The Growth and End of Groups

1 Basic Definitions

Let G be a finitely generated group, unless otherwise mentioned. We will be interested in finite generating sets S of G, not containing 1, such that if $s \in S$ then $s^{-1} \in S$. We define a norm $|\ |_S$ of $g \in G$ to be the length of a reduced word (in terms of elements of S) that represents g. Any two such reduced words have the same length, whence this induces a metric d_S on G as follows :

$$d_S(g,h) := |g^{-1}h|_S,\ \ g,h \in G.$$

This is symmetric since S has inverses. It is also left-invariant. The metric d_S is called the *word metric* on G given by S.

We may form the associated *Cayley graph* $\Gamma(G,S)$ of G, whose vertices are the elements of G. We say $g \in G$ is *adjacent* to g' if $g' = gs$ for some $s \in S$. Since S contain inverses, the adjacency relation is symmetric. Induced with the metric d_S, the Cayley graph becomes a metric space. We observe that $\Gamma(G,S)$ is *proper*, i.e., a subset is compact if and only if it is closed and bounded, since the valency of each vertex is finite and equals $|S|$. Moreover, if $\varphi : G \overset{\cong}{\to} G'$ and S generates G as before, then $S' = \varphi(S)$ is a generating set for G'. The induced map $\tilde{\varphi} : (\Gamma(G,S), d_S) \to (\Gamma(G',S'), d_{S'})$ is an isometry.

It is natural to ask what happens when we change the generating set S for G. In preparation, we need

Definition 1.1. Two increasing functions $f,g : \mathbb{N} \to \mathbb{R}^+$ are equivalent if there exists $A,B,C > 0$ such that

$$\frac{1}{A}f(\lceil x/C\rceil) - B < g(x) < Af(\lceil Cx\rceil) + B\ \ \forall x \in \mathbb{R}.$$

It follows very easily from the definition that

Proposition 1.2. (Growth of functions)
(1) *$f(R) = R^m$ and $g(R) = R^n$ are equivalent if and only if $k = m$. More generally, two polynomials are equivalent if and only if they have the same degree.*
(2) *$f(R) = R^k$ and $g(R) = c^R, c > 1$ are not equivalent.*
(3) *For $A,B > 1$, $f(R) = A^R$ and $g(R) = B^R$ are equivalent.*

For a generating set S of G, let $B_S(R) = \{g \in G | d_S(g,1) \leq R\}$. We define the *growth function* f_S to be

$$f_S(R) = |B_S(R)|.$$

We are ready to prove :

Theorem 1.3. *For two finite generating sets S,S' of G, the growth functions f_S and $f_{S'}$ are equivalent.*

Proof Let $c_1 = \max\{d_S(s',1) | s' \in S'\}, c_2 = \max\{d_{S'}(s,1) | s \in S\}$ and $c = \max\{c_1, c_2\}$. Then

$$\frac{1}{c} d_S(g,1) \le d_{S'}(g,1) \le c\, d_S(g,1)$$

for any $g \in G$. This implies the equivalence of f_S and $f_{S'}$. □

Thus, the growth of a group is independent of the (finite) generating set chosen.

However, the growth function is not enough to distinguish between groups. We may define one more coarse invariant which is helpful.

Definition 1.4. Assume that X is a connected, locally path connected, locally compact topological space. We say X has *one end* if given a compact subset $K \subseteq X$, there is a compact set $L \supseteq K$ such that for any $x, y \in X \setminus L$ there is a path in $X \setminus K$ joining x and y.

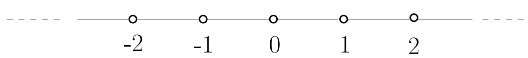
It follows from this definition that $\mathbb{Z}, \mathbb{R}$ does not have one end while $\mathbb{Z}^2, \mathbb{R}^2$ has one end. Since any homeomorphism of spaces preserve this property, it immediately implies that the $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$. The statement of importance for groups is :

Proposition 1.5. *Suppose $\Gamma(G,S)$ has one end. Then so does $\Gamma(G,S')$ for any other finite generating set S'.*

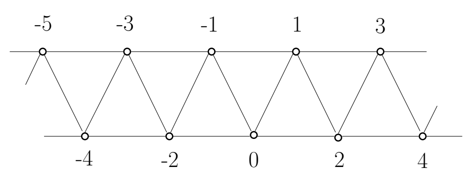
In particular, this implies that $\mathbb{Z}$ is not ismorphic to $\mathbb{Z}^2$.

Examples

(1) Let $G = \mathbb{Z}$ and $S = \{\pm 1\}, S' = \{\pm 1, \pm 2\}$ be two generating sets. The Cayley graph $\Gamma(G,S)$ is



It does not have one end and $f_S(R) = 2R+1$. $\Gamma(G,S')$ has a growth function $f_{S'}(R) = 4R+1$ and looks like



More generally, set $G = \mathbb{Z}^n, S = \{\pm e_i \,|\, i = 1, \ldots, n\}$. Think of $\Gamma(G,S)$ as the grid on $\mathbb{Z}^n$ inside $\mathbb{R}^n$. It has one end and the growth function can be calculated recursively. Let $f_n(R)$ denote the volume of the positive quadrant of G. Then

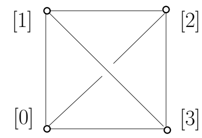
$$f_n(R) = \sum_{i=0}^{R} \binom{n+i-1}{n-1} = \binom{n+R}{n}$$

is a polynomial of degree n. A simple counting yields

$$f_S(R) = \sum_{i=0}^{n} (-1)^{n-i} 2^i \binom{n}{i} f_i(R),$$

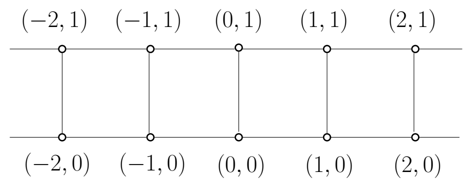
which is a polynomial of degree n.

(2) Let $G = \mathbb{Z}_n, S = \{[\pm 1]\}$. $\Gamma(G,S)$ is just the cyclic graph on n vertices, having one end. For $R \ge [n/2]$, $f_S(R) = n$ is just the constant function. For example, $G = \mathbb{Z}_4$ with $S = \{[\pm 1], [\pm 2]\}$. Then $\Gamma(G,S)$ is just K_4, the complete graph on 4 vertices :



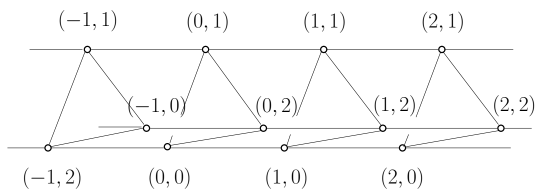
The growth function is again constant for $R \geq 1$ and the graph has one end.

(3) Let $G = \mathbb{Z} \oplus \mathbb{Z}_2, S = \{\pm(1,0), (0,1)\}$. $\Gamma(G,S)$ is just the *infinite ladder* graph :



This does not have one end. The associated growth function is $f_S(R) = 4R$ for $R \geq 1$.

(4) Let $G = \mathbb{Z} \oplus \mathbb{Z}_3, S = \{\pm(1,0), \pm(0,1)\}$. $\Gamma(G,S)$ looks like



The graph does not have one end and $f_S(R) = 6R - 1$ for $R \geq 1$.

(5) Let G be any finitely generated abelian group. We write $G = \mathbb{Z}^k \oplus \text{Tor}$, where k is the rank of G. Let S be a finite generating set. Since torsion doesn't change the growth rate of the growth function, $f_S(R) \sim R^k$ (refer example $2, 3, 4$ above).

(6) Let $G = \mathbb{Z}_2 * \mathbb{Z}_2, S = \{\alpha, \beta\}$. $\Gamma(G,S)$ looks exactly like $\Gamma(\mathbb{Z}, \{\pm 1\})$. Therefore, this graph has a growth function $f_S(R) = 2R + 1$ and does not have one end. Thus, there are non-isomorphic graphs that doesn't have one end and equivalent growth functions.

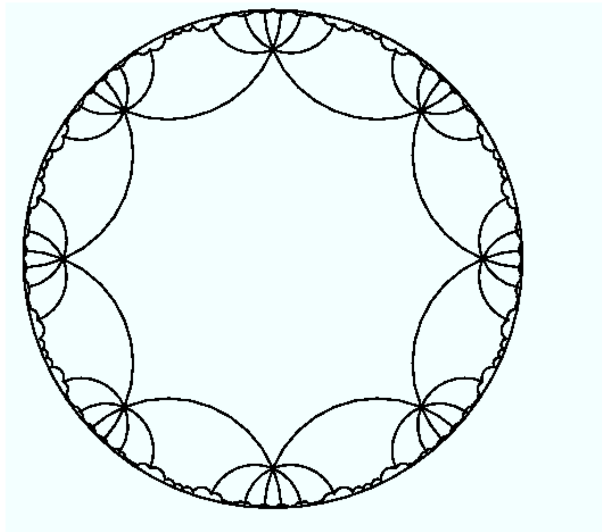
(7) Let $G = \mathbb{Z} * \mathbb{Z}, S = \{\pm\alpha, \pm\beta\}$. $\Gamma(G,S)$ is just a regular tree of degree 4. It is easy to see that $\Gamma(G,S)$ is just the universal cover of $S^1 \vee S^1$. A nice way to view the cover is to embed it in the Poincaré disc. This does not have one end and the growth function is $f_S(R) = 2 \cdot 3^R - 1, R \geq 1$ is exponential.

In general, set $G = *_{i=1}^k \mathbb{Z}$, the free group on k generators. Let $S = \{\alpha_i^{\pm 1} \mid i = 1, \ldots, k\}$ generate G. Then (for $R \geq 1$)

$$f_S(R) = 1 + 2k(2k-1) + 2k(2k-1)^2 + \ldots + 2k(2k-1)^{R-1} = (\frac{k}{k-1})(2k-1)^R - \frac{1}{k-1}$$

has exponential growth.

(8) Let G be the fundamental group of a closed oriented surface Σ of genus 2, i.e, $G = \{a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1}$ $1\}$ with $S = \{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}\}$. The universal cover of Σ is the Poincaré disc, which can be tiled by octagons in a regular fashion.



The dual to the 1-skeleton of this tiling gives us an embedding of $\Gamma(G,S)$. The growth function $f_S(R)$ is bounded above by the growth function of a free group on 4 generators. Since $\mathbb{Z}*\mathbb{Z}$ is a subgroup of G, f_S is bounded below by the growth function of $\mathbb{Z}*\mathbb{Z}$. Thus,

$$2\cdot 3^R - 1 \leq f_S(R) \leq \frac{4\cdot 7^R - 1}{3}$$

and G has exponential growth.

(9) Let G be the *integer Heisenberg group*, consisting of 3×3 upper triangular matrices with 1's on the diagonal. This group is nilpotent and it can be shown that the growth function is quartic.

As seen in the examples, the growth function helps in distinguishing between free groups and abelian groups. If H is a subgroup of finite index in G and hence necessarily finitely generated, then the growth functions of H and G are equivalent. Using the fact that f.g nilpotent groups are of polynomial growth, we conclude that if a f.g group G has a nilpotent subgroup H of finite index then G has polynomial growth.

Theorem 1.6. (Gromov '81)
If a f.g group G has polynomial growth then it contains a nilpotent subgroup of finite index.

It also follows that for a generating set $S=\{s_i^{\pm 1} | i=1,\ldots,k\}$ of G,

$$f_S(R) \leq \sum_{i=0}^{R} (2k)^i \leq (2k+1)^R.$$

Thus, for any infinite group G

$$\lim_{R\to\infty} \frac{\ln f_S(R)}{R} \text{ exists.}$$

Hence, G either grows exponentially or subexponentially.

Definition 1.7. A growth function $f:\mathbb{N}\to\mathbb{R}$ is called
(i) *polynomial* if $\lim_{R\to\infty} \frac{\ln f(R)}{\ln R} = \alpha > 0$,
(ii) *superpolynomial* if $\lim_{R\to\infty} \frac{\ln f(R)}{\ln R} = \infty$,

(iii) *exponential* if $\lim_{R\to\infty} \frac{\ln f(R)}{R} = \beta > 0$,
(iv) *subexponential* if $\lim_{R\to\infty} \frac{\ln f(R)}{R} = 0$,
(v) *intermediate* if *(ii)* and *(iv)* holds.

By 1968 it was known that all known classes of groups were of polynomial growth or of exponential growth. Milnor then asked the question if groups of intermediate growth exist. This was answered in the affirmative by Grigorchuk in 1983.

2 Švarc-Milnor Theorem

From the viewpoint of coarse geometry, one tends to visualize metric spaces from a distance, i.e., one studies large-scale behaviour. In this asymptotic sense, a bounded metric space should be coarsely equivalent to a point. Naively, the integral lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$ looks like $\mathbb{R}^n$ as we move further away from it. This point of view is effective because it is often true that relevant geometric properties of metric spaces are determined by their coarse geometry. Two important examples of its use are Gromov's beautiful notion of hyperbolic group and Mostow's proof of his famous rigidity theorem.

Definition 2.1. A map $f : (X, d_X) \to (Y, d_Y)$ is called *coarse Lipschitz* if there exists $L, c > 0$ such that

$$d_Y(f(x_1), f(x_2)) \le L d_X(x_1, x_2) + c$$

for $x_1, x_2 \in X$.

We say $f, g : (X, d_X) \to (Y, d_Y)$ are *equivalent* if there exists $c > 0$ such that

$$d_Y(f(x), g(x)) < c, \; x \in X.$$

An isomorphism in the category of metric spaces and coarse Lipschitz maps up to equivalence is called a *quasi-isometry*, abbreviated *q.i.*

Thus, two spaces (X, d_X) and (Y, d_Y) are q.i if there are coarse Lipschitz maps $f : X \to Y$, $g : Y \to X$ such that $f \circ g, g \circ f$ are bounded distance away from $\mathrm{Id}_Y, \mathrm{Id}_X$ respectively.

Examples

(1) $\iota : \mathbb{Z} \to \mathbb{R}$ is a q.i with the inverse given by the greatest integer function. More generally, $\iota : \mathbb{Z}^n \to \mathbb{R}^n$ is a q.i for any $n > 0$.

(2) Any map $f : \mathbb{Z}_2 \to S^2$ is a q.i since both spaces, being compact, are q.i to a point.

(3) Let G be as in example 9 of §1 with the given embedding of $\Gamma(G, S)$ into the Poincaré disc. This map is a quasi-isometric embedding. It will follow from 2.3 that it is a q.i.

(4) $\mathrm{Id} : (G, d_S) \to (G, d_{S'})$ is a q.i (refer to the proof of 1.3) for any two finite generating sets S, S' of G.

The first three examples can all be rewritten in the form

$$\pi_1(M) \xrightarrow{q.i} \widetilde{M},$$

where $\widetilde{M}$ is the universal cover of M for suitable compact manifolds M. This is indeed true in general and follows from the Švarc-Milnor theorem. However, to prove it we need :

Definition 2.2. A metric space (X, d_X) is called a *geodesic metric space* if for any $x_1, x_2 \in X$, there exists an isometric embedding $\gamma : [a, b] \to (X, d_X)$ such that $\gamma(a) = x_1, \gamma(b) = x_2$.

The space (X, d_X) is called *proper* if the closed ball of any radius around any point is compact.

Observe that in a proper metric space, a subset K is compact if and only if it is closed and bounded. In fact, this property is equivalent to the definition above. It also follows from the Hopf-Rinow theorem that any complete Riemannian manifold M is proper.

Let $\pi : \widetilde{X} \to X$ be a connected cover of a geodesic metric space. We define a metric on $\widetilde{X}$, induced by d_X :

$$d_{\widetilde{X}}(x_1, x_2) := \inf\{l(\pi \circ \gamma) | \gamma : [a, b] \to \widetilde{X}, \gamma(a) = x_1, \gamma(b) = x_2\}.$$

It can be verified that this a local isometry. The space $(\widetilde{X}, d_{\widetilde{X}})$ is not a geodesic metric space in general. However, for a manifold M equipped with a Riemannian metric g (which induces d_M), the induced metric on any connected cover $\pi' : M' \to M$ is just the pull-back $\pi'^* g$ (which induces $d_{M'}$). Hence, M' is complete and proper if M is complete. In particular, the universal cover $\widetilde{M}$ is a proper geodesic space with the induced metric and $\pi_1(M)$ acts by isometries.

Theorem 2.3. (Švarc-Milnor)
Let (X, d_X) be a geodesic metric space. Let G be a group of isometries of X acting properly and co-compactly. Then G is finitely generated and $g \mapsto gx_0$ induces a quasi-isometry $G \xrightarrow{q.i} X$ for any $x_0 \in X$.

There are several implications of this statement, listed as examples below.

Examples

(1) Two geodesic metric spaces X, X' are q.i if there exists a group G that acts properly and co-compactly on both.

(2) For any compact manifold M, $\pi_1(M)$ acts on the universal cover $\widetilde{M}$ properly and co-compactly via isometries. Thus, $\pi_1(M) \xrightarrow{q.i} \widetilde{M}$ for any compact Riemannian manifold.

(3) Since any locally finite graph with edges of length 1 is proper, $\Gamma(G, S)$ is a proper space for any f.g group G. The group G acts on $\Gamma(G, S)$ by isometries and properly discontinuously. The action is compact since the orbit of 1 is co-bounded. Hence, $G \xrightarrow{q.i} \Gamma(G, S)$.

Milnor had also proved results relating the curvature of a manifold to the volume growth of the manifold and its universal cover. This can further be compared with the growth of the fundamental group.

Theorem 2.4. (Milnor '68)
If M^n is a non-compact Riemannian manifold with non-negative Ricci curvature then the growth of any f.g subgroup of $\pi_1(M)$ satisfies $f(R) \leq kR^n$ for some constant k.

Theorem 2.5. (Milnor '68)
If M is a compact Riemannian manifold with negative sectional curvature then its fundamental group $\pi_1(M)$ has exponential growth rate. However, the fundamental group of a compact manifold of negative scalar curvature need not be of exponential growth.

In general most complete, non-compact manifolds have exponential growth. For examples of manifolds with polynomial growth, one can look at complete manifolds with non-negative Ricci curvature, real algebraic submanifolds in $\mathbb{R}^n$ or nilpotent Lie groups. However, the most famous problem in geometric group theory for the last 30 years still remains open.

Conjecture (Milnor)
If M is a complete non-compact manifold with non-negative Ricci curvature then $\pi_1(M)$ is finitely generated.