Homotopy groups of Lie groups

1 Review of Lie groups

A *Lie group* G is a smooth manifold equipped with a compatible smooth group structure. Classical examples are the various subgroups of the general linear groups. In fact, any compact Lie group appear as a subgroup of $GL(n)$ for large enough n. It follows from the theory of covering spaces that any Lie group G has a universal cover $\widetilde{G}$ which is also a Lie group. The covering map $\pi : \widetilde{G} \to G$ is a smooth homomorphism with kernel $\pi_1(G)$, which is a discrete central subgroup of $\widetilde{G}$. The tangent spaces at the identities T_1G and $T_1\widetilde{G}$ are isomorphic. It is called the *Lie algebra* of G and denoted by $\mathfrak{g}$. Conversely, given a finite dimensional Lie algebra it is the Lie algebra of some Lie group. As with manifolds, exponentiating a vector x at 1, i.e., $x \in \mathfrak{g}$, we get an element of G. This map $\exp : \mathfrak{g} \to G$ is a local diffeomorphism at $0 \in \mathfrak{g}$. For compact groups, this map is surjective but isn't in general. A Lie group will be called *semisimple* if its Lie algebra $\mathfrak{g}$ has no proper abelian ideals. Equivalently, $\mathfrak{g}$ is semisimple if and only if the Killing form

$$K(x, y) = \mathrm{trace}(\mathrm{ad}(x)\mathrm{ad}(y))$$

is non-degenerate. The notion of semisimplicity of $\mathfrak{g}$, as Lie algebras over $\mathbb{R}$ or $\mathbb{C}$, is also equivalent to the complete reducibility of its finite dimensional representations.

A *Lie subgroup* H of G is a subgroup which is also a submanifold. It is known that a subgroup H of G is a Lie subgroup if and only if it is closed. An *integral subgroup* of G is a connected subset H which is a subgroup and an immersed submanifold. It is classically known that there is a bijective correspondence between Lie subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ and connected integral subgroups H of G such that $T_1H = \mathfrak{h}$. Note that if H is a normal Lie subgroup then G/H has a canonical structure of a Lie group. For example, the connected component of the identity G^0 is a normal Lie subgroup and G/G^0 is discrete. Moreover, any neighbourhood U of the identity generates G^0.

A morphism $f : G_1 \to G_2$ of Lie groups induces a linear map $f_* : \mathfrak{g}_1 \to \mathfrak{g}_2$. If f_* is surjective then by the inverse function theorem, $f : V \to U$ is a projection around suitable neighbourhoods of the identity. If G_2 is connected then U generates G_2, whence f is a surjection. Moreover, if G_1 is connected and simply connected then

$$\mathrm{Hom}(G_1, G_2) = \mathrm{Hom}(\mathfrak{g}_1, \mathfrak{g}_2).$$

Therefore, if $f_* : \mathfrak{g}_1 \to \mathfrak{g}_2$ is an isomorphism then $f : G_1 \to G_2$ is a covering of Lie groups.

For any connected abelian Lie group G of dimension n, its Lie algebra $\mathfrak{g}$ has $\mathbb{R}^n$ as the underlying vector space with the trivial commutator. Since $\mathbb{R}^n$ is simply connected, the isomorphism of Lie algebras lift to a morphism of Lie groups $\varphi : \mathbb{R}^n \to G$, which is also a covering projection. Thus, any simply connected abelian Lie group of dimension n is isomorphic to $\mathbb{R}^n$. One can similarly show that any connected abelian Lie group is isomorphic to $\mathbb{T}^r \times \mathbb{R}^{n-r}$ for some r. Thus, any connected abelian group has $\mathbb{R}^n$ as its universal cover. For any general compact group G one can study its connected maximal abelian subgroups. These are naturally called *maximal tori* and are conjugates of each other. The dimension of a maximal tori is defined to be the *rank* of G. For example, the only compact groups of rank 1 are $S^1, SO(3)$ and S^3.

2 Homotopy groups of Lie groups

We shall be following parts of [2] for this section

Let G be a connected compact Lie group. Recall that the maximal connected abelian subgroups of G are tori and due to a classical result of É. Cartan, any two such maximal tori are conjugates of each other. The centre Z of G is the intersection of all maximal tori in G. To describe the universal covering group $\widetilde{G}$ better we need :

Lemma 2.1. *Let G be a connected Lie group and C be a discrete central subgroup of G such that G/C is compact. Then there exists a compact neighbourhood D of identity such that $\mathrm{int}(D) \cdot C = G$ and C is finitely generated.*

Proof Choose an open neighbourhood U of 1 in G such that $\overline{U}$ is compact. Since the natural projection $p : G \to G/C$ is open, $p(U)$ is an open neighbourhood of 1 in G/C and its translates by elements of G/C cover G/C. By the compactness of G/C, finitely many translates cover G/C. Choose preimages for these translates in G, say $g_1, \ldots, g_k$, and set D as the union of $g_i\overline{U}$. This is a compact set and $p(\mathrm{int}(D)) \supseteq G/C$, whence $\mathrm{int}(D) \cdot C = G$.

Let D be as chosen before. Since D^2 is compact, it is contained in finitely many of translates of D by elements of C, i.e.,

$$D^2 \subset Dc_1 \cup Dc_2 \cup \cdots \cup Dc_m.$$

If Γ is the subgroup generated by $c_1, \ldots, c_m$ then $D^2 \subset D \cdot \Gamma$. An easy induction shows that $D^n \subset D \cdot \Gamma$. Since G is connected, the union of powers of D exhaust G. Consequently, $G = D \cdot \Gamma$ and any $c \in C$ is of the form db where $d \in D \cap C, b \in \Gamma$. Since D is compact and C is discrete, $D \cap C$ is finite and C is finitely generated. □

Recall that the fundamental group of any H-space, in particular of a Lie group, is known to be abelian. Using the lemma for the covering projection $\pi : \widetilde{G} \to G$ of a connected compact Lie group G with $C = \ker \pi \cong \pi_1(G)$, we get :

Corollary 2.2. *The fundamental group of a connected compact Lie group is a finitely generated abelian group.*

This can be deduced independently by observing that a compact smooth manifold has a triangulation with finitely many cells in each dimension. Thus, the fundamental group is just given by the number of 1-cycles modulo the relations given by the 2-cells. This is clearly finitely presented.

There exists an invariant inner product on $\mathfrak{g}$ since G is compact, viz., take any inner product on $\mathfrak{g}$ and average over the group action using the Haar measure. Using this invariant product it is easy to show that any ideal $\mathfrak{h}$ in $\mathfrak{g}$ has an orthogonal complementary ideal $\mathfrak{h}^\perp$. This fact is used to show that

$$\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}] \,,$$

where $\mathfrak{z}$ is the Lie algebra of Z, the centre of G.

Proposition 2.3. *Let G be a connected compact Lie group. Then the following are equivalent :*
(i) *Z is finite;*
(ii) *the universal cover $\widetilde{G}$ is compact.*

Proof Assume (i). This implies that $\mathfrak{z}$ is zero and $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. It suffices to show that $\pi_1(G)$, which is a finitely generated abelian group, is finite. If not, then it contains $\mathbb{Z}$ as a subgroup, thereby providing a morphism $\varphi : \pi_1(G) \to \mathbb{Z}$. It can be shown using the compactness of $\widetilde{G}/\pi_1(G) \cong G$ and Lemma 2.1 that there is an extended morphism $\varphi : \widetilde{G} \to \mathbb{R}$ of Lie groups. Then the kernel of $\varphi_* : \mathfrak{g} \to \mathbb{R}$ is of codimension 1 and contains $[\mathfrak{g}, \mathfrak{g}]$, which contradicts $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

Conversely, assume that Z is infinite and hence non-discrete. Z has finitely many components since it is compact. Therefore the identity component of Z is infinite and has non-zero dimension, denoted by q. Let K be the integral subgroup of G corresponding to $[\mathfrak{g}, \mathfrak{g}]$ and $\widetilde{K}$ be the universal covering group of K. Then $\widetilde{K} \times \mathbb{R}^q$ is isomorphic to $\widetilde{G}$ since the corresponding algebras are. Hence $\widetilde{G}$ is not compact, a contradiction. □

Since $\mathfrak{z}$ is a maximal abelian ideal, the decomposition $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$ implies that $[\mathfrak{g}, \mathfrak{g}]$ is semisimple. Let K be the integral subgroup of G corresponding to $[\mathfrak{g}, \mathfrak{g}]$. Let $H = G/Z_0$, where Z_0 is the connected component of Z. Then there is a covering map $\varphi : K \to H$. Since H has a finite centre, the universal covering group $\widetilde{H}$ is compact. This means K is compact and therefore, closed in G. So K is a semisimple Lie group.

Now consider the compact Lie group $K \times Z_0$ and define $\varphi : K \times Z_0 \to G$ by $\varphi(k, z) = kz$. This is a morphism of Lie groups and φ_* is an isomorphism of Lie algebras. Therefore it is a covering projection. The kernel

$$\ker \varphi = \{(k, z) \mid kz = 1\} = \{(c, c^{-1}) \mid c \in K \cap Z_0\}$$

is a finite central subgroup of $K \times Z_0$. Thus, we have proved :

Proposition 2.4. *Let G be a connected compact Lie group. Then φ induces an isomorphism of $(K \times Z_0)/D$ with G, where $D = \{(c, c^{-1}) \mid c \in K \cap Z_0\}$.*

This reduces the classification of connected compact Lie groups to connected compact semisimple Lie groups. Moreover, the universal cover $\widetilde{K}$ of K is a connected, simply connected compact semisimple Lie group and the cover of Z_0 is $\mathbb{R}^q$, where $q = \dim Z_0$. Thus, the universal of G is isomorphic to $\widetilde{K} \times \mathbb{R}^q$. The groups $\pi_j, j \geq 2$ don't change when we quotient by a discrete subgroup.

Proposition 2.5. *Let G be a connected compact semisimple Lie group. Then its universal covering group $\widetilde{G}$ is compact. Moreover, the higher homotopy groups satisfy $\pi_j(G) = \pi_j(K), j \geq 2$.*

If K is connected, compact and semisimple then $\mathfrak{k}$ is semisimple and can be written as the direct sum of its minimal ideals $\mathfrak{k}_i, i = 1, \ldots, p$, which are simple Lie algebras. Let K_i be the integral subgroup of K corresponding to $\mathfrak{k}_i$ and $\widetilde{K}_i$ be the universal covering group. Then $\widetilde{K}_1 \times \cdots \times \widetilde{K}_p$ is isomorphic to $\widetilde{K}$, which is compact. Thus, $\widetilde{K}_i$ and hence K_i are both compact. Therefore the map $\varphi : K_1 \times \cdots \times K_p \to K$ given by $\varphi(k_1, \ldots, k_p) = k_1 \cdots k_p$ is a covering projection (use the fact that the exponential map is surjective for compact groups). This discussion establishes the following result :

Proposition 2.6. *Any connected compact semisimple Lie group K is a quotient by a finite central subgroup of a product $K_1 \times \cdots \times K_p$ of connected simple Lie groups.*

In view of proposition 2.5 and 2.6, for $j \geq 2$ we have

$$\pi_j(G) = \pi_j(K) = \pi_j(K_1) \times \ldots \times \pi_j(K_p),$$

where K_i's are connected and compact. From the classification theorem of simple Lie groups, the possibilities for connected compact simple Lie groups are :

(i) $SU(n)$ - simply connected,
(ii) $SO(n)$ ($n \neq 4$) - has a universal double cover Spin(n) (if $n > 2$).

Remark We have purposely left out the exceptional groups G_2, F_4, E_6, E_7 and E_8 since it is beyond the scope of this discussion. It was shown by A. Borel and later by R. Bott that π_2 is zero and π_3 is $\mathbb{Z}$ for these groups.

Using the long exact sequence of the fibration $SU(n-1) \to SU(n) \to S^{2n-1}$ we see that $\pi_2(SU(n-1)) \cong \pi_2(SU(n))$. Since $\pi_2(SU(2)) = 0$, it follows that

$$\pi_2(SU(n)) = 0, \ n \geq 2.$$

Exactly the same way, it can be shown that

$$\pi_3(SU(n)) = \mathbb{Z}, \ n \geq 2.$$

Similarly, using $SO(n-1) \to SO(n) \to S^{n-1}$ and the fact that $SO(4) \cong SO(3) \times S^3$, it is easily shown that

$$\pi_2(SO(n)) = 0 \ n \geq 2,$$

and we also get

$$\pi_3(SO(n)) = \mathbb{Z}, \ n \geq 3, \, n \neq 4.$$

A famous result of Iwasawa states that any connected Lie group deformation retracts to any of its maximal compact subgroup. Thus, for topological purposes it is enough to consider compact groups and we conclude :

Theorem 2.7. *Let G be any connected Lie group. Then $\pi_1(G)$ is a finitely generated abelian group, $\pi_2(G) = 0$ and $\pi_3(G)$ is a finitely generated free abelian group.*

3 Applications and further remarks

Let E be a principal G-bundle over a 3-manifold M for any simply connected Lie group G. We would like to construct a section of this bundle. Recall that for a G-bundle, the existence of a section is equivalent to the triviality of the bundle. Now consider a simplicial decomposition of M into tetrahedrons, giving rise to $M_0 \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow M_3 = M$, where M_i is the skeleta of this decomposition consisting of cells of dimension at most i. Moreover assume that each tetrahedron is in some local chart, i.e., the fibre over the tetrahedron can be trivialized. To start with, one can define a section s over M_0 by arbitrarily assigning points in the corresponding fibre. To extend s to M_1 we need to lift edges in M_1 to E which is tantamount, using the trivialization, to the connectivity of the fibre G. Therefore, $s : M_1 \to E$ can be defined. To define s over M_2, we need to lift triangles in M_2 to E. Again, this is equivalent to the contractibility of the lift of the boundary of the triangles. Notice that the trivialization of the bundle over the triangle implies that the fibre over the triangle deformation retracts to a fibre over any point inside the triangle. In effect, we associate for each triangle in the base an element of $\pi_1(G) = 0$. Therefore, we can always lift triangles. Similar considerations show that the possibility of extending s to $M_3 = M$ is given by an obstruction which associates to each tetrahedron an element of $\pi_2(G) = 0$. Therefore, a section exists and $E \to M$ is trivializable.

Using ideas from rational homotopy theory one can show that for a simply connected, compact Lie group G, all its even rational homotopy groups are zero. This is not true over the integers as the example $\pi_4(S^3) = \mathbb{Z}_2$ suggests. In fact, one has the classical statement that G, of rank r, is rational homotopic to a product of odd spheres, i.e., there exists a map

$$\varphi : \prod_{i=1}^{r} S^{i_r} \to G$$

which induces an isomorphism of rational homotopy groups. This map also induces an isomorphism of rational (or real) cohomology rings. A few examples of classical groups in this light are :

$$\begin{aligned} S^3 \times S^5 \times \ldots \times S^{2n-1} &\cong_{\mathbb{Q}} SU(n) \\ S^3 \times S^7 \times \ldots \times S^{4n-1} &\cong_{\mathbb{Q}} Sp(n) \\ S^3 \times S^7 \times \ldots \times S^{4n-1} &\cong_{\mathbb{Q}} SO(2n+1) \\ S^{2n+1} \times S^3 \times S^7 \times \ldots \times S^{4n-1} &\cong_{\mathbb{Q}} SO(2n+2). \end{aligned}$$

Moreover, the ring structure of G over integers is usually more complicated. For example, as proved in [1], page 309,

$$H^*(SO(5);\mathbb{Z}) = \mathbb{Z}[x,y,z]/(2x, x^4, y^4, z^2, x^3 - y^2, xz),$$

where $|x| = 2, |y| = 3$ and $|z| = 7$.

We conclude by a quick proof of Theorem 2.7 for π_2 and π_3. There are isomorphisms

$$\pi_i(\Omega G) \cong \pi_{i+1}(G),$$

where ΩG is the space of all based loops in G. One can use Morse theory to analyze the CW-structure of ΩG for a simply connected and compact group G. It was shown ([3], Theorem 21.7) by R. Bott that ΩG consists of only even dimensional cells, whence $\pi_2(G) = \pi_1(\Omega G) = 0$. By Hurewicz theorem, $\pi_3(G) = \pi_2(\Omega G) = H_2(\Omega G)$, which is a freely generated abelian group of rank equal to the number of 2-cells in ΩG.

References

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[2] D. Miličić, *Lectures on Lie Groups*, http://www.math.utah.edu/∼milicic/lie.pdf

[3] J. Milnor, *Morse Theory*, Annals of Mathematics Studies, No. 51, Princeton University Press.