What is the bar construction?

1 Introduction

The bar construction appears under different names and in seemingly different contexts. For any topological group G one can associate a space BG, unique up to homotopy equivalence, which classifies principal G-bundles. It is called the *classifying space* of G and was originally constructed by Milnor [2]. However, this is a particular case of constructing a *classifying space* of a category enriched over sets or topological spaces. One simply thinks of G as a category with one object and elements of G as morphisms. And G can be replaced by H-spaces or based loop spaces. Even more generally, one can talk about the *classifying space* of (suitable) categories. On the algebraic side, the *bar construction* assigns a coalgebra to an algebra. This construction is important in its own right and can be carried out in an appropriate setting of properads as well. We'll see in the course of this discussion that the basic conceptual construction is combinatorial in nature and involves basic properties of standard simplices. In fact, given a simplicial set there is a notion of *geometric realization* which is perhaps the primordial example of a bar construction. This will be the essential tool in our understanding and in the process, we'll be analyzing the combinatorics of simplices.

There is an *adjoint* construction to this, often referred to as the *cobar* construction. It's denoted by $\Omega(\cdot)$ and resonates with the fact that in the specific example of topological spaces, Ω maps spaces to H-spaces or, at the chain level, maps differential graded coalgebras $(C_*(X), \partial)$ to $C_*(\Omega X)$ which is a differential graded algebra. However, we'll focus on the bar construction in this talk and as may be inferred, we use $B(\cdot)$ to denote the bar construction following the notation used to denote classifying spaces.

Before we delve into the details, we would like to briefly mention the significance. The classifying space is evidently important for classifying bundles. Certain cohomology theories, for example Hochschild homology, arise as a bar construction. Any topological space can be realized as the geometric realization of a simplicial set. In fact, any smooth manifold can be realized, up to homeomorphism, by the bar construction of a topological category associated to a generic Morse function on M.

2 Simplicial Objects & Combinatorics of Simplices

A *simplicial set* is a contravariant functor $\mathcal{F} : \Delta_{\mathbf{simp}} \to \mathbf{Set}$ from the *simplicial category* $\Delta_{\mathbf{simp}}$ to the category **Set** of sets. Similarly, a *Delta set* is a contravariant functor $\mathcal{F} : \Delta \to \mathbf{Set}$ from the *delta category* to **Set**. More specifically, a simplicial set is a collection of finite sets $\mathcal{K} := \{K_n\}_{n\geq 0}$ equipped with maps

$$d_i : K_n \longrightarrow K_{n-1},\ i = 0, \ldots, n$$

$$s_i : K_n \longrightarrow K_{n+1},\ i = 0, \ldots, n.$$

These maps satisfy the relations

$$d_i d_j = d_{j-1} d_i,\ i < j \tag{2.1}$$

$$s_i s_j = s_{j+1} s_i,\ i \leq j \tag{2.2}$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & i < j \\ \mathrm{Id} & i = j, j+1 \\ s_j d_{i-1} & i > j+1. \end{cases} \tag{2.3}$$

A *Delta set* is a collection of sets $\mathcal{K} := \{K_n\}_{n\geq 1}$ such that we have operators d_i as above satisfying (2.1). We do not require the degeneracy maps s_i's. Note that the categories $\Delta_{\mathbf{simp}}$ and Δ have the same objects but there are more morphisms in the simplicial category and there are more operators, viz., the degeneracy maps, in $\Delta_{\mathbf{simp}}$.

How does one interpret these equations? For that one has to see what the analogous maps are for simplices. Consider the standard n-simplex

$$\Delta_n := \{(t_0, t_1, \ldots, t_n, t_{n+1}) \in \mathbb{R}^{n+2} \,|\, 0 = t_0 \leq t_1 \leq \cdots \leq t_n \leq t_{n+1} = 1\}.$$

The face and degeneracy maps

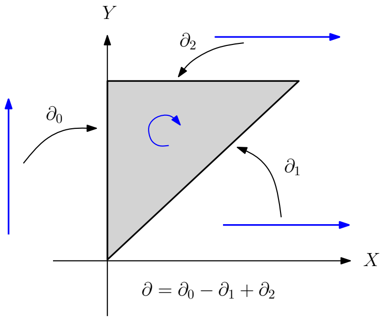
$$\partial_i : \Delta_{n-1} \longrightarrow \Delta_n, \;\; s_i : \Delta_{n+1} \longrightarrow \Delta_n$$

for $i = 0, \ldots, n$ are given by

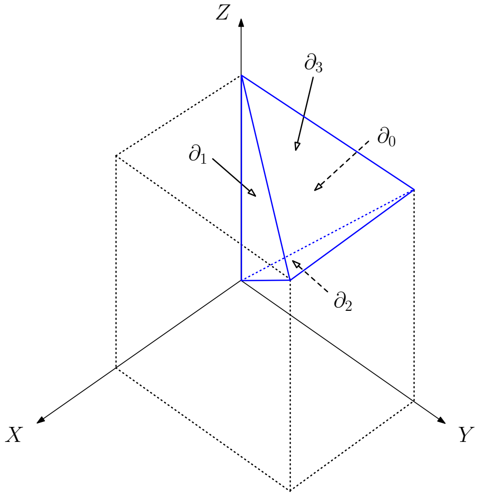
$$\partial_i(t_0, t_1, \ldots, t_n) = (t_0, t_1, \ldots, t_i, t_i, \ldots, t_n) \tag{2.4}$$

$$s_i(t_0, t_1, \ldots, t_{n+2}) = (t_0, t_1, \ldots, t_i, t_{i+2}, \ldots, t_{n+2}). \tag{2.5}$$

For example, the face maps ∂_i's for a standard 2-simplex is given below while the degeneracy maps s_0, s_1 are projection to the x and y-axis respectively.



For the 3-simplex one has the following picture :



Evidently these maps satisfy relations adjoint to the relations above, viz.,

$$\partial_i \partial_j = \partial_j \partial_{i-1}, \;\; i > j \tag{2.6}$$

$$s_i s_j = s_j s_{i+1}, \;\; i \geq j \tag{2.7}$$

$$s_i \partial_j = \begin{cases} \partial_j s_{i-1} & i > j \\ \text{Id} & i = j, j-1 \\ \partial_{j-1} s_i & i < j-1. \end{cases} \tag{2.8}$$

The significance of this construction arises from the idea of *geometric realization* of a simplicial set. Given $\mathcal{K}$ with the discrete topology form the topological sum

$$\overline{\mathcal{K}} := (K_0 \times \Delta_0) + (K_1 \times \Delta_1) + \cdots + (K_n \times \Delta_n) + \cdots$$

We generate the equivalence relation

$$\begin{array}{rcl} (d_i k, x) & \sim & (k, \partial_i x) \\ (s_i k, x') & \sim & (k, s_i x') \end{array}$$

where $k \in K_n, x \in \Delta_{n-1}, x' \in \Delta_{n+1}$. The identification space $|\mathcal{K}| := \overline{\mathcal{K}}/\sim$ will be called the *geometric realization* of $\mathcal{K}$. This a CW-complex with exactly one n-cell for each non-degenerate n-simplex in $\mathcal{K}$. Conversely, to any CW-complex X one can associate a simplicial set $\mathcal{S}(X) = \{\mathrm{Hom}_{\mathbf{Top}}(\Delta_n, X)\}_{n\geq 0}$. This is precisely the singular complex of X, originally due to Eilenberg. For all of this and more, Milnor [3] is, as usual, a great read!

One can study simplicial objects or Delta objects in other categories as well, i.e., contravariant functors $\mathcal{F} : \Delta_{\mathbf{simp}} \to \mathscr{C}$ or $\mathcal{F} : \Delta \to \mathscr{C}$. In particular, for the *geometric realization* procedure to go through we need a canonical covariant functor $\mathscr{I} : \Delta_{\mathbf{simp}} \to \mathscr{C}$. We also require the category to admit operations analogous to disjoint union $(+)$, cartesian product $(\times)$ and quotients for $\mathscr{C} = \mathbf{Set}, \mathbf{Top}$. In technical terms we need $\mathscr{C}$ to be a *monoidal abelian category.* We shall focus our attention $\mathscr{C} = \mathbf{Ch}_{\Bbbk}$, the category of chain complexes over a ring $\Bbbk$. In this case $\mathscr{I}$ assigns to a simplex Δ its chain complex $C_*(\Delta_n)$.

Definition 2.1. A simplicial object $\mathscr{K}$ in $\mathbf{Ch}_{\Bbbk}$ is a collection of chain complexes $\mathscr{K} := \{(\mathscr{K}(n)_*, \partial)\}_{n\geq 0}$ equipped with maps d_i, s_i's satisfying (2.1),(2.2) and (2.3) and these operators commute with the internal boundary map ∂.

A Delta object is a collection $\mathscr{K} := \{(\mathscr{K}(n)_*, \partial)\}_{n\geq 0}$ of chain complexes equipped with maps s_i's such that these commute with ∂ and (2.1) holds.

The maps d_i, s_i have degree zero. Given a simplicial (or Delta) object $\mathscr{K}$ we define a new chain complex

$$\overline{\mathscr{K}} := \oplus_{i=0}^{\infty}\big(\mathscr{K}(n)_* \otimes_{\Bbbk} C_*(\Delta_n)\big)$$

The differential $\overline{D}$ for $\overline{\mathscr{K}}$ is given by the usual Leibnitz rule, i.e., for $\alpha \otimes \beta \in \mathscr{K}(n)_* \otimes_{\Bbbk} C_*(\Delta_n)$ define

$$\overline{D}(\alpha \otimes \beta) := (\partial\alpha \otimes \beta) + (-1)^{|\alpha|}(\alpha \otimes \partial\beta).$$

Let $\mathscr{L}$ be the $\Bbbk$-module generated by the elements

$$(d_i\alpha) \otimes \beta - \alpha \otimes (\partial_i\beta),\ (s_i\alpha) \otimes \beta' - \alpha \otimes (s_i\beta'),$$

where $\alpha \in \mathscr{K}(n)_*, \beta \in C_*(\Delta_{n-1}), \beta' \in C_*(\Delta_{n+1})$. Let $\mathscr{L}_\Delta \subset \mathscr{L}$ be the $\Bbbk$-submodule generated by $(d_i\alpha) \otimes \beta - \alpha \otimes (\partial_i\beta)$. Notice that

$$\begin{array}{rcl} \overline{D}\big((d_i\alpha) \otimes \beta - \alpha \otimes (\partial_i\beta)\big) & = & \big(d_i(\partial\alpha) \otimes \beta - \partial\alpha \otimes \partial_i\beta\big) + (-1)^{|\alpha|}\big(d_i\alpha \otimes \partial\beta - \alpha \otimes \partial_i(\partial\beta)\big) \in \mathscr{L}_\Delta \\ \overline{D}\big((s_i\alpha) \otimes \beta' - \alpha \otimes (s_i\beta')\big) & = & \big(s_i(\partial\alpha) \otimes \beta' - \partial\alpha \otimes s_i\beta'\big) + (-1)^{|\alpha|}\big(s_i\alpha \otimes \partial\beta' - \alpha \otimes s_i(\partial\beta')\big) \in \mathscr{L}. \end{array}$$

Therefore, both $\mathscr{L}_\Delta$ and $\mathscr{L}$ are closed under $\overline{D}$ whence $(\overline{\mathscr{K}}/\mathscr{L}, \overline{D})$ and $(\overline{\mathscr{K}}/\mathscr{L}_\Delta, \overline{D})$ are both chain complexes.

Definition 2.2. (Bar construction)
Given a simplicial object $\mathscr{K}$ in $\mathbf{Ch}_{\Bbbk}$ we call $(\overline{\mathscr{K}}/\mathscr{L}, \overline{D})$ the *realization* (or bar construction) of $\mathscr{K}$ and denote it by $B\mathscr{K}$. If $\mathscr{K}$ is a Delta object in $\mathbf{Ch}_{\Bbbk}$ then we call $B\mathscr{K} := (\overline{\mathscr{K}}/\mathscr{L}_\Delta, \overline{D})$ the bar construction of $\mathscr{K}$.

3 Examples of the Bar Construction

We shall analyze various examples of the bar construction to understand better the underlying combinatorial nature of things.

Example 3.1. (Classifying spaces)

Let G be a topological group which is a CW complex. One can associate a simplicial object taking values in **Top** by assigning $K_n := G^n$ and defining maps

$$d_i : K_n \to K_{n-1}, \;\; d_i(g_1, \dots, g_n) := \begin{cases} (g_2, \dots, g_n) & \text{if } i = 0 \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n) & \text{if } 1 \leq 1 \leq n-1 \\ (g_1, \dots, g_{n-1}) & \text{if } i = n. \end{cases}$$

$$s_i : K_n \to K_{n+1}, \;\; s_i(g_1, \dots, g_n) := (g_1, \dots, g_i, e, g_{i+1}, \dots, g_n).$$

The bar construction of $\{K_n\}_{n \geq 0}$ is denoted by BG. However, Segal [4] provided a slightly different approach where we *ignore* the maps s_i's and showed that for reasonable groups[1] the two constructions are the same. The non-requirement of maps s_i's buys us the following nice fact :

Theorem 3.2. (Segal)

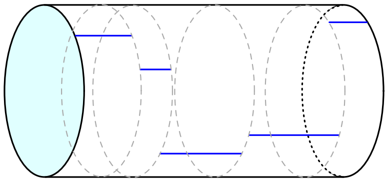
If A is an abelian topological group[1] *then a model for its classifying space BA can be chosen so that BA is an abelian group.*

In fact, he shows that one can replace

$$\Big\{ \coprod_{i \geq 0}^{n} (G^i \times \Delta_i) \Big\} / \sim$$

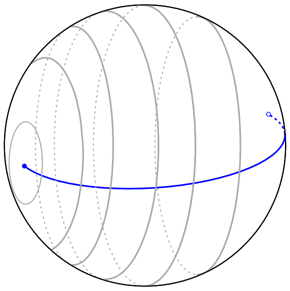
with $\mathcal{S}f_n(G)/G$, where $\mathcal{S}f_n(G)$ is the space of step functions on $[0,1]$ with at most $(n+1)$-steps and taking values in G. G acts naturally on this space. Of course, putting a natural topology on $\mathcal{S}f_n(G)$ is a bit tricky and therefore, to simplify matters, we assume that G is metrizable.

A picture of BS^1 via Segal's approach is as follows : *consider all step functions on* $[0,1]$ *taking values in* S^1 *up to rotation as appropriate markings on a cylinder up to rotation.*



This is clearly an abelian group! In fact, for any group G that admits a metric one can put a natural metric on the space $\mathcal{S}f(G)$ of all step functions. This space has a natural action of G which is free. Moreover, it is not hard to see that this space is contractible and therefore provides a model for EG.

Let's move on to the picture of BS^1 according to Milnor. Let $(BS^1)_n$ denote the nth stage in the process of building the classifying space. We are assuming that $(BS^1)_0 = \{1\}$. A picture of $(BS^1)_1$ is given below, where we have to *collapse* the blue line to a point (due to s_0).



How does one see this? Take $S^1 \times [0,1]$ and identity $[0,1]$ with $\Delta_1 = \{(0,t,1) \,|\, t \in [0,1]\}$. Then

$$(e^{i\theta}, (0,0,1)) = (e^{i\theta}, \partial_0(0,1)) \sim (1, (0,1)) \sim (e^{i\phi}, \partial_1(0,1)) = (e^{i\phi}, (0,1,1))$$

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[1] A must be a local neighbourhood retract.

collapses the two circles over the boundary of $[0,1]$. Moreover,

$$(1,(0,t,1)) = (s_0(1),(0,t,1)) \sim (1,(0,1))$$

implies that we must collapse half of the equator (the blue line in the picture). The picture is analogous to that of reduced suspension.

What about the second stage of BS^1? We first have $(S^1 \times S^1) \times \Delta_2$ which has the following identifications on the boundary :

$$\begin{array}{rcl} ((e^{i\theta_1}, e^{i\theta_2}),(0,0,t,1)) & \sim & (e^{i\theta_2},(0,t,1)) \\ ((e^{i\theta_1}, e^{i\theta_2}),(0,t,1,1)) & \sim & (e^{i\theta_1},(0,t,1)) \\ ((e^{i\theta_1}, e^{i\theta_2}),(0,t,t,1)) & \sim & (e^{i(\theta_1+\theta_2)},(0,t,1)). \end{array}$$

In pictures, this means that the torus collapses to either the meridian circle or the longitudinal circle or the diagonal circle on the three boundary intervals. Of course, further boundary identifications impose that the torus collapses to a point on the three vertices of Δ_2. There are further collapses due to the maps s_0, s_1. It is a good exercise to see why $(BS^1)_2$ is homotopy equivalent to $\mathbb{CP}^2$. In general, this combinatorial description persists down the ladder and $(BS^1)_n \cong \mathbb{CP}^n$ for $n \geq 1$.

Question *Let G be a topological group and set $K_n := G^n/G$ where G acts diagonally on G^n by conjugation. We define the maps d_i, s_i's similar to the ones defined for BG. What is the bar construction of this simplicial object given by $\{K_n\}_{n\geq 0}$?*

Example 3.3. (Looping & delooping)

Fix a connected CW complex X and let ΩX denote its based loop space. One can consider ΩX as a topological category with one object and ΩX as its morphisms equipped with the appropriate topology. In fact, one can consider ΩX as a topological group due to a (technical) result of Milnor. It is well known that $B\Omega X$ is homotopy equivalent to X.

Let us apply this to $X = S^1$. We know that $\Omega S^1 \simeq \mathbb{Z}$ since homotopy classes of based maps are in bijective correspondence with the winding number. Therefore, $[X, B\mathbb{Z}]$ classifies principal $\mathbb{Z}$-bundles over X, i.e., covering spaces of X with countable index and a free action of $\mathbb{Z}$ on the fibres. But $B\mathbb{Z} \simeq S^1$ tells us that $[X, B\mathbb{Z}] \cong [X, S^1]$ which classifies $H^1(X;\mathbb{Z})$.

Example 3.4. (Eilenberg complex)

As we've briefly seen before, given a topological space X one can associate its *singular complex* $\mathcal{S}(X) := \{\mathrm{Hom}_{\mathbf{Top}}(\Delta_n, X)\}_{n\geq 0}$. This is a simplicial set and its geometric realization is weakly homotopy equivalent to X. This fact is the backbone of simplicial homology - with the knowledge that we can recover the space from its simplices, it let's us work with simplices which are inherently combinatorial.

Example 3.5. (Bar construction of an algebra)

The classical notion of a bar construction of a differential graded algebra produces a coalgebra. There is an adjoint construction known as the *cobar construction* and it is an adjoint to B under suitable assumptions. Our definition produces the same construction when we consider the natural simplcial object in $\mathbf{Ch}_{\Bbbk}$ associated with an algebra.

As an example, let $\mathcal{A} = \Bbbk[x]/(x^2)$ be a differential graded algebra with $d \equiv 0$ and $|x| = 1$. One of the natural choice for a simplicial object (taking values in chain complexes over $\Bbbk$) associated with $\mathcal{A}$ is to set $K_n := \mathcal{A}^{\otimes n}$. Unravelling the definition of the bar construction and calculating, one is led to the conclusion that

$$B\mathcal{A} \cong \Bbbk \oplus \overline{\mathcal{A}}[1] \oplus (\overline{\mathcal{A}}[1])^{\otimes 2} \oplus \cdots$$

where $\overline{\mathcal{A}} := \mathcal{A}/\Bbbk$ and $V[1]$ has V as the underlying vector space with elements shifted up in degree by 1. In other words,

$$B\left(\Bbbk[x]/(x^2)\right) \cong \Bbbk[\alpha], \;\; \alpha = x[1], \;\; d \equiv 0.$$

Notice that $\mathcal{A} = C_*(S^1;\Bbbk)$ while $B\mathcal{A}$ gives a model for $C_*(BS^1;\Bbbk)$. As seen in the first example, $BS^1 \simeq \mathbb{CP}^\infty$, whence $H_*(\mathbb{CP}^\infty;\Bbbk) \cong \Bbbk[\alpha]$. One can calculate the coalgebra structure via this model as well.

Example 3.6. (Morse theory)

We shall illustrate the following result [1] by examples.

Theorem 3.7. (Cohen-Jones-Segal)

Let $f : M \to \mathbb{R}$ be a Morse function defined on a closed Riemannian manifold M. Then associated to f is a topological category $\mathcal{C}_f$ whose objects are the critical points of f and whose space of morphisms between critical points a and b is the space $\overline{\mathcal{M}}(a, b)$ of piecewise flow lines, of the gradient flow of f, joining a to b.

(1) *If f is a generic Morse function, i.e., one whose Morse flow satisfies the Morse-Smale transversality condition, there is a homeomorphism*

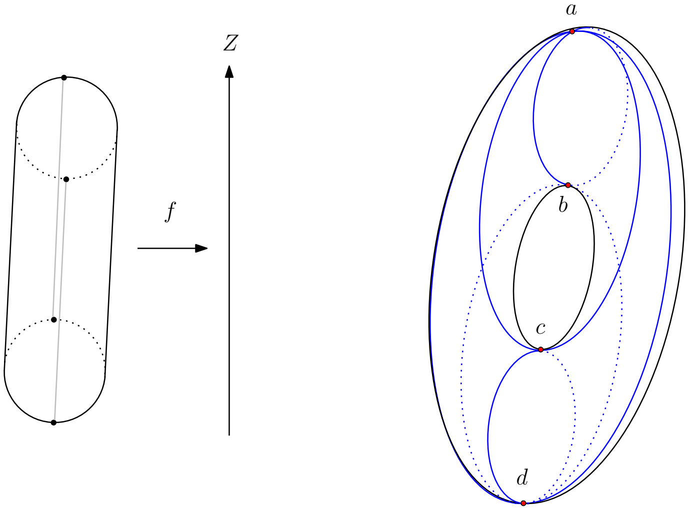
$$B\mathcal{C}_f \cong M.$$

(2) *For any Morse function there is a homotopy equivalence*

$$B\mathcal{C}_f \cong M.$$

We shall briefly explain what the Morse-Smale transversality condition is.

Let's do an example. Take a generic Morse function $f : S^1 \times S^1 \to \mathbb{R}$. The genericity rules out the usual height function (which is Morse-Smale) but if one imagines the torus standing but slightly tilted then this height function satisfies the Morse-Smale transversality condition. It is evident from the picture below.



There are four critical points and the flow lines are shown above. Notice that $\overline{\mathcal{M}}(x, y)$ is the closure of the space of piecewise flow lines up to reparametrization. Therefore, $\overline{\mathcal{M}}(a, b), \overline{\mathcal{M}}(a, c), \overline{\mathcal{M}}(b, c)$ and $\overline{\mathcal{M}}(c, d)$ have two points each. The space $\overline{\mathcal{M}}(a, d)$ is the union of four disjoint closed intervals as the picture depicts below :

CJS_torus2.pdf

In the bar construction, we have four vertices (corresponding to the critical points which are objects of f) and we have an interval for each element of $\overline{\mathcal{M}}(x, y)$. Therefore, we have 8 intervals and 4 squares (due to $\overline{\mathcal{M}}(a, d) = \coprod_{i=1}^{4} [0, 1]$) and lastly we have a 2-simplex for each composable morphism of which there are four of them. One unravels the formulas and makes the necessary identifications to realize the picture as above.

References

[1] R. Cohen, J. D. S. Jones, G. Segal, Morse theory and classifying spaces, *Warwick University Preprint*, `math.stanford.edu/~ralph/morse.ps`, (1995).

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[3] J. Milnor, The realization of a semi-simplicial complex, *Ann. of Math.* **65**, No. 2 (1957), 357-362.

[4] G. Segal, Classifying spaces and spectral sequences, *Publications mathématiques de l' I.H.É.S.*, tome **34** (1968), 105-112.