'\nu} * p) \otimes (\eta', \nu, p). \end{aligned}$$

To show that it is a coalgebra we proceed by extending $\partial : P_*(X,R) \to P_*(X,R)$ to $\partial_\otimes : P_*(X,R)^{\otimes 2} \to P_*(X,R)^{\otimes 2}$ by the (graded) Leibnitz rule. We carry out the following computations :

$$\begin{aligned}
\partial(\eta,\nu,p) &= (-1)^{|\eta|-1} \sum_{\eta\subseteq\eta'\subseteq\nu} [\eta,\eta'](\eta',\nu,p) + (-1)^{|\eta|} \sum_{\eta\subseteq\nu_i\subseteq\nu} [\nu_i,\nu](\eta,\nu_i,p_{\nu_i\nu} * p),\\
\Delta\partial(\eta,\nu,p) &= (-1)^{|\eta|-1} \sum_{\eta\subseteq\eta'\subseteq\eta''\subseteq\nu} [\eta,\eta'](\eta',\eta'',p_{\eta''\nu} * p) \otimes (\eta'',\nu,p)\\
&\quad +(-1)^{|\eta|} \sum_{\eta\subset\nu_i'\subseteq\nu_i\subseteq\nu} [\nu_i,\nu](\eta,\nu_i',p_{\nu_i'\nu_i} * p_{\nu_i\nu} * p) \otimes (\nu_i',\nu_i,p_{\nu_i\nu} * p)\\
&= (-1)^{|\eta|-1} \sum_{\eta\subseteq\eta'\subseteq\eta''\subseteq\nu} [\eta,\eta'](\eta',\eta'',p_{\eta''\nu} * p) \otimes (\eta'',\nu,p)\\
&\quad +(-1)^{|\eta|} \sum_{\eta\subset\nu_i'\subseteq\nu_i\subseteq\nu} [\nu_i,\nu](\eta,\nu_i',p_{\nu_i'\nu} * p) \otimes (\nu_i',\nu_i,p_{\nu_i\nu} * p).
\end{aligned}$$

On the other hand (skipping a steps exactly as the calculations for $P_*(X)$) we have

$$\begin{aligned}
\partial_\otimes\Delta(\eta,\nu,p) &= \underbrace{\sum_{\eta\subseteq\eta'\subseteq\tilde\eta\subseteq\nu} (-1)^{|\eta|-1}[\eta,\eta'](\eta',\tilde\eta,p_{\tilde\eta\nu} * p) \otimes (\tilde\eta,\nu,p)}_{I_1}\\
&\quad + \sum_{\eta\subseteq\tilde\eta\subseteq\nu} (-1)^{|\eta|}(\eta,\partial\tilde\eta,p_{(\partial\tilde\eta)\tilde\eta} * p) \otimes (\tilde\eta,\nu,p)\\
&\quad + \underbrace{\sum_{\eta\subseteq\tilde\eta\subseteq\nu_i\subseteq\nu} (-1)^{|\eta|}[\nu_i,\nu](\eta,\tilde\eta,p_{\tilde\eta\nu} * p) \otimes (\tilde\eta,\nu_i,p_{\nu_i\nu} * p)}_{I_2}\\
&\quad + \sum_{\eta\subseteq\tilde\eta\subseteq\nu} (-1)^{|\eta|-|\tilde\eta|}(\eta,\tilde\eta,p_{\eta(\delta\eta)} * p) \otimes (\delta\tilde\eta,\nu,p).
\end{aligned}$$

Since $I_1 + I_2 = \Delta\partial(\eta,\nu,p)$ we just need to show that

$$I = \sum_{\eta\subseteq\tilde\eta\subseteq\nu} (-1)^{|\eta|}(\eta,\partial\tilde\eta) \otimes (\tilde\eta,\nu) + \sum_{\eta\subseteq\tilde\eta\subseteq\nu} (-1)^{|\eta|-|\tilde\eta|}(\eta,\tilde\eta) \otimes (\delta\tilde\eta,\nu) = 0.$$

Again (exactly as in $(2b(ii))$) a simple calculation shows that

$$\begin{aligned}
I &= \sum_{\eta\subseteq\tilde\eta_i\subseteq\tilde\eta\subseteq\nu} (-1)^{|\eta|}[\tilde\eta_i,\tilde\eta](\eta,\tilde\eta_i,p_{\tilde\eta_i\nu} * p) \otimes (\tilde\eta,\nu,p)\\
&\quad + \sum_{\eta\subseteq\tilde\eta\subseteq\tilde\eta'\subseteq\nu} (-1)^{|\eta|-|\tilde\eta|}[\tilde\eta,\tilde\eta'](-1)^{|\tilde\eta|-1}(\eta,\tilde\eta,p_{\tilde\eta\nu} * p) \otimes (\tilde\eta',\nu,p)\\
&= 0.
\end{aligned}$$

Hence Δ is a chain map and $P_*(X,R)$ is a coassociative coalgebra.

Problem 3 Construct an associative DGA over R associated to the pair subdivision by considering the dual complex of R module maps into R and check that you get a differential graded associative algebra with differential of degree plus one.

Solution We have seen in (2) that $(P_*(X,R),\partial)$ is a coassociative coalgebra, i.e., the following diagram commutes

$$\begin{array}{ccc} P_*(X,R) & \xrightarrow{\Delta} & P_*(X,R)^{\otimes 2} \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta\times\mathrm{I}} \\ P_*(X,R)^{\otimes 2} & \xrightarrow{\mathrm{I}\times\Delta} & P_*(X,R)^{\otimes 3}. \end{array}$$

We have also noticed that ∂ is of degree -1. In $(2b)$ we had extended ∂ over $P_*(X,R)^{\otimes 2}$ by the Leibnitz rule to form a chain complex $(P_*(X,R)^{\otimes 2},\partial_\otimes)$. For any R-modules A,B we have

$$\mathrm{Hom}_R(A\otimes B,R)\cong \mathrm{Hom}_R(A,R)\otimes \mathrm{Hom}_R(B,R).$$

Dualizing Δ and combining this observation we get a map $\circ$ and this commuting square

$$\begin{array}{ccc} \mathrm{Hom}_R(P_*(X,R),R)^{\otimes 2} & \xrightarrow{\circ} & \mathrm{Hom}_R(P_*(X,R),R) \\ \uparrow{\scriptstyle\circ\times\mathrm{I}} & & \uparrow{\scriptstyle\circ} \\ \mathrm{Hom}_R(P_*(X,R),R)^{\otimes 3} & \xrightarrow{\mathrm{I}\times\circ} & \mathrm{Hom}_R(P_*(X,R),R)^{\otimes 2}. \end{array}$$

This is exactly the associativity relation, i.e., $(a\circ b)\circ c = a\circ(b\circ c)$ for $a,b,c\in \mathrm{Hom}_R(P_*(X,R),R)$. Notice that $a\circ b=(a\otimes b)\Delta$. We also observe that the dual of ∂ is δ of degree $+1$ and that $\partial^2=0$ implies $\delta^2=0$. Now

$$\delta(a\circ b)=\partial^*(a\circ b)=(a\circ b)\partial=(a\otimes b)\Delta\partial.$$

Since Δ was a chain mapping (as seen in $(2b)$) $\Delta\partial=\partial_\otimes\Delta$. Thus

$$\delta(a\circ b)=(a\otimes b)\partial_\otimes\Delta=(\delta a\otimes b)\Delta+(-1)^{|a|}(a\otimes\delta b)\Delta=\delta a\circ b+(-1)^{|a|}a\circ\delta b.$$

This proves that δ is a differential. The DGA $(\mathrm{Hom}_R(P_*(X,R),R),\delta)$ is the required associative DGA over R.

Problem 4 Let $X \hookrightarrow Y$ be an inclusion of finite connected complexes preserving the base points and inducing an isomorphism between the fundamental groups and an isomorphism of homology with coefficients in R, namely for the homology of the complex mentioned in $(1a)$.

Remarks We have seen in (1) that the homology of the chain complex as defined in $(1a)$ is just the usual $\mathbb{Z}$ homology of $\tilde{X}$, the universal cover of X. The given hypothesis states that $X \overset{\iota}{\hookrightarrow} Y$ induces (i) $\iota_* : \pi_1(X) \cong \pi_1(Y)$ and (ii) isomorphisms $H_*(\tilde{X}) \cong H_*(\tilde{Y})$. For the associated sequence $\tilde{X} \hookrightarrow \tilde{Y} \to (\tilde{Y}, \tilde{X})$ the induced long exact sequence

$$\cdots \to H_{i+1}(\tilde{Y}, \tilde{X}) \to H_i(\tilde{X}) \xrightarrow{\cong} H_i(\tilde{Y}) \to H_i(\tilde{Y}, \tilde{X}) \to \cdots$$

yields $H_*(\tilde{Y}, \tilde{X}) = 0$. Using the (relative) Hurewicz theorem in this case to the pair $(\tilde{Y}, \tilde{X})$ (ref *A. Hatcher - Algebraic Topology*, pgs. $366 - 367$)

Theorem *If a pair (X, A) is $(n-1)$-connected, $n \geq 2$, with A simply connected and non-empty, then $H_i(X, A) = 0$ for $i < n$ and $H_n(X, A) \cong \pi_n(X, A)$.*

one concludes that $(\tilde{Y}, \tilde{X})$ is ∞-connected. The long exact sequence of homotopy groups then yield

$$\cdots \to 0 \to \pi_i(\tilde{X}) \cong \pi_i(\tilde{Y}) \to 0 \to \cdots .$$

Thus $\tilde{X}$ and $\tilde{Y}$ are homotopy equivalent by Whitehead's theorem. Consequently all the homotopy groups of X and Y are isomorphic, the isomorphism induced via the inclusion $X \hookrightarrow Y$. Again by Whitehead X and Y are homotopy equivalent. The converse is easily seen to be true since homotopy equivalent spaces have homotopy equivalent universal covers (their path spaces) and hence isomorphic homology. In conclusion, the hypothesis given here is equivalent to demanding that $\iota : X \hookrightarrow Y$ be a homotopy equivalence.

(a) Show that the DGA of (3) for X and the DGA of (3) for Y are related by the fact that there is a DGA map between them inducing an isomorphism on homology.

Solution Since $\pi_1(X) \cong \pi_1(Y)$ the group rings are the same and we denote it by the common symbol R. Since cells of X are naturally cells of Y, there is a natural inclusion map of pair complexes (also denoted ι by abuse of notation) and the following commutative diagram

$$\begin{array}{ccc} P_*(X,R) & \xrightarrow{\iota} & P_*(Y,R) \\ \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\ P_{*-1}(X,R) & \xrightarrow{\iota} & P_{*-1}(Y,R). \end{array}$$

Dualizing this we get an induced commutative diagram on the R-dual complexes

$$\begin{array}{ccc} \mathrm{Hom}_R(P_*(Y,R),R) & \xrightarrow{\iota^*} & \mathrm{Hom}_R(P_*(X,R),R) \\ \downarrow{\scriptstyle \delta=\partial^*} & & \downarrow{\scriptstyle \delta} \\ \mathrm{Hom}_R(P_{*+1}(Y,R),R) & \xrightarrow{\iota^*} & \mathrm{Hom}_R(P_{*+1}(X,R),R). \end{array}$$

Recall that $P_*(Y,R)$ is a coassociative coalgebra with a map $\Delta : P_*(Y,R) \to P_*(Y,R)^{\otimes 2}$. There is a naturally induced square

$$\begin{array}{ccc} P_*(Y,R) & \xrightarrow{\Delta} & P_*(Y,R) \otimes P_*(Y,R) \\ \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota\times\iota} \\ P_*(X,R) & \xrightarrow{\Delta|_X} & P_*(X,R) \otimes P_*(X,R). \end{array}$$

Let $\alpha, \beta : P_*(Y,R) \to R$ be R-module homomorphisms. Then $\alpha \cdot \beta := (\alpha \otimes \beta)\Delta$. This means

$$\iota^*(\alpha \cdot \beta) = \iota^*((\alpha \otimes \beta)\Delta) = (\alpha \otimes \beta)\Delta\iota$$

which equals, by the commutativity of the previous square, to

$$(\alpha \otimes \beta)(\iota \times \iota)\Delta|_X = (\iota^*\alpha \otimes \iota^*\beta)\Delta|_X = \iota^*\alpha \cdot \iota^*\beta.$$

Thus ι^* is a DGA map. If $\alpha \in \mathrm{Hom}_R(P_*(X,R),R)$ is a boundary, i.e., $\alpha = \delta_Y\beta$ then

$$\iota^*(\alpha) = \iota^*(\delta_Y\beta) = \delta_X(\iota^*\beta)$$

is also a boundary. Similarly, if $\delta_Y\alpha = 0$ then $\delta_X(\iota^*\alpha) = 0$. To simplify notations we set $\tilde{H}_*(X) = H_*(\mathrm{Hom}_R(P_*(Y,R),R))$. Thus we have a well-defined induced map on homology :

$$[\iota^*] : \tilde{H}_*(Y) \to \tilde{H}_*(X).$$

This map is surjective because any homomorphism $\phi : P_*(X,R) \to R$ can be extended to $P_*(Y,R)$ by setting it to be zero on pairs of cells not contained in X. We need only check injectivity. $P_*(X,R)$ can be thought of as the usual homology of $\tilde{X}$ with a subdivision of the given cell structure. Since the homology groups of $\tilde{X}$ and $\tilde{Y}$ are isomorphic, we conclude that $\iota : P_*(X,R) \to P_*(Y,R)$ induces an isomorphism on homology. Dualizing this we have that $[\iota^*]$ is an isomorphism.

(b) Use this to show that homotopy equivalent finite complexes have quasi isomorphic DGA's where quasi isomorphic means the equivalence relation generated by the relation of $(4a)$.

(**Hint** : *Enlarge Y to Y' a neighborhood of Y in a large euclidean space which deformation retracts to Y. Then approximate the homotopy equivalence between X and Y by a cellular embedding of X into Y'. Now apply* (4*a*) *twice.*)

Solution Since X and Y are compact Hausdorff spaces, a cellular embedding of X into another space is just a continuous injective map. Let

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y$$

be a homotopy equivalence, i.e., $f \circ g \sim \mathrm{Id}_Y$ and $g \circ f \sim \mathrm{Id}_X$. One can consider the mapping cylinders M_f and M_g respectively. It is clear that M_f (resp. M_g) deformation retracts to Y (resp. X) and $Y \hookrightarrow M_f$ (resp. $X \hookrightarrow M_g$) is a cellular embedding. On the other hand the map $f : X \to M_f$ (resp. $g : Y \to M_g$) can be homotoped to make it a cellular embedding $\tilde{f} : X \hookrightarrow M_f$ (resp. $\tilde{g} : Y \hookrightarrow M_g$). Observe that $X \sim M_f$ (resp. $Y \sim M_g$). By $(4a)$ we have a DGA map

$$\tilde{f}^* : \mathrm{Hom}_R(P_*(M_f, R), R) \to \mathrm{Hom}_R(P_*(X, R), R)$$

which induces an isomorphism on homology. The restriction

$$\tilde{f}^*|_Y : \mathrm{Hom}_R(P_*(Y, R), R) \to \mathrm{Hom}_R(P_*(X, R), R)$$

is also a DGA map and induces an isomorphism on homology since $\tilde{H}_*(X) \cong \tilde{H}_*(M_f) \cong \tilde{H}_*(Y)$. Similarly we have

$$\tilde{g}^*|_X : \mathrm{Hom}_R(P_*(X, R), R) \to \mathrm{Hom}_R(P_*(Y, R), R)$$

inducing isomorphism on homology, which proves quasi isomorphism.

We discuss this a bit further. First we need to label a few of the maps :

$$X \xrightarrow{f} Y \underset{r_Y}{\overset{\iota_Y}{\rightleftarrows}} M_f \qquad , \qquad Y \xrightarrow{g} X \underset{r_X}{\overset{\iota_X}{\rightleftarrows}} M_g$$

where r_X, r_Y are the respective retractions. We note that $r_Y \iota_Y \sim \mathrm{Id}_Y, r_X \iota_X \sim \mathrm{Id}_X, \tilde{f} \sim \iota_Y f$ and $\tilde{g} \sim \iota_X g$. Assuming the notion of homotopy of DGA's, since

$$\tilde{f}^*|_Y \sim (\iota_Y f)^*|_Y = (r_Y \iota_Y f)^* \sim f^*$$

and similarly $\tilde{g}^*|_X \sim g^*$, we conclude that

$$\tilde{g}^*|_X \tilde{f}^*|_Y : \mathrm{Hom}_R(P_*(Y, R), R) \to \mathrm{Hom}_R(P_*(Y, R), R)$$

is homotopic to $g^* f^* \sim (f \circ g)^* \sim \mathrm{Id}$. Similarly one can prove that

$$\tilde{f}^*|_Y \tilde{g}^*|_X \sim \mathrm{Id} : \mathrm{Hom}_R(P_*(X, R), R) \to \mathrm{Hom}_R(P_*(X, R), R).$$

Thus both these maps induce the corresponding identity map at the level of homology. This completes the proof.

Problem 5 **(a)** Define the notion of a free associative graded differential algebra with unit (T, d) over a field k with a triangular differential of degree plus one.

Solution A *differential graded algebra* (or simply DGA) A is a graded algebra equipped with a degree $+1$ map $d : A \to A$ that satisfies :
(i) $d^2 = 0$
(ii) $d(a \cdot b) = (da) \cdot b + (-1)^{|a|} a \cdot (db)$.
The first condition says that d gives A the structure of a chain complex. Condition (ii) says that the differential d respects the graded Leibnitz rule.
A *free associative DGA* with unit (T, d) over k having a 'triangular' d is a DGA which is associative and freely generated as an algebra over k by $\{x_\alpha | \alpha \in J\}$ such that

$$dx_\alpha = \sum_{\text{finite};|x_{\beta_j}|<|x_\alpha|} c_I x_{\beta_1}^{i_1} \cdots x_{\beta_n}^{i_n}.$$

(b) Define the notion of a free DGA resolution of an arbitrary DGA A. State an existence theorem for free resolutions of DGA's. Sketch a proof.

Solution A *free DGA resolution* of (A, d') is a free DGA (T, d) with a triangular differential d and a DGA map $\phi : T \to A$ which is a chain map of algebras, i.e.,

$$\begin{array}{ccc} T & \xrightarrow{\phi} & A \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d'} \\ T & \xrightarrow{\phi} & A \end{array}$$

such that ϕ induces an isomorphism on homology.

Existence Theorem There exists free DGA resolutions for any DGA A.

Very sketchy proof Let us assume that $H_0(A) = k$, the ground field and that $H_*(A)$ is finite dimensional over k. In other words A is graded by non-negative integers and A contains the base field. Let

$$\{1\} \bigcup_{i=1}^{\infty} \{a(i)_\alpha | \alpha \in J_i, |J_i| < \infty\}$$

be a list of graded ($a(i)_\alpha$ is of degree i) representatives of $H_*(A)$. Choose symbols

$$\bigcup_{i=1}^{\infty} \{x(i)_\alpha | \alpha \in J_i, |J_i| < \infty\}$$

and define $\phi : x(i)_\alpha \to a(i)_\alpha$ and extend it to the tensor algebra $T(\{x(i)_\alpha | \alpha \in J_i, i \in \mathbb{N}\})$ and set $dx(i)_\alpha = 0$ for all i, α. Then ϕ induces a (graded) surjection on the homology groups of T to A. Since the representatives of $H_*(A)$ may have relations among themselves, say $a(1)_1 a(1)2 - a(1)_3 a(1)_4 = d_A(w)$ then define $\phi : y_w \to w$ and

$d(y_w) := x(1)_1 x(1)_2 - x(1)_3 x(1)_4$. We do this each relation. Thus taking the tensor algebra T generated by the x's and the new y's we have the extended DGA map $\phi : (T, d) \to (A, d_A)$. This process might have created relations in (T, d). Since we want a free (associative) DGA as our prototype for T we have to introduce new generators to kill these relations to make it free. In general, we may have to keep doing this process and the limiting algebra should be a free DGA resolution of A.

(c) Describe a best possible uniqueness property of free resolutions of DGA's using an undefined term called homotopy between DGA maps. State a uniqueness theorem about resolutions which you think may be true. Tell what property of constructing homotopies should be true for this undefined notion in order that one may prove this theorem by induction over the triangular ordering of generators.

Solution Assuming the existence of the notion of homtopy between DGA maps, we hope to have only one free resolution for a given DGA A upto homotopy equivalence. In other words, if (T, d) and (T', d') are two free resolutions of A then there should be DGA maps $f : T \to T'$ and $g : T' \to T$ such that $f \circ g \sim \mathrm{Id}_{T'} : T' \to T'$ and $g \circ f \sim \mathrm{Id}_T : T \to T$. This should imply(?), in particular, that f (resp. g) induces isomorphism on the homology of T and T'. If f_*, g_* denote the induced maps at the homology level then we should have

$$\begin{aligned} f_* \circ g_* &= \mathrm{Id}_{H_*(T')}, \\ g_* \circ f_* &= \mathrm{Id}_{H_*(T)}. \end{aligned}$$

We also expect the basic properties of usual homotopy of maps to hold for this notion of homotopy -

(i) homotopy equivalence should be an equivalence relation and

(ii) the resulting maps got by post/pre composing two homotopic DGA maps by another DGA map are also homotopic.

So in conclusion we expect :

Uniqueness Theorem Any free resolution of a DGA is unique upto homotopy equivalence.

Problem 6 An augmentation of a DGA (T, d) is by definition a DGA map $\varepsilon : (T, d) \to (k, 0)$. Let T' be the kernel of the augmentation map ε which is a maximal ideal and DGA subalgebra. Let $L = T'/T'^2$. Lift L to a free generating set of T'. So we can write

$$T' = \bigoplus_{i \geq 1} L^{\otimes i}.$$

Show that d is completely determined by maps $d_1 : L \to L$, $d_2 : L \to L^{\otimes 2}$, $d_3 : L \to L^{\otimes 3}$, etc. Deduce from $d^2 = 0$ equations for these components, in particular the first three equations.

Solution (*We shall assume throughout that the base field k is of characteristic not equal to* 2) There is a short exact sequence of k-modules

$$0 \to T' \to T \xrightarrow{\varepsilon} k \to 0.$$

One has the obvious inclusion $\iota : k \hookrightarrow T$ which satisfies $\varepsilon\iota = \mathrm{Id}_k : k \to k$. Thus the sequence splits : $T = k \oplus T'$. Set $L = T'/T'^2$ and lift L to a free generating set of T'. Then

$$T = k \oplus L \oplus L^{\otimes 2} \oplus L^{\otimes 3} \oplus \cdots$$

and the differential $d : T \to T$ can be thought of as a map from L to T and extended to T via the Leibnitz rule. Post composing with the projection maps $\pi_i : T \to L^{\otimes i}$ the differential d splits into maps $d_i := L \to L^{\otimes i}$. Each of these maps can be extended to T by Leibnitz rule and let d_i denote the extension also. For example, $d_j : L^{\otimes i} \to L^{\otimes i+j-1}$ by applying d_j to each coordinate of a i-tuple and summing over all such possibilities with appropriate signs. Consequently

$$d = d_0 + d_1 + d_2 + \cdots$$

is determined by the maps d_i's. Observe that d is of degree 1 and so $[d, d] = 2d^2$ is also a differential. The equation $d^2 = 0$ is equivalent to $d^2|_L = 0$. This breaks up into

$$\pi_i \circ d^2 : L \xrightarrow{0} L^{\otimes i}.$$

We list below the corresponding equations for the first few components :

$$\begin{aligned}
d_1^2 &= 0,\ i = 1\\
d_1 d_2 + d_2 d_1 &= 0,\ i = 2\\
d_2^2 + d_1 d_3 + d_3 d_1 &= 0,\ i = 3\\
d_1 d_4 + d_4 d_1 + d_2 d_3 + d_3 d_2 &= 0,\ i = 4\\
d_3^2 + d_1 d_5 + d_5 d_1 + d_2 d_4 + d_4 d_2 &= 0,\ i = 5\\
d_1 d_6 + d_6 d_1 + d_2 d_5 + d_5 d_2 + d_3 d_4 + d_4 d_3 &= 0,\ i = 6.
\end{aligned}$$

Problem 7 Let ch denote the homology of the complex (L, d_1). Show that ch is defined and has the structure of a coassociative coalgebra using the first, second and third equations in (6).

Solution The first equation reads $d_1^2 = 0$ where $d_1 : L \to L$. Thus the homology of the complex ch is well defined. We shall denote by $(ch =)H_*(L)$ the homology groups $(\mathrm{ker} d_1/\mathrm{Im} d_1)_*$. The second equation translates into the commutative diagram :

$$\begin{array}{ccc} L & \xrightarrow{d_2} & L \otimes L \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle d_1} \\ L & \xrightarrow{d_2} & L \otimes L. \end{array}$$

Thus d_2 is a chain map between chain complexes and induces a map at the homology level which we denote by Δ. Observe that

$$\Delta : H_*(L) \to H_*(L) \otimes H_*(L)$$

because if $d_1(\alpha) = 0$ then write $d_2(\alpha) = \sum \beta_i \otimes \beta_i'$. Then

$$0 = -d_2 d_1(\alpha) = d_1(\sum \beta_i \otimes \beta_i') = \sum d_1(\beta_i) \otimes \beta_i' + \sum (-1)^{|\beta_i|} \beta_i \otimes d_1(\beta_i').$$

Since L is free with the trivial algebra action (i.e., $\alpha \circ \beta = 0$ for $\alpha, \beta \in L$) this implies that the terms in the RHS of the equation above are either zero or must cancel in pairs. Consequently each β_i (resp. β_i') is either a boundary or closed. Thus the target space of Δ is $H_*(L)^{\otimes 2}$ and $([\alpha]) = [d_2(\alpha)]$. Towards proving that $H_*(L)$ is a coassociative coalgebra it suffices (and is necessary) to prove that the following diagram commutes :

$$\begin{array}{ccc} H_*(L) & \xrightarrow{\Delta} & H_*(L) \otimes H_*(L) \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta \times \mathrm{I}} \\ H_*(L) & \xrightarrow{\mathrm{I} \times \Delta} & H_*(L)^{\otimes 3}. \end{array}$$

Let $[\alpha] \in H_*(L)$. Then combining $d_2^2 = -d_1 d_3 - d_3 d_1$ and $d_1(\alpha) = 0$ we get

$$((\Delta \times \mathrm{I})\Delta - (\mathrm{I} \times \Delta)\Delta)[\alpha] = [d_2^2(\alpha)] = [-d_1 d_3(\alpha)] = [0].$$

Thus $H_*(L)$ is indeed a coassociative coalgebra.

Problem 8 If (T, d) is a free resolution of the DGA A then ch of (7) is called the *cohomotopy coalgebra* of the DGA A. Show ch is a quasi-isomorphism invariant of A using (5) and (6). Deduce a corollary about homotopy invariants of finite cell complexes using (1), (2), (3) and (4).

Solution We shall make use of :

Lemma Let $(\tilde{T}, d, \varepsilon)$ and $(\tilde{T}', d', \varepsilon')$ be two free resolutions of a DGA A with its respective augmentations. Let ch and ch' denote the respective cohomotopy coalgebra. Then $ch \cong ch'$.

Proof Since any two free resolutions are homotopic (via a homotopy of DGA maps) we have

$$\tilde{T} \underset{g}{\overset{f}{\rightleftarrows}} \tilde{T}'$$

such that $f \circ g \sim \mathrm{Id}_{\tilde{T}'}$ and $g \circ f \sim \mathrm{Id}_{\tilde{T}}$. The fact that any DGA map between DGA's with unit is identity on the scalar field and $\tilde{T} = k \oplus T$, $\tilde{T}' = k \oplus T'$ imply that $f|_T : T \to T'$ and $g|_{T'} : T' \to T$. This induces $\tilde{f} : L \to L'$ and $\tilde{g} : L' \to L$. The maps f, g commute with the differentials and hence $g \circ f : T \to T$ commutes with d. Similarly $f \circ g : T' \to T'$ commutes with d'. This forces the following commuting squares :

$$\begin{array}{ccccc} L & \xrightarrow{\tilde{f}} & L' & \xrightarrow{\tilde{g}} & L \\ \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_1'} & & \downarrow{\scriptstyle d_1} \\ L & \xrightarrow{\tilde{f}} & L' & \xrightarrow{\tilde{g}} & L. \end{array}$$

Since $g \circ f : T \to T$ is homotopic to identity so is $\tilde{g} \circ \tilde{f}$ and thus $ch \xrightarrow{\tilde{g}_*} ch' \xrightarrow{\tilde{f}_*} ch$ is just the identity map. Similarly $\tilde{f}_* \circ \tilde{g}_* = \mathrm{Id} : ch' \to ch'$. Therefore $ch \cong ch'$. □

Observation If $\tilde{T} = \tilde{T}'$ then actually $L \cong L'$. Thus we have an isomorphism at the chain level. This trivially implies isomorphism on homology.

With this lemma in hand we proceed in proving quasi-isomorphism invariance. Let $(T, d, \varepsilon), (T', d', \varepsilon')$ be free resolutions of DGA's A, B respectively. If A and B are quasi-isomorphic then there exists DGA maps $f : A \to B$ and $g : B \to A$ inducing isomorphisms on homology. Thus post composing $\pi : T \to A$ with $f : A \to B$ one can think of B having two free resolutions - (T, d, ε) and (T', d', ε'). By lemma 2, $ch \cong ch'$. This completes the proof that ch is a quasi-isomorphism invariant.

Remark To relate this to DGA's associated with finite regular cell complexes we need to extend this theory for DGA's over the group ring R since the associative DGA (as defined in (3)) is only defined over R.

Problem $8'$ Carry through the discussion of (5), (6) and (7) for the pair complex DGA over R, the group ring. Obtain candidates for the homology of non-simply connected spaces.

References and further comments

Problem 1 Discussed with Luoy; used class notes.

Problem 2 Discussed with Luoy; used class notes and homeworks.

Problem 3 None.

Problem 4 Discussed with Chris (the mapping cylinder part); results used are referred to *A. Hatcher - Algebraic Topology*.

Problem 5 Used class notes and referred *P. A. Griffiths, J. W. Morgan - Rational Homotopy Theory and Differential Forms*.

Problem 6 Used class notes.

Problem 7 Used class notes.

Problem 8 None.

Problem $8'$ None; had some ideas but half-baked.