What is an Eilenberg-MacLane space?

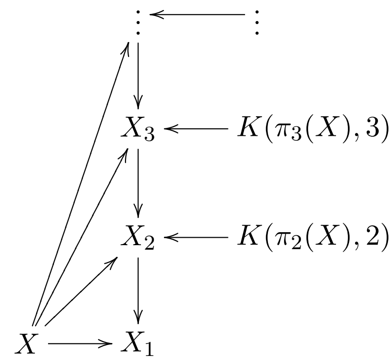
1 Introduction

To each topological space X we can associate a sequence of groups, for e.g., homotopy, homology or cohomology groups. One can then naively ask if the (co)homology groups determine a space up to homeomorphism? Since homology is also homotopy invariant, X and $X \times \mathbb{R}$ has the same homology but are not homeomorphic in general. What about homotopy groups? Does it determine a space up to homotopy equivalence? Again, the answer is negative as the spaces $\mathbb{RP}^2 \times S^3$ and $S^2 \times \mathbb{RP}^3$ suggest. They have $S^2 \times S^3$ as the universal cover but one is orientable while the other isn't! In what follows, we shall be concerned with topological spaces that are CW complexes, i.e., spaces that are built out of attaching cells and toplogized in a nice way. It is worthwhile to note the classic result of J. H. C. Whitehead :

Theorem 1.1 *Let $f : X \to Y$ be a continuous maps between CW complexes. If f induces an isomorphism on homotopy groups then f is a homotopy equivalence. Moreover, if f is the inclusion map then Y deformation retracts onto X.*

A connected, pointed space is called an *Eilenberg-MacLane space* if it has only one non-trivial homotopy group. It is denoted by $K(G, n)$ where G is the nth homotopy group of the space. As we will see, this notation is well defined since there is only one such space up to homotopy equivalence. Moreover, for a group G, one can define $K(G, 0)$ as just the group G with the discrete topology. The first example that comes to mind is $S^1 = K(\mathbb{Z}, 1)$. Such spaces appear naturally in the context to be described forthwith.

Any (connected) space X comes equipped with its homotopy groups $\{\pi_i(X)\}_{i\geq 1}$, which are modules over the fundamental group. The algebraic data describing the space can be put together in the form of a Postnikov tower. Construct a space X_1 from X by attaching 3-cells to kill $\pi_2(X)$ and keeping $\pi_1(X)$ unchanged. Subsequently attach cells of higher dimension to kill higher homotopy groups. The space X_1 resulting out of this inductive process is a $K(\pi_1(X), 1)$. In general, define X_n to be the space constructed out of X by attaching $(n+2)$-cells and higher so that X_n has trivial homotopy groups above dimension n. Moreover, the inclusion $X \hookrightarrow X_n$ induces an isomorphism on the homotopy groups up to order n. There is also an inclusion $X_n \to X_{n-1}$. Start with X and kill $\pi_{n+i}(X), i \geq 1$ to get X_n. Form a space $\widetilde{X}$ by attaching $(n+1)$-cells to X_n to kill $\pi_n(X_n) = \pi_n(X)$. This intermediate space $\widetilde{X}$ has the same homotopy groups as X up to dimension $n-1$. Consequently, kill the higher homotopy groups of $\tilde{X}$ and get X_{n-1}. Since any map between spaces can be thought of as fibrations, the map $X_n \to X_{n-1}$, as described before, has fibre $K(\pi_n(X), n)$. All these fit together in the following way :



This Postnikov data determines X up to homotopy equivalence. Spaces with exactly one homotopy group have trivial Postnikov tower and hence these are classified by the homotopy group, i.e., there is only one space (up to homotopy equivalence). In homotopy theory, these are the basic building blocks for any topological space. In fact, as the tower suggests, any space (up to homotopy equivalence) can be built out of twisted products of Eilenberg-MacLane spaces $K(\pi_n(X), n)$ arising from its homotopy groups. Note that a space, in general, is not the product of such Eilenberg-MacLane spaces since the ambient space could be finite dimensional CW complex while the product space is typically infinite dimensional and has non-zero homology in infinitely many dimensions. A good example to keep in mind is S^2.

The existence of such spaces is easy to prove. To construct a $K(G, 1)$, choose a presentation of G and take a bouquet of circles representing the generators of G and add 2-cells to this for each relation in the presentation. This gives a 2-complex X_2 and $\pi_1(X_2) = G$ by construction. Take a presentation of $\pi_2(X_2)$ and attach 3-balls along its boundary to X_2 to kill each generator of $\pi_2(X_2)$. The resulting space X_3 has no π_2 and π_1 remains unchanged since we are adding 3-cells. Iterate this process with $\pi_3(X_3)$ and so on to get a space X. Technically, this is the direct limit of the spaces

$$X = \varinjlim X_2 \hookrightarrow X_3 \hookrightarrow X_4 \hookrightarrow \cdots$$

with the direct limit topology. By construction, X is a $K(G, 1)$ space. To construct $K(G, n)$ we proceed similarly. Take a presentation for G and start with a wedge of n-spheres, one each for each generator for G. Then attach $(n+1)$-cells for each relation to get a space X_{n+1} with $\pi_n(X_{n+1}) = G$. As before, kill higher homotopy elements by attaching higher cells. An useful result in this direction is :

Proposition 1.2 *The connected component of the trivial loop in the based loop space $\Omega K(G, n+1)$ is $K(G, n)$.*

The proof relies on a more general observation that $f : (S^i, p) \to (\Omega X, 1_x)$ can be thought of as $\tilde{f} : S^i \times S^1 \to X$ such that $\tilde{f}$ maps $(\{p\} \times S^1) \cup (S^i \times \{1\})$ to $x \in X$. This is tantamount to saying $\tilde{f} : S^i \wedge S^1 \to X$. Since homotopies commute with this transfer map and $S^i \wedge S^1 = S^{i+1}$, we conclude that

$$\pi_{i+1}(X) = \pi_i(\Omega X).$$

This means that

$$K(G, n) = \Omega K(G, n+1) = \Omega^2(K(G, n+2)) = \cdots = \Omega^k(K(G, n+k)) = \cdots$$

and secretly we are dealing with infinite loop spaces when we are working with Eilenberg-MacLane spaces. It is useful to keep in mind the basic example of the integers, i.e.,

$$\mathbb{Z} = \Omega S^1 = \Omega^2 \mathbb{CP}^\infty = \Omega^3 K(\mathbb{Z}, 3) = \cdots .$$

One can think of based loop space as a functor from spaces to H-spaces. In particular, we have seen that $K(G, n) \approx \Omega K(G, n+1)$. There is a reverse functorial construction to this, viz., start with any topological group G and form it's classifying space BG. There are two related constructions for this, the classical one is due to Milnor where he constructs a space

$$EG := \varinjlim \underbrace{G * G \cdots * G}_{n}.$$

It can be shown that EG is contractible and G acts freely on it. The quotient is defined to be the classifying space BG. One may also think of EG as the union of simplices labelled by elements of G with suitable identifications according to edge degeneracies. It turns out that $G \hookrightarrow EG \to BG$ is a principal G-bundle and classifies G-bundles, whence the name for BG. Any other model for EG, in the category of CW-complexes, would be homotopy equivalent to this one. However, it is not clear why this model has, if any, a group structure. For instance, with $G = S^1$ this produces $ES^1 = S^\infty$ which has no obvious group structure.

The construction due to Segal is to start with G and work with the same direct limit but also allowing degeneracies in the G-variable, i.e., collapse (g, g, t) in $G * G$ and likewise. We shall call this model $\mathcal{E}G$. This model versus Milnor's is exactly analogous to reduced suspension versus usual suspension. An alternate way to view $\mathcal{E}G$ is to visualize it as the space of piecewise constant functions on $[0, 1]$ with values in G. This has a natural group structure and $G \hookrightarrow \mathcal{E}G$ as the constant functions. When G is abelian, G sits inside $\mathcal{E}G$ as a normal subgroup, whence $\mathcal{B}G$ is also a group. Directly visualizing the topology on $\mathcal{E}G$ as a function space is a little harder. If (G, d) is a metric space then one can define a metric on $\mathcal{E}G$ by setting

$$d(f, g) := \int_0^1 d(f(t), g(t))\, dt.$$

The contractibility of $\mathcal{E}G$ is then very easy to see via

$$H : \mathcal{E}G \times [0, 1] \longrightarrow \mathcal{E}G$$

$$H(f, t)(s) := \begin{cases} f(\frac{s}{1-t}) & \text{if } s \leq 1 - t, \\ 1_G & \text{if } 1 - t < s \leq 1. \end{cases}$$

One may think of H as "combing" simultaneously all the step functions from right to left. The only difference between EG and $\mathcal{E}G$ is that the former is locally trivial while the later needn't be! However, if G is an absolute neighbourhood retract then $\mathcal{E}G$ works fine.

For a suitable abelian group A, one iterates either Milnor's or Segal's procedure to get $B^n A$. From the long exact sequence arising from $A \hookrightarrow EA \to BA$ we see that BA is a $K(A, 1)$. In general, $B^n A$ is a $K(A, n)$. This is known as *delooping* and also called the *bar construction* in a purely algebraic context. The Segal model of $K(A, n)$ has the structure of an abelian group while the Milnor model may not!

2 Examples

We shall illustrate our previous discussion with a few examples.

Example 1 Take any connected finite graph Γ. Contracting its maximal tree, this becomes a wedge sum of circles. Therefore, $\pi_1(\Gamma)$ is a free group and Γ is a $K(\pi_1(\Gamma), 1)$. In particular, S^1 is a $K(\mathbb{Z}, 1)$.

Example 2 Any closed oriented surface of genus $g > 0$ is covered by the Poincaŕe disk or $\mathbb{R}^2$. Therefore, such surfaces are examples of $K(\pi_1, 1)$. Since non-oriented closed surfaces have a connected double cover that is orientable, such surfaces are also $K(\pi_1, 1)$. Moreover, non-closed surfaces deformation retracts onto graphs and hence give $K(G, 1)$ for some free group G.

Example 3 Let S^∞ be the infinite sphere, thought of as the unit sphere in $\mathbb{C}^\infty$. This can also be thought of as the direct limit of S^n sitting inside S^{n+1} as the equator. The space S^∞ is contractible and admits a free $\mathbb{Z}_m$ action via multiplication by the mth roots of unity. Then the quotient $S^\infty/\mathbb{Z}_m$ is a $K(\mathbb{Z}_m, 1)$ and is called an *infinite Lens space.* It is named so since such spaces arise as direct limits of finite dimensional Lens spaces. In particular, the infinite real projective space $\mathbb{RP}^\infty$ is a $K(\mathbb{Z}_2, 1)$.

Example 4 For any topological group G, there is a principal G-bundle $G \hookrightarrow EG \to BG$ with contractible total space EG and classifying space BG. Using the long exact sequence of homotopy groups we conclude that BG is a $K(G, 1)$ if G is a discrete group. This construction produces an infinite dimensional space and BG has an infinite number of cells in each positive dimension if G is infinite. For example, $B\mathbb{Z}$ is much bigger than S^1, the most efficient model for $K(\mathbb{Z}, 1)$.

Example 5 Consider the free action of S^1 on S^∞. The resulting fibration $S^1 \hookrightarrow S^\infty \to S^\infty/S^1 =: \mathbb{CP}^\infty$ gives a base which is a $K(\mathbb{Z}, 2)$. One can also define $\mathbb{CP}^\infty$ to be the direct limit of finite complex projective spaces. Moreover, we see (refer Proposition 1.2) that $\Omega\mathbb{CP}^\infty$ is homotopic to S^1.

Example 6 To construct $K(\mathbb{Z}, n)$ one can start with S^n which has $\pi_n(S^n) = \mathbb{Z}$ and attach $(n+2)$-cells to kill $\pi_{n+1}(S^n)$. Then one iterates this process of attaching higher cells to kill higher homotopy groups. The resulting space is a $K(\mathbb{Z}, n)$. Alternatively, using the Dold-Thom theorem (refer 3.1) one can take a model for $K(\mathbb{Z}, n)$ to be the configuration space of finite sets of unordered points in S^n.

Example 7 Let P_n (resp. B_n) denote the fundamental group of the configuration space of n distinct ordered (resp. unordered) points in $\mathbb{C}$. B_n is called the *braid group* on n strings while P_n is called the *pure braid group.* Notice that $\mathbb{C}^n \setminus \Delta$, the configuration space of n distinct ordered points in $\mathbb{C}$, is a $K(P_n, 1)$ and can be shown by induction. Moreover, this covers the space $(\mathbb{C}^n \setminus \Delta)/S_n$, whence this is a $K(B_n, 1)$. When $n = 2$, the braid group $B_2 = \mathbb{Z} = P_2$ and

$$K(\mathbb{Z}, 1) = \mathbb{C}^2 \setminus \Delta \approx \Delta \times S^1 \approx S^1.$$

Example 8 Let K be a knot in the 3-sphere. Then it follows from 3-manifold theory that $S^3 \setminus K$ is a $K(G, 1)$. As the simplest example, take K to be the unknot in S^3 and observe that K can be thought of as the unit circle in the xy-plane in $\mathbb{R}^3$. Slightly thickening this we get a solid torus, the complement of which in S^3 is also a solid torus. Therefore, $S^3 \setminus K$ is a $K(\mathbb{Z}, 1)$.

Example 9 Start with a Hilbert space $\mathcal{H}$ and let $U(\mathcal{H})$ denote the group of unitary operators. Kuiper's theorem from mid-sixties tells us that $U(\mathcal{H})$ is contractible. If we quotient by the free action of the circle we get $PU(\mathcal{H})$, the *projective unitary operators.* This is a model for $K(\mathbb{Z}, 2)$ and admits a group structure which is non-abelian! However, this is abelian up to homotopy. Now consider the space $\mathcal{H}_{HS}$ of *Hilbert-Schmidt operators*. A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is in $\mathcal{H}_{HS}$ if for some orthonormal basis $\{e_i\}$ of $\mathcal{H}$, the sum

$$\|T\|_{HS} := \sum_{i=1}^{\infty} \|Te_i\|^2$$

is finite. It can be shown that the number above is just $\mathrm{tr}(TT^*)$ and therefore independent of the orthonormal basis chosen. The space $\mathcal{H}_{HS}$ contain adjoints. Moreover, if $T \in \mathcal{H}_{HS}$ then T is compact, of trace class and $\|T\| \leq \|T\|_{HS}$. If we define

$$\langle S, T\rangle := \mathrm{tr}(ST^*)$$

then with this norm $\mathcal{H}_{HS}$ forms a Hilbert space. It then follows that $PU(\mathcal{H})$ acts on $U(\mathcal{H}_{HS})$ by conjugation. This gives us a principal $PU(\mathcal{H})$-bundle with a contractible total space whence the Banach manifold $U(\mathcal{H}_{HS})/PU(\mathcal{H})$ is a $K(\mathbb{Z}, 3)$.

Example 10 Let X be the topological monoid of all maps of the circle to itself with positive degree. Form the group completion G of X, i.e., formally adjoin inverses and make all right and left compositions into equivalences. Let G^0 denote the connected component of the identity of the topological group G. It can be shown that G^0 is a $K(\mathbb{Q}, 1)$.

3 Applications

The Eilenberg-MacLane spaces are useful in giving a homotopical description of singular cohomology theory. The spaces $K(\mathbb{Z}, n)$ fit together via the loop space functor

$$K(\mathbb{Z}, 0) \xleftarrow{\Omega} K(\mathbb{Z}, 1) \xleftarrow{\Omega} K(\mathbb{Z}, 2) \xleftarrow{\Omega} \cdots$$

to form a *spectrum*, called the *Eilenberg-MacLane spectrum*. Given a connected space X, one can define a cohomology theory by setting

$$h^n(X; \mathbb{Z}) := [X, K(\mathbb{Z}, n)],$$

where $[X, Y]$ denotes the pointed homotopy classes of maps from X to Y. In our case, since $K(\mathbb{Z}, n)$ has no fundamental group if $n > 1$ and has abelian π_1 when $n = 1$, pointed homotopy classes and free homotopy classes are in bijective correspondence. Hence forth, whenever the target is $K(\mathbb{Z}, n)$, by $[X, K(\mathbb{Z}, n)]$ we will mean free homotopy classes. Given maps $f, g : X \to K(\mathbb{Z}, n) = \Omega K(\mathbb{Z}, n+1)$ one can define the sum $f + g$ as the following composite map

$$X \xrightarrow{(f,g)} \Omega K(\mathbb{Z}, n+1) \times \Omega K(\mathbb{Z}, n+1) \xrightarrow{\times} \Omega K(\mathbb{Z}, n+1) = K(\mathbb{Z}, n).$$

The multiplication map $\times$ is commutative up to homotopy, whence $[f + g]$ is a well defined abelian binary operation on $[X, K(\mathbb{Z}, n)]$.

To explain the cup product structure we first observe that there is a map

$$K(\mathbb{Z}, m) \wedge K(\mathbb{Z}, n) \xrightarrow{\mu} K(\mathbb{Z}, m+n)$$

inducing an isomorphism on $\pi_{m+n} = \mathbb{Z}$. Recall that $K(\mathbb{Z}, n)$ has a model (refer example 2.6) which starts with S^n and has cells of dimension $(n + 2)$ and higher attached to it. In that model

$K(\mathbb{Z},m)\wedge K(\mathbb{Z},n)$ has $S^m \wedge S^n = S^{m+n}$ and has higher cells attached to it. Map this $(m+n)$-sphere by the identity map to the corresponding sphere of $K(\mathbb{Z},m+n)$. The higher cells of $K(\mathbb{Z},m)\wedge K(\mathbb{Z},n)$ can be then mapped to $K(\mathbb{Z},m+n)$ since $\pi_i(K(\mathbb{Z},m+n)) = 0$ for $i > m+n$. Moreover, up to homotopy, this map is unique. Now given maps $f : X \to K(\mathbb{Z},m), g : X \to K(\mathbb{Z},n)$ we define the map $f \cup g : X \to K(\mathbb{Z},m+n)$ to be the composite

$$X \xrightarrow{f\times g} K(\mathbb{Z},m)\times K(\mathbb{Z},n) \xrightarrow{q} K(\mathbb{Z},m)\wedge K(\mathbb{Z},n) \xrightarrow{\mu} K(\mathbb{Z},m+n).$$

Since all the above maps respect homotopies, we get a well defined object $[f \cup g] \in [X, K(\mathbb{Z},m+n)]$.

There are natural isomorphisms between $h^n(X;\mathbb{Z})$ and $H^n(X;\mathbb{Z})$, whence one may define the usual singular integral cohomology in this way. Note that all of this goes through for any abelian group A, i.e., the singular cohomology $H^n(X;A)$ can be defined as $[X, K(A,n)]$. With this viewpoint in mind, let us see what the first few integral cohomology groups are. When $n = 1$, we get

$$H^1(X;\mathbb{Z}) = [X, S^1]$$

and for $n = 2$ we get

$$H^2(X;\mathbb{Z}) = [X, \mathbb{CP}^\infty].$$

This equation is the familiar fact that complex line bundles are classified by elements of $H^2(X;\mathbb{Z})$.

Let us change tracks and discuss the famous theorem of Dold-Thom in the late 50's which gave a homotopy theoretic definition of homology. To simplify our discussion we assume our space X is connected and we have chosen a base point x_0. One can define $\mathrm{Symm}^n(X)$ to be the quotient of the S_n-action on n copies of X. There is a natural inclusion $X^n \hookrightarrow X^{n+1}$ by putting x_0 in the $(n+1)$th coordinate. This induces an inclusion

$$\mathrm{Symm}^n(X) \hookrightarrow \mathrm{Symm}^{n+1}(X)$$

and $\mathrm{Symm}^\infty(X)$ is defined to be the direct limit of these spaces. It is an H-space which is commutative and has a strict identity.

Theorem 3.1 (Dold-Thom)
For connected CW complexes X, there is a natural isomorphism

$$\pi_*(\mathrm{Symm}^\infty(X)) \cong \widetilde{H}_*(X;\mathbb{Z}).$$

For the case of S^2, we know that $\mathrm{Symm}^n(S^2) = \mathbb{CP}^n$. We recover that $\mathbb{CP}^\infty = \mathrm{Symm}^\infty(S^2)$ is a $K(\mathbb{Z},2)$. Similarly, for $X = S^n$ we see that $\mathrm{Symm}^\infty(S^n)$ is a $K(\mathbb{Z},n)$. Therefore, $K(\mathbb{Z},n)$ can be interpreted as the configuration space of finite sets of unordered points in S^n.

We conclude with a neat little observation. Recall that $K(\mathbb{Z}_n,1)$ can be modelled by the infinite Lens space. It can be defined as a cellular space with one cell in each dimension and the boundary maps between cellular chain groups alternating between multiplication by 0 and n. Then it follows that the homology of $K(\mathbb{Z}_n,1)$ is $\mathbb{Z}_n$ in each odd dimension. Consequently, no finite CW complexes can model $K(\mathbb{Z}_n,1)$. More generally, if a group G has torsion then $K(G,1)$ is never a finite dimensional CW complex. Armed with this observation, we can conclude that the fundamental group of any surface (or of any aspherical manifold) has no torsion.

References

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