An Introduction to Chern-Simons Theory

1 Basic Definitions

Recall that a principal G-bundle E over B is a fibration with fibre being a Lie group G and there is a free right (resp. left) action on the total space by elements of G such that this is free and transitive on each fibre. The map $\pi : E \to B$ induces $d\pi : TE \to TB$, where T_eE locally splits into $T_{\pi(e)}B \oplus T_1G$ by local triviality. Thus, $\ker d\pi$ is T_1G which can be canonically identified with $\mathfrak{g}$. For such a bundle, a connection is a 1-form θ on E taking values in $\mathfrak{g}$ and satisfying

$$R_g^*\theta = (\mathrm{ad}_g)^{-1} \circ \theta \tag{1.1}$$

$$\theta(v) = v_{\mathfrak{g}},\ v \in \ker d\pi. \tag{1.2}$$

Here the right action R_g acts by g on the right and $v_{\mathfrak{g}}$ refers to the identification of v with an element $v_{\mathfrak{g}}$ of $\mathfrak{g}$ as mentioned before. In particular, this means that if θ_x denote the Maurer-Cartan form on $E_x = G$ and $i_x : E_x \hookrightarrow E$, then $i_x^*(\theta) = \theta_x$. Also recall the structure equation

$$d\theta_x + \frac{1}{2}[\theta_x, \theta_x] = 0. \tag{1.3}$$

It follows from elementary obstruction theory that

Proposition 1.1. *Let $E \to B$ be as above such that G is connected, simply connected and compact. Let B be a manifold of dimension at most 3. Then the bundle is trivial.*

In what follows, let M be a closed, connected and oriented 3-manifold and let E be a principal $SU(2)$-bundle on M. We fix a trivialization on E. Let

$$\Omega^i(M, su_2) = \Gamma(M, \wedge^i T^*M \otimes (M \times su_2))$$

denote the space of forms on M with values in su_2. There are two obvious ways to extend $a \in \Omega^1(M, su_2)$ to a 1-form on $E = M \times SU(2)$:

(1) using (1.1) and declaring it to be the Maurer-Cartan form on each fibre,

(2) just extend to each copy of $M \times \{g\}$ by (1.1).

Thus, the space of connections on any (principal) $SU(2)$-bundle over M (making use of (1) above) is just $\mathcal{A}(M) = \Omega^1(M, su_2)$ - the space of su_2-valued 1-forms on M. The curvature of a connection $a \in \mathcal{A}(M)$ is defined to be

$$F_a = da + a \wedge a \in \Omega^2(M, su_2). \tag{1.4}$$

We observe here that

$$(a \wedge a)(X, Y) = \frac{1}{2}\big(a(X)a(Y) - a(Y)a(X)\big) = \frac{1}{2}[a(X), a(Y)]$$

and

$$R_g^* F_a = \mathrm{ad}_{g^{-1}} F_a \tag{1.5}$$

$$i_x^* F_a = 0. \tag{1.6}$$

The last equation follows from (1.3). We call a connection *flat* if the curvature is zero. The *Bianchi identity* follows by differentiating F_a :

$$dF_a + [a, F_a] = 0. \tag{1.7}$$

Introduce the covariant derivative

$$d_a = d + \mathrm{ad}(a). \tag{1.8}$$

This induces the complex

$$\Omega^0(M, su_2) \xrightarrow{d_a} \Omega^1(M, su_2) \xrightarrow{d_a} \Omega^2(M, su_2) \xrightarrow{d_a} \Omega^3(M, su_2). \tag{1.9}$$

The failure of the above to be a chain complex, i.e., $d_a^2 = 0$, is measured by the flatness of a since $d_a^2 = F_a$.

2 The Chern-Simons functional

Definition 2.1. For $a \in \mathcal{A}(M)$, choose an oriented 4-manifold W which bounds M and collar neighbourhood of the boundary

$$\partial W \subset U = M \times [0,1].$$

Take any $\tilde{a} \in \mathcal{A}(W)$ such that it is the pullback of a near the boundary. The **Chern-Simons functional** is then defined to be

$$CS(a) = \frac{1}{8\pi^2}\int_W tr(F_{\tilde{a}} \wedge F_{\tilde{a}}). \tag{2.1}$$

This is well defined for two reasons :

(1) Since any closed oriented 3-manifold is cobordant to a point, such a W (as above) always exist. This is a result by R. Thom who originally proved it by calculating the stable homotopy groups of Thom spaces up to 3 using the cohomology structure and homotopy theory. Later Likorish gave another proof in the 60's using Heegard decompositions and the fact that homeomorphisms of surfaces are generated by Dehn twists.

(2) The Chern-Simons functional is well defined because if we take two such 4-manifolds W_1, W_2 with collar neighbourhoods U_1, U_2 and connections a_1, a_2 respectively, then gluing these two give a closed oriented 4-manifold $\tilde{W}$ with a connection $\tilde{a}$. Since $F_{\tilde{a}} \in \Omega^2(M, su_2)$, let

$$F_{\tilde{a}} = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix},\ w_1 + w_4 = 0,\ w_2 + \overline{w_3} = 0,$$

where w_i's are complex valued 2-forms on M. The equivalence class of the coefficient of t^j in the expansion of $\det(I + \frac{ti}{2\pi}F_{\tilde{a}})$ is the Chern class c_j. The expansion is

$$1 + \frac{it}{2\pi}(w_1 + w_4) + \frac{t^2}{4\pi^2}(w_2 \wedge w_3 - w_1 \wedge w_4),$$

whence $c_2 = [(w_2 \wedge w_3 - w_1 \wedge w_4)/4\pi^2]$. On the other hand,

$$\begin{aligned} \frac{1}{8\pi^2}tr(F_{\tilde{a}} \wedge F_{\tilde{a}}) &= \frac{1}{8\pi^2}tr\begin{pmatrix} w_1 \wedge w_1 + w_2 \wedge w_3 & w_1 \wedge w_2 + w_2 \wedge w_4 \\ w_3 \wedge w_1 + w_4 \wedge w_3 & w_3 \wedge w_2 + w_4 \wedge w_4 \end{pmatrix} \\ &= \frac{w_1 \wedge w_1 + w_2 \wedge w_3 + w_3 \wedge w_2 + w_4 \wedge w_4}{8\pi^2} \\ &= \frac{w_2 \wedge w_3 - w_1 \wedge w_4}{4\pi^2}. \end{aligned}$$

Since the $SU(2)$-bundle is trivial c_2 is exact, i.e., $(w_2 \wedge w_3 - w_1 \wedge w_4)/4\pi^2 = \delta\alpha$. Therefore

$$CS(a_1) - CS(a_2) = \int_{\tilde{W}} c_2 = \int_{\partial\tilde{W}} \alpha = 0.$$

Actually, there is an explicit expression for the Chern-Simons functional :

Lemma 2.2.

$$CS(a) = \frac{1}{8\pi^2}\int_M tr(a \wedge da + \frac{2}{3}a \wedge a \wedge a). \tag{2.2}$$

Proof Let $(a_t)_{t\in[0,1]}$ be a path of connections; it determines a connection $a \in \mathcal{A}(M \times [0,1])$. It follows easily from definition that

$$CS(a_1) - CS(a_0) = \frac{1}{8\pi^2}\int_{M\times[0,1]} tr(F_a \wedge F_a).$$

Now take $a_t = ta$. Then

$$F_a = dt \wedge a + t\, da + t^2 a \wedge a$$

and

$$\begin{aligned} tr(F_a \wedge F_a) &= tr(t\, dt \wedge a \wedge da + t^2\, dt \wedge a \wedge a \wedge a + t\, dt \wedge da \wedge a + t^2\, da \wedge da) \\ &\quad + tr(t^3\, da \wedge a \wedge a + t^2\, dt\, a \wedge a \wedge a + t^3\, a \wedge a \wedge da) \\ &= 2t\, dt \wedge tr(a \wedge da) + t^2\, tr(da \wedge da) + 2t^2 dt \wedge tr(a \wedge a \wedge a) \\ &\quad + 2t^3 tr(da \wedge a \wedge a) \\ &= d\big(tr(t^2 a \wedge da + \frac{2}{3}t^3 a \wedge a \wedge a)\big). \end{aligned}$$

The proof is complete by using Stokes' theorem. □

Reverting back to thinking of a as a connection on the trivial $SU(2)$-bundle on M, we shall see that $\frac{1}{8\pi^2}tr(a\, da + \frac{2}{3}a \wedge a \wedge a)$ is a 3-form on the total space which restricts to the volume form on each fibre. This will be called the Chern-Simons form and denoted by $CS_3(a)$. It is also true (as seen by simply differentiating) that this form is the anti derivative of $\frac{1}{8\pi^2}tr(F_a \wedge F_a)$. We remark in passing that the trace appears since

$$tr : su_2 \otimes su_2 \to \mathbb{R}, \quad A \otimes B \mapsto tr(AB)$$

is the (non-degenerate) Killing form on $SU(2)$.

The differential of the Chern-Simons functional is

$$\begin{aligned} d\, CS_a(b) &= \frac{d}{dt}\Big|_{t=0} CS(a + tb) \\ &= \lim_{t\to 0}\frac{1}{8\pi^2 t}\int_M tr\Big(\frac{2t}{3}(a \wedge b \wedge a + a \wedge a \wedge b + b \wedge a \wedge a)\Big) \\ &\quad + \lim_{t\to 0}\frac{1}{8\pi^2 t}\int_M tr\Big(\frac{2t^2}{3}(a \wedge b \wedge b + b \wedge a \wedge b + b \wedge b \wedge a)\Big) \\ &\quad + \lim_{t\to 0}\frac{1}{8\pi^2 t}\int_M tr\Big(\frac{2t^3}{3} b \wedge b \wedge b\Big) + \lim_{t\to 0}\frac{1}{8\pi^2 t}\int_M tr\big(t(a \wedge db + b \wedge da)\big) \\ &= \frac{1}{8\pi^2}\int_M tr(b \wedge da + a \wedge db + 2b \wedge a \wedge a) \end{aligned}$$

For any two connections a, b, it follows from definition that $tr(a \wedge b) = 0$; then

$$0 = d\, tr(a \wedge b) = tr(da \wedge b - a \wedge db),$$

which implies $tr(a \wedge db) = tr(da \wedge b) = tr(b \wedge da)$. Thus

$$d\, CS_a(b) = \frac{1}{4\pi^2}\int_M tr(b \wedge F_a). \tag{2.3}$$

In particular, the critical points of the Chern-Simons functional $\mathrm{Crit}(CS) = \mathcal{A}^{\text{flat}}(M)$ is the space of flat connections, i.e., those with curvature $F_a = 0$. The Hessian is

$$\begin{aligned}
\mathrm{Hess}(CS)_a(b,c) &= \frac{\partial^2\, CS_a}{\partial b\, \partial c} \\
&= \frac{\partial}{\partial b}\Big(\frac{1}{4\pi^2}\int_M tr(c \wedge F_a)\Big) \\
&= \lim_{t\to 0} \frac{1}{4\pi^2 t}\int_M tr\big(c \wedge (F_{a+tb} - F_a)\big) \\
&= \lim_{t\to 0} \frac{1}{4\pi^2 t}\int_M tr\big(c \wedge (t\, db + t\, b \wedge a + t\, a \wedge b + t^2 b \wedge b)\big) \\
&= \frac{1}{4\pi^2}\int_M tr(c \wedge db + c \wedge b \wedge a + c \wedge a \wedge b),
\end{aligned}$$

Since $d_a b = db + b \wedge a + a \wedge b$ (as defined in (1.8)), we get

$$\mathrm{Hess}(CS)_a(b,c) = \frac{1}{4\pi^2}\int_M tr(c \wedge d_a b). \tag{2.4}$$

It is easily verified that $\mathrm{Hess}(CS)_a$ is symmetric.

Since $\ker d_a$ is infinite dimensional, the Hessian is degenerate and CS is not Morse. This is due to the presence of an infinite dimensional symmetry group called the group of gauge transformations (see §3).

3 Gauge Symmetry

The group of *gauge transformations* $\mathcal{G}(M)$ is the space of smooth maps from M to $SU(2)$, with point wise multiplication as the group action. An element $g \in \mathcal{G}(M)$ of the gauge group defines a diffeomorphism of the trivial $SU(2)$-bundle E over M by the right action on each fibre; we shall denote this diffeomorphism also by g. Thus g acts on $a \in \mathcal{A}(M)$ by $g \cdot a = g^*a$. Actually, there is an explicit formula for the pullback :

$$g \cdot a = gag^{-1} - dg\, g^{-1}. \tag{3.1}$$

It is clear that g^* is natural, i.e., commutes with wedge products and d. In particular, the pullback of the CS 3-form arising from a is just the CS 3-form generated by $g \cdot a$.

Two elements $g_1, g_2 \in \mathcal{G}(M)$ are (path) connected if and only if $\deg g_1 = \deg g_2$ since the target space for both maps $g_i : M \to SU(2)$ is S^3. Consequently, the connected component of $\mathcal{G}(M)$ is $\mathcal{G}_0(M)$, consisting of elements of degree 0. The cosets of $\mathcal{G}_0(M)$ are the elements of $\pi_0(\mathcal{G}(M))$ and there is a natural injective homomorphism $\deg : \pi_0(\mathcal{G}(M)) \to \mathbb{Z}$ such that

$$[g] \to \deg g.$$

This map is an isomorphism since there exist maps of any given degree. It is also known that that

$$CS(g \cdot a) = CS(a) + \deg g. \tag{3.2}$$

There is a nice geometric way to see this - one can think of M as the identity section, i.e., $M \times \{1\}$ in the total space E. The CS form coming from a is integrated over this section to give $CS(a)$. The map $g : E \to E$ maps the homology class $[M]$ corresponding to the identity section to $g_*[M]$. Since $CS_3(g \cdot a) = g^*(CS_3(a))$,

$$CS(g \cdot a) = \int_M CS_3(g \cdot a) = \int_{g_*(M)} CS_3(a).$$

We can perturb g (degree remaining fixed) such that $g_*(M) \pitchfork M$. Since $\dim E = 6$, $S = g_*(M) \cap M$ is of dimension 0 and consists of finitely many points. Let $S = \{x_1, \ldots, x_k\}$ and for ε sufficiently small, fix mutually disjoint balls $B_\varepsilon(x_i)$ in M. Then the section given by g on M on the complement of these neighbourhoods can be homotoped to $M \times \{h\}, h \neq 1$ since $SU(2) \setminus \{1\}$ is path connected. Homotope the g-section on $B_\varepsilon(x_i) \setminus B_{\varepsilon/2}(x_i)$ to a section which is given by g near $B_{\varepsilon/2}(x_i)$ and is given by $M \times \{g\}$ near $B_\varepsilon(x_i)$. Denote this new section by $\tilde{g}$. Then $\tilde{g}_*M \pitchfork M$ and

$$\int_{g_*M} CS_3(a) = \int_{\tilde{g}_*M} CS_3(a).$$

But as $\varepsilon \to 0$ this process splits the (smooth) integrating space g_*M into the union of the trivial section $M \times \{h\}$ (which is homotopic to $M \times \{1\}$) and $\{x_i\} \times E_{x_i}$, in the limit. Thus

$$\begin{aligned} CS(g \cdot a) &= \int_M CS_3(a) + \sum_{i=1}^{k} \int_{\pm E_{x_i}} CS_3(a) \\ &= CS(a) + \sum_{i=1}^{k} \pm 1 \\ &= CS(a) + \deg g. \end{aligned}$$

A few explanations are in order. Here $\pm E_{x_i}$ means that the CS 3-form, which is the volume form when restricted to a fibre, is integrated on $E_{x_i} = G$ with the orientation preserved/reversed respectively. Hence the integral contributes ± 1 accordingly. Thus, changing the section by a map into the group $SU(2)$ changes the integral by the degree of the map of M into $SU(2)$.

It follows from (3.2) that CS is not well-defined on $\mathcal{B}(M) := \mathcal{A}(M)/\mathcal{G}(M)$ but it is on its infinite cyclic cover $\tilde{\mathcal{B}}(M) := \mathcal{A}(M)/\mathcal{G}^0(M)$. Observe that

$$F_{g\cdot a} = gF_a g^{-1}$$

and hence flat connections are preserved by the gauge group. Thus, we have a function

$$CS : \tilde{\mathcal{B}}(M) \to \mathbb{R} \tag{3.3}$$

whose critical points are the equivalence classes of flat connections. This function is not Morse for reasons discussed before. This leads naturally to questions on suitable perturbations of this function so that it is Morse, allowing us to do Floer homology.

Caution : The quotient spaces $\tilde{\mathcal{B}}(M)$ etc. are actually the quotients of completions of $\mathcal{A}(M), \mathcal{G}(M)$ etc. under suitable Sobolev norms.