Cohomology of Lie groups and Lie algebras

1 Introduction

The aim of this expository essay is to illustrate one example of a local-to-global phenomenon. What we mean by that is better illustrated by explaining the topic at hand. We aim to understand the de Rham cohomology groups of a Lie group. But instead of doing it using actual differential forms, we shall use the properties of a Lie group (especially the fact that it is a group) to reduce calculations on the de Rham complex to calculations involving the Lie algebra and its tensor powers. On the one hand, this reduces the problem to a local one while on the other, makes it easier to solve by virtue of being a linear problem. In what follows, we explain the passage from global to local in §2 and then exhibit a few computations of the cohomology groups in §3.

2 Cohomology of Lie groups

Let G be a connected compact Lie group of dimension n with a normalized bi-invariant measure μ on it. Let $L_g : G \to G$ denote the left multiplication by $g \in G$ and $m : G \times G \to G$ the multiplication. We will be working with real coefficients throughout this section unless specified otherwise.

Definition 2.1. Let $(C(G), d)$ denote the cochain complex of de Rham differential forms on G. An element $\alpha \in C(G)$ is called a *left-invariant form* if $L_g^*\alpha = \alpha$ for any $g \in G$. The space of all left-invariant forms will be denoted by $C_L(G)$.

Observe that $C_L(G)$ is a graded d-closed subalgebra and the inclusion map $\iota : C_L(G) \to C(G)$ induces a map of graded algebras

$$\iota_* : H_L^*(G;\mathbb{R}) \to H^*(G;\mathbb{R}).$$

Notice that $C_L^1(G) = \mathfrak{g}^*$ is the dual to the left-invariant vector fields and $C_L(G)$ is the exterior algebra over $\mathfrak{g}^*$. We have an averaging map $\rho : C(G) \to C_L(G)$ defined by

$$\alpha \mapsto \int_G L_g^*\alpha \, d\mu.$$

This is a map of cochain complexes which is identity on $C_L(G)$ and $\rho \circ \iota$ is the identity on $C_L(G)$. This implies that ι_* is injective. We claim that

Proposition 2.2. *The map $\iota_* : H_L^*(G;\mathbb{R}) \to H^*(G;\mathbb{R})$ is an isomorphism.*

Proof Suppose we have constructed a chain map $h : C^i(G) \to C^{i-1}(G)$ of degree -1 such that

$$\iota \circ \rho - \mathrm{Id} = dh + hd$$

on $C(G)$. Since $\iota_* \circ \rho_* = \mathrm{Id}$, ι_* is surjective. It is injective from the previous discussion, whence it is an isomorphism. We construct h as a composition of $h_G \circ m^*$ where $h_G : C(G \times G) \to C(G)$ is homogeneous of degree -1.

Let $\pi_1 : G \times G \to G$ denote the projection of the trivial G-bundle to G. We have a map $\int^G : C(G \times G) \to C(G)$ called the *fibre integral* and is defined at $g \in G$ by integrating it over the fibre at g. It is a homogeneous map of degree $-n$ and commutes with d. We now define a degree 0 map

$$I_\Omega : C(G \times G) \to C(G)$$

by setting

$$I_\Omega(\omega)(g) := \int^G \omega \wedge \pi_1^* \Omega,$$

where Ω is the normalized left-invariant volume form on G. For any $\alpha \in C(G)$

$$[(I_\Omega \circ m^*)\alpha]\,(g) = \int^G m^* \alpha \wedge \pi_1^*(\Omega) = \int_G (L_g^* \alpha)(g)\, d\mu.$$

This proves that $I_\Omega \circ m^* = \rho$. Let $i : G \to G \times G$ denote the map sending g to $(g, 1)$. Then $m \circ i = \mathrm{Id}$ and consequently $i^* \circ m^* = \mathrm{Id}$. If we construct

$$h_G : C(G \times G) \to C(G)$$

such that $I_\Omega - i^* = dh_G + h_G d$ then it follows that

$$\iota \circ \rho - \mathrm{Id} = I_\Omega \circ m^* - i^* \circ m^* = (dh_G + h_G d) \circ m^* = d(h_G m^*) + (h_G m^*)d,$$

where the last equality holds since L^* is a cochain map.

If we change the the volume form Ω to another n-form Ψ supported in a contractible local chart $U \ni 1$ of G such that $\int_G \Psi = 1$, then $\Omega - \Psi = d\eta$ for some $(n-1)$-form η. Then the maps I_Ω and I_Ψ are chain homotopic. The homotopy is given by

$$h_\eta(\alpha) = (-1)^i \int^G \alpha \wedge \pi_1^* \eta, \quad \alpha \in C^i(G \times G).$$

Choosing Ψ has the advantage that $I_\Psi : C(U \times G) \to C(G)$ and clearly $U \times G$ deformation retracts to G. Thus, I_Ψ and i^* are chain homotopic, whence I_Ω and i^* are also chain homotopic. $\square$

Remark 2.3. *The same proof, with slight modifications, works well for a G-action on a manifold M by a compact, connected Lie group. We can prove that the inclusion of the subcomplex of G-invariant forms on M into the complex of all forms on M is an isomorphism in cohomology.*

We shift our focus to *invariant forms*, i.e., forms invariant under the left and right actions L_g and R_g respectively. In particular, these are invariant under the adjoint action $\mathrm{Ad}_g = L_g \circ R_{g^{-1}}$. These forms are invariant under d. If we define the action I, of $G \times G$ on G, by

$$I_{g_1, g_2}(g) = g_1 g g_2^{-1}$$

then the algebra of differential forms that are invariant under this action is precisely the space of invariant forms, denoted $C_I(G)$.

Lemma 2.4. *$C_I(G)$ consists of closed forms.*

Proof First observe that if $\tau : G \to G$ denotes the inverse map, then

$$d\tau_g = -(R_{g^{-1}})_* \circ (L_{g^{-1}})_*,\ g \in G$$

and $\tau^*\alpha = (-1)^p\alpha$ for $\alpha \in C_I^p(G)$. Since $d\alpha \in C_I^{p+1}(G)$,

$$(-1)^{p+1}d\alpha = \tau^* d\alpha = d\tau^*\alpha = (-1)^p d\alpha,$$

whence $d\alpha = 0$. □

Since $C_I(G)$ is closed, $H_I^*(G) = C_I(G)$ and by the remark, it is isomorphic to $H^*(G)$. We have isomorphisms

$$C_I(G) \cong H_L^*(G) \cong H^*(G). \tag{2.1}$$

If G is semisimple, this isomorphism is just a manifestation of the Hodge theorem. More precisely, it is known that for any semisimple group one can find a bi-invariant Riemannian metric on G. Hodge had proved that the harmonic forms with respect to such a metric are exactly $C_I(G)$.

We have the multiplication $m : G \times G \to G$ and $m^* : C(G) \to C(G \times G)$ which induces a map

$$\Delta : H^*(G) \longrightarrow H^*(G \times G) \xrightarrow{\cong} H^*(G) \otimes H^*(G).$$

of degree 0. Let $i_1, i_2 : G \to G \times G$ be the inclusion maps opposite 1. If $\gamma \in H^*(G \times G)$ then

$$\gamma = i_1^*\gamma \otimes 1 + \beta + 1 \otimes i_2^*\gamma,$$

where $\beta \in H^+(G)^{\otimes 2}$. Since $m \circ i_1 = m \circ i_2 = \mathrm{Id}$,

$$\Delta(\alpha) = \alpha \otimes 1 + \beta + 1 \otimes \alpha,\ \alpha \in H^*(G), \beta \in H^+(G)^{\otimes 2}.$$

Definition 2.5. An element $\alpha \in H^+(G)$ is called *primitive* if

$$\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha. \tag{2.2}$$

Remark 2.6. *It is classically known that any compact connected Lie group is rationally homotopy equivalent to a product of odd spheres. The volume forms of these spheres generate the primitive elements of $H^*(G)$.*

The primitive elements form a graded subspace, P_G, of $H^*(G)$. Notice that there are no even primitives because if α was one such then $1 \otimes \alpha$ and $\alpha \otimes 1$ would commute, both being even. Now let k be the least positive number such that $\alpha^k = 0$. Then

$$0 = \Delta(\alpha^k) = (\alpha \otimes 1 + 1 \otimes \alpha)^k = \sum_{i=1}^{k-1} \alpha^i \otimes \alpha^{k-i}.$$

In particular, $\alpha \otimes \alpha^{k-1} = 0$, whence $\alpha = 0$. Since every homogeneous element of P_G is odd, it's square is zero. Thus, the inclusion $P_G \hookrightarrow H^*(G)$ extends to a homomorphism

$$\lambda_G : \Lambda P_G \to H^*(G) \tag{2.3}$$

of graded algebras. It can be shown using properties of power maps and its eigenspaces that dim $P_G = \operatorname{rank} G$ and λ_G is an isomorphism. Thus, $H^*(G)$ is of dimension $2^{\operatorname{rank} G}$.

3 Cohomology of Lie algebras

Let $\mathfrak{g}$ be a finite dimensional Lie algebra. By Lie's theorem, it corresponds to a simply connected Lie group G. To each $\mathfrak{g}$-module M we can associate a cochain complex $C^k(\mathfrak{g}; M)$, whose cohomology is defined to be the *Lie algebra cohomology of* $\mathfrak{g}$ with values in M. We define

$$C^k(\mathfrak{g}; M) := \mathrm{Hom}(\Lambda^k \mathfrak{g}, M),\ \ k = 0, 1, \ldots, \dim \mathfrak{g}, \tag{3.1}$$

the vector space of real valued multilinear, skew maps with values in M. The coboundary operator $\delta : C^k(\mathfrak{g}; M) \to C^{k+1}(\mathfrak{g}; M)$ is defined by

$$\begin{aligned} (\delta\omega)(x_0, \ldots, x_k) \ &:= \ \sum_{i=0}^{k} (-1)^i x_i \cdot \omega(\ldots, \hat{x_i}, \ldots) \\ &+ \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([x_i, x_j], \ldots, \hat{x_i}, \ldots, \hat{x_j}, \ldots). \end{aligned} \tag{3.2}$$

It is easily verified, using Jacobi and the properties of the $\mathfrak{g}$-action on M, that $\delta \circ \delta = 0$.

Since our main object of interest is cohomology with values in $\mathbb{R}$, we set $M = \mathbb{R}$ with the trivial action of $\mathfrak{g}$. We will also abbreviate notation and denote $C^k(\mathfrak{g}; \mathbb{R})$ by $C^k(\mathfrak{g})$ and the corresponding cohomology groups $H^k(\mathfrak{g}; \mathbb{R})$ by $H^k(\mathfrak{g})$. Observe that the cohomology groups so obtained are just the the cohomology group of left-invariant forms on G and δ is exactly d. By definition, $C^0(\mathfrak{g}) = \mathbb{R}$ and $C^1(\mathfrak{g}) = \mathfrak{g}^* \cong \mathfrak{g}$. The first three coboundary maps are :

$$(\delta\alpha)(x) = 0, \tag{3.3}$$

$$(\delta\beta)(x, y) = -\beta([x, y]), \tag{3.4}$$

$$(\delta\gamma)(x, y, z) = -\gamma([x, y], z) - \gamma([y, z], x) - \gamma([z, x], y). \tag{3.5}$$

where $x, y, z \in \mathfrak{g}$ and α, β, γ are $0, 1$ and 2-cochains.

For small values of k, the cohomology groups have certain interesting interpretations. The first equation (3.3) implies that

$$H^0(\mathfrak{g}) = \mathbb{R}. \tag{3.6}$$

Using (3.4) we see that $H^1(\mathfrak{g})$ is exactly the kernel of $\delta : C^1(\mathfrak{g}) \to C^2(\mathfrak{g})$ since the map $\delta : C^0(\mathfrak{g}) \to C^1(\mathfrak{g})$ is zero. Elements α in the kernel are precisely the ones that vanish on commutators, i.e., $\alpha([x, y]) = 0$ for any $x, y \in \mathfrak{g}$. Alternatively, these can be viewed as maps from $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ to $\mathbb{R}$, whence

$$H^1(\mathfrak{g}) \cong \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]. \tag{3.7}$$

In particular, the first cohomology vanishes for a semisimple Lie algebra.

To interpret $H^2(\mathfrak{g})$ we need to understand the kernel of (3.5), i.e., 2-cochains ω such that

$$\omega(([x, y], z) + \omega([y, z], x) + \omega([z, x], y) = 0. \tag{3.8}$$

The restraint above is called the *cocycle condition* and is equivalent to ω being closed. Any such ω defines a central extension

$$0 \to \mathbb{R} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

with the Lie bracket on $\tilde{\mathfrak{g}}$ given by

$$[(x,s),(y,t)] := ([x,y]\,,\omega(x,y)). \tag{3.9}$$

The bracket satisfies the Jacobi identity due to (3.8) and is skew since ω is. Conversely, given a central extension, the bracket on $\tilde{\mathfrak{g}}$ is defined as in (3.9) and ω must satisfy (3.8). Thus, the central extensions of $\mathfrak{g}$ by $\mathbb{R}$ are in bijective correspondence with the 2-cocycles.

We try to see what relations are forced on the 2-cocycles ω, ω' if the corresponding central extensions $\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}'$ are equivalent. Recall that two extensions $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}'$ are equivalent if there exists a map $\varphi : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}'$ of Lie algebras such that the following commutes :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \tilde{\mathfrak{g}} & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \tilde{\mathfrak{g}}' & \longrightarrow & \mathfrak{g} & \longrightarrow & 0. \end{array}$$

Both the extensions are $\mathfrak{g} \oplus \mathbb{R}$ as vector spaces and $\varphi : \mathfrak{g} \oplus \mathbb{R} \to \mathfrak{g} \oplus \mathbb{R}$ is the identity when restricted to $\mathbb{R}$. Moreover, φ is an isomorphism (by the five-lemma) and $\varphi(x,0) = x + \alpha(x)$ where $\alpha \in C^1(\mathfrak{g})$. We have

$$[\varphi(x,0), \varphi(y,0)] = [(x, \alpha(x)), (y, \alpha(y))] = ([x,y]\,, \omega'(x,y))$$

and we also have

$$\varphi([(x,0),(y,0)]) = \varphi(([x,y]\,, \omega(x,y)) = ([x,y]\,, \alpha([x,y]) + \omega(x,y)).$$

Thus, the 2-cocycles are cohomologous via α.

Proposition 3.1. *Equivalence classes of central extensions of $\mathfrak{g}$ by $\mathbb{R}$ are in bijective correspondence with elements of $H^2(\mathfrak{g})$.*

It can be deduced that if $\mathfrak{g}$ is semisimple then there are no non-trivial central extensions.

Remark 3.2. *If G is simply connected then $H^2(G;\mathbb{Z}) \hookrightarrow H^2(G;\mathbb{R})$ is an injection. Recall that isomorphism classes of circle bundles over G correspond to $H^2(G;\mathbb{Z})$ and the total space of any such bundle can be made into a group, i.e., there is a short exact sequence of groups*

$$1 \to S^1 \to \widetilde{G} \to G \to 1$$

realizing such a bundle. The map of Lie algebras then give us the integral central extensions.

To discuss $H^3(\mathfrak{g})$, we shall restrict ourselves to algebras such that $H^1(\mathfrak{g}) = 0 = H^2(\mathfrak{g})$. The Lie algebras of any connected compact semisimple Lie group G satisfies this property. It follows from (3.4) that the negative of the dual of δ is a map

$$\delta^* : \Lambda^2\mathfrak{g} \to \mathfrak{g}, \;\; x \wedge y \mapsto [x,y]\,. \tag{3.10}$$

Since $\delta : \Lambda\mathfrak{g}^* \to \Lambda\mathfrak{g}^*$ satisfies $\delta^2 = 0$, the map δ^* extends to $\Lambda\mathfrak{g}$ and satisfies $\delta^* \circ \delta^* = 0$. The resulting homology groups will be called the *homology groups* of $\mathfrak{g}$ and denoted by $H_i(\mathfrak{g})$. By our assumption that the first two cohomology groups vanish, it follows from the duality of δ and δ^* that $H_1(\mathfrak{g}) = 0 = H_2(\mathfrak{g})$. In fact, the explicit formula of δ^* is

$$x_0 \wedge \cdots \wedge x_p \xrightarrow{\delta^*} \sum_{i<j} (-1)^{i+j+1} [x_i, x_j] \wedge x_0 \wedge \cdots \hat{x_i} \cdots \hat{x_j} \cdots \wedge x_p. \tag{3.11}$$

Notice that δ^* may not be a derivation.

Since $\mathfrak{g} \cong \mathfrak{g}^*$ as $\mathfrak{g}$-modules, the space of (symmetric) invariant bilinear forms on $\mathfrak{g}$, $\mathrm{Bil}(\mathfrak{g}) = (S^2\mathfrak{g})^{\mathfrak{g}}$, is isomorphic to $(S^2\mathfrak{g}^*)^{\mathfrak{g}}$. With this identification, define a map

$$\varphi : (S^2\mathfrak{g}^*)^{\mathfrak{g}} \to (\Lambda^3\mathfrak{g}^*)^{\mathfrak{g}}$$

$$B \mapsto \varphi(B) : (x \wedge y \wedge z) \to B([x,y], z) = B(\delta^*(x \wedge y), z). \tag{3.12}$$

The 3-form $\varphi(B)$ is anti-symmetric since B is invariant and symmetric and $[,]$ is skew. The invariance follows from the Jacobi identity and the invariance of B, viz,

$$\begin{aligned} & \varphi(B)([w,x] \wedge y \wedge z) + \varphi(B)(x \wedge [w,y] \wedge y) + \varphi(B)(x \wedge y \wedge [w,z]) \\ = \ & B([[w,x], y], z) + B([[y,w], x], z) + B([x,y], [w,z]) \\ = \ & -B([[x,y], w], z) + B([x,y], [w,z]) \\ = \ & 0. \end{aligned}$$

Let $\omega \in (\Lambda^3\mathfrak{g}^*)^{\mathfrak{g}}$. Since ω is closed, we have

$$\begin{aligned} 0 \ = \ & \underbrace{\omega([x_0,x_1] \wedge x_2 \wedge x_3) - \omega([x_0,x_2] \wedge x_1 \wedge x_3) + \omega([x_0,x_3] \wedge x_1 \wedge x_2)}_{=0 \text{ by invariance}} \\ & +\omega([x_1,x_2] \wedge x_0 \wedge x_3) - \omega([x_1,x_3] \wedge x_0 \wedge x_2) + \omega([x_2,x_3] \wedge x_0 \wedge x_1) \\ = \ & \underbrace{\omega([x_1,x_2] \wedge x_0 \wedge x_3) - \omega([x_1,x_3] \wedge x_0 \wedge x_2) + \omega([x_1,x_0] \wedge x_3 \wedge x_2)}_{=0 \text{ by invariance}} \\ & +\omega([x_2,x_3] \wedge x_0 \wedge x_1) - \omega([x_0,x_1] \wedge x_2 \wedge x_3) \\ = \ & \omega([x_2,x_3] \wedge x_0 \wedge x_1) - \omega(x_2 \wedge x_3 \wedge [x_0,x_1]). \end{aligned}$$

This implies

$$\omega(u \wedge \delta^* v) = \omega(\delta^* u \wedge v) \tag{3.13}$$

$$\omega(\delta^* w \wedge y) = 0 \tag{3.14}$$

for $u, v \in (\Lambda^2\mathfrak{g})^{\mathfrak{g}}, w \in (\Lambda^3\mathfrak{g})^{\mathfrak{g}}$. We are now prepared to prove the following proposition which provides the connection between $\mathrm{Bil}(\mathfrak{g})$ and $H^3(G) \cong (\Lambda^3\mathfrak{g}^*)^{\mathfrak{g}}$.

Proposition 3.3. *The map $\varphi : (S^2\mathfrak{g}^*)^{\mathfrak{g}} \to (\Lambda^3\mathfrak{g}^*)^{\mathfrak{g}}$ is an isomorphism for any semisimple Lie algebra $\mathfrak{g}$.*

Proof Injectivity of φ follows from $H^1(\mathfrak{g}) = 0$ (equivalently $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$). To prove surjectivity, let $\omega \in (\Lambda^3\mathfrak{g}^*)^{\mathfrak{g}}$. Define $B \in (S^2\mathfrak{g}^*)^{\mathfrak{g}}$ by

$$B(x,y) = \omega(u \wedge y), \text{ where } \delta^* u = x.$$

This is well defined since if $\delta^* v = x$ then $\delta^*(u-v) = 0$. Since $H_2(\mathfrak{g}; \mathbb{R}) = 0$, there exists $w \in (\Lambda^3\mathfrak{g})^{\mathfrak{g}}$ such that $\delta^* w = u - v$. Then $\omega(\delta^* w \wedge y) = 0$ by (3.14). Using (3.13) and the surjectivity of $\delta^* : \Lambda^2\mathfrak{g} \to \mathfrak{g}$,

$$B(\delta^* u, \delta^* v) = \omega(u \wedge \delta^* v) = \omega(v \wedge \delta^* u) = B(\delta^* v, \delta^* u),$$

the symmetry of B follows. By definition $\varphi(B) = \omega$. Since

$$B([x,w]\,,y) = \omega(x \wedge w \wedge y) = \omega(w \wedge y \wedge x) = B(x,[w,y]),$$

B is invariant. □

In view of this result and the discussion preceding it, we conclude that $\mathrm{Bil}(\mathfrak{g})$ is isomorphic to $H^3(G;\mathbb{R})$. If G is simple, then it is 1-dimensional since any such bilinear form is a multiple of the Killing form on $\mathfrak{g}$.