The maximal tori theorem of É. Cartan

1 Introduction : Basic definitions

For an abelian group, all the irreducible representations are 1 dimensional and therefore are characters. In particular, for $S^1 = U(1)$ the characters are labelled by integers, i.e.,

$$\chi_n(e^{i\theta}) = e^{in\theta},\ n \in \mathbb{Z}.$$

A group $T = T^k$ consisting of a product of k copies of S^1 will be called a **torus**.

Definition *A maximal torus $T \subset G$ is a subgroup which is a torus and is maximal among all the tori contained in G.*

The reason for working with the quotient spaces is the following : for a compact, connected Lie group G, the space G/T helps one relate the representation theory of a maximal torus T and that of G. Since for T^k the representations are labelled by k-tuples of integers, this is completely known. Moreover, while studying characters of G, it suffices to work with its restriction to T since characters are class functions and the theorem states that T intersects any conjugacy class. The space G/T is also interesting, by itself, in a number of ways. These are Kähler manifolds and projective algebraic varieties as well. Sometimes it can also be thought of as "flag manifolds". For instance, when $G = U(n)$ and $T =$Diag$(e^{i\theta_1}, \ldots, e^{i\theta_n})$, the space G/T is the space of complex flags in $\mathbb{C}^n$, i.e., chains of inclusions

$$\mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^n.$$

Remark For G non-compact there may not be any non-trivial tori.

Recollect that a connected abelian Lie group A is of the form $T^k \times \mathbb{R}^l$. Thus any compact connected abelian Lie group must be a torus. Let G be a connected compact Lie group. Choose a vector in $\mathfrak{g}$ and exponentiate to get a 1 parameter subgroup H of G which is connected and abelian. Now $\overline{H}$ is compact, connected and abelian and hence a torus. Thus maximal torus exists. Let T be a maximal torus in what follows. It will be shown later that

Proposition *A maximal torus is a maximal abelian subgroup.*

The converse is not true in general. For example, let $G = SO(n)$, $n \geq 3$ and consider the subgroup of matrices

$$\begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}.$$

This forms a maximal abelian subgroup that is not a torus. The reason for the failure is connectedness. And in fact, there is a bijection between maximal tori and connected maximal abelian subgroups.

We define

Definition *The Weyl group $W(G,T)$ is the group of automorphisms of T which are the restrictions of inner automorphisms of G.*

Any such automorphism has the form $t \mapsto ntn^{-1}$, $n \in N(T)$. $N(T)$, being a closed subgroup of G, is compact. Let $Z(T)$ be the centralizer of T. This is also closed and $T \subseteq Z(T) \subset N(T)$. $N(T)$ maps onto $N(T)/Z(T) \cong W(G,T)$. Now we claim that $N(T)_e = T$. This will show two things
(i) the number of cosets of $N(T)_e$ in $N(T)$ is finite by compactness of $N(T)$, and
(ii) the Weyl group is finite.
We define a continuous map

$$\varphi : N \to \mathrm{Aut}(T),\ \varphi(n)(t) = ntn^{-1}$$

arising from the map $N \times T \to T$, which is a restriction of the map $G \times G \to G$, $(g,h) \mapsto ghg^{-1}$. Since $\mathrm{Aut}(T)$ is discrete, $N(T)_e$ acts trivially. If $N(T)_e$ properly contains T then choose a 1-parameter subgroup H not contained in T. The subgroup (T,H) generated by H and T is connected and abelian. Since $T \subset (T,H) \subseteq \overline{(T,H)} \subseteq N(T)_e$ this forces $\overline{(T,H)}$ to be a compact, connected and abelian group and hence a torus, contradicting the maximality of T. Thus $N(T)_e = T$. As we shall see later $W(G,T)$ is independent of the choice of T since any two maximal tori are conjugates in G. Consequently, the resulting Weyl groups are isomorphic. Later, it will follow from the topological proof of the theorem that the Euler characteristic of G/T is $|W(G,T)|$.
A torus $T \subset G$ (not necessarily maximal) acts on $\mathfrak{g}$ by the restriction of the map $Ad : G \to \mathrm{Aut}\,\mathfrak{g}$. Choose a positive definite symmetric form on $\mathfrak{g}$ bi-invariant under G (and hence Ad invariant). Then $\mathfrak{g}$ splits into orthogonal irreducible representations of T, which are of dimesnions 1 and 2. The ones of dimension 1 are trivial. For the ones of dimension of 2, choose an orthonormal basis and represent T by $SO(2)$. Thus,

$$\mathfrak{g} = V_0 \oplus (\oplus_1^m V_j),$$

where T acts trivially on V_0 and it acts on V_j ($\dim V_j = 2$, $j > 0$) as

$$\begin{pmatrix} \cos 2\pi\theta_j(t) & -\sin 2\pi\theta_j(t) \\ \sin 2\pi\theta_j(t) & \cos 2\pi\theta_j(t) \end{pmatrix},$$

where $\theta_j(t) : T \to \mathbb{R}/\mathbb{Z}$ is to be thought of as a non-zero map $\theta_j : \mathfrak{t} \to \mathbb{R}$ taking integer values on the kernel of the (surjective) exponential map $Exp : \mathfrak{t} \to T$. Let

$$\phi_j : T \to SO(2)$$

denote the surjective map above. The kernel is a codimension 1 closed Lie subgroup of T. Denote the complement of the union of $\ker(\phi_j), 1 \leq j \leq m$, which is an open dense submanifold of T, by M. Of course any (topological) generator of T lies in M since any ϕ_j vanishing on a generator vanishes on T, contradicting the non-triviality of the representation of T on V_j.

Definition For a maximal torus T, the functions $\pm\theta_j$ are called the roots of G.

It's easy to see that this is well-defined. If we change to another orthonormal basis of V_j then the change of basis matrix $P \in O(2)$. Since T acts on V_j by conjugation, a simple calculation shows that

$$P \begin{pmatrix} \cos 2\pi\theta_j(t) & -\sin 2\pi\theta_j(t) \\ \sin 2\pi\theta_j(t) & \cos 2\pi\theta_j(t) \end{pmatrix} P^{-1} = \begin{pmatrix} \cos 2\pi\theta_j(t) & -\sin 2\pi\theta_j(t) \\ \sin 2\pi\theta_j(t) & \cos 2\pi\theta_j(t) \end{pmatrix}.$$

This proves that the roots are well defined. We will see that they are independent of T. We also have the easy

Proposition T is maximal if and only if $V_0 = \mathfrak{t}$.

Proof Since T is abelian, it acts trivially on $\mathfrak{t}$ via Ad. Hence $\mathfrak{t} \subset V_0$. Suppose $\mathfrak{t} = V_0$ and $T \subset T'$. Then

$$\mathfrak{t} \subset \mathfrak{t}' \subset V_0' \subset V_0$$

implies $\mathfrak{t} = \mathfrak{t}'$ and $T = T'$. Conversely suppose $V_0 \neq \mathfrak{t}$ with T maximal. Then choose $X \in V_0, X \notin \mathfrak{t}$. The one parameter subgroup $H = \{exp(tX)|t \in \mathbb{R}\}$ acts trivially and is not contained in T. Therefore the closure $\overline{(T, H)}$ of the subgroup (T, H) generated by T and H is a compact, connected abelian subgroup strictly containing T, a contradiction. □

As a corollary we see that dim G - dim $T = 2m$ is even. Let $F(T, G)$ denote the fixed point set of the adjoint action of T on G. We note that $V_0 = \mathfrak{t}$ holds if and only if $F(T, G)$ has T as one of its connected component. Before embarking on the proof of the main result, let's study some examples of maximal tori.

2 Examples

The unitary groups, the spin groups and the orthogonal groups will be our main class of examples.

Example 1 Let $G = U(n)$ and $T = \mathrm{Diag}(e^{2\pi i x_1}, \ldots, e^{2\pi i x_n})$. The Lie algebra $\mathfrak{u}_n$ can be decomposed into a direct summand of

1. $\mathfrak{t}$ - matrices in $\mathrm{Diag}(id_1, \ldots, id_n)$ with d_j's real.
2. $M_{rs}, r < s$ - matrices with z in the (r,s)th entry, $-\bar{z}$ in the (s,r)th entry and zero elsewhere.

One can easily check that $D = \mathrm{Diag}(e^{2\pi i x_1}, \ldots, e^{2\pi i x_n})$ maps the linear space spanned by M_{rs} to itself. More precisely, $DM_{rs}(z)D^{-1} = M_{rs}(w)$ with $w = e^{2\pi i(x_r - x_s)}z$. So the roots are $\pm\theta_{rs}$ where $\theta_{rs} = x_r - x_s, r < s$. This also shows that T is maximal since $V_0 = \mathfrak{t}$. The corresponding Weyl group is the symmetric group Σ_n, acting on T by permuting the x_j's and consequently the roots. Therefore $\chi(G/T) = n!$.

Example 2 Let $G = SU(n)$ and $T = \mathrm{Diag}(e^{2\pi i x_1}, \ldots, e^{2\pi i x_n})$ with $\sum_1^n x_i = 0$. Everything works as in the previous example but with the restriction that the trace be zero for elements of $\mathfrak{su}_n$. The functions $x_r - x_s$ are still non-trivial, so $V_0 = \mathfrak{t}$, T is maximal and the roots are $(x_r - x_s)$. If we specialize to :

Example 3 $G = SU(2)$ and T to be the $U(1)$ subgroup of matrices of the form

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix},$$

one can easily check that $Z(T) = T$. The space G/T is $\mathbb{C},\nVdash\mathbb{P}^{=}S^2$ and the map $T \hookrightarrow G \to G/T$ is the Hopf fibration. One also verifies that

$$N(T) = T \cup \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T.$$

Thus the connected component of identity is T and $W(G,T)$ is the group of two elements and $\chi(S^2) = 2$, which is already known.

Example 4 Let $G = Sp(n)$ and $T = \mathrm{Diag}(e^{2\pi i x_1}, \ldots, e^{2\pi i x_n})$. The Lie algebra $\mathfrak{sp}_n$ can be decomposed into a direct summand of

1. $\mathfrak{t}$ - matrices in $\mathrm{Diag}(id_1, \ldots, id_n)$ with d_j's real.
2. $M_{rs}, r < s$ - matrices with z in the (r,s)th entry, $-\bar{z}$ in the (s,r)th entry and zero elsewhere.
3. N_r - matrices with zj in the (r,r)th entry, i.e., $zj\delta_{ir\,jr}$.
4. $P_{rs}, r < s$ - matrices with zj in the (r,s) and (s,r)th entry and zero elsewhere.

As before, one can check that $D = \mathrm{Diag}(e^{2\pi i x_1}, \ldots, e^{2\pi i x_n})$ satisfies the following :

$$\begin{aligned} DM_{rs}D^{-1} &= e^{2\pi i(x_r - x_s)}M_{rs}, \\ DN_rD^{-1} &= e^{4\pi i x_r}N_r, \\ DP_{rs}D^{-1} &= e^{2\pi i(x_r + x_s)}P_{rs}. \end{aligned}$$

Thus $V_0 = \mathfrak{t}$, T is a maximal torus and the roots are $\pm 2x_r$, $(x_r - x_s)$ and $\pm(x_r + x_s)$ for $r \neq s$.

Example 5 Let $G = SO(2n)$ and T to be the image of the maximal torus we had in $U(n)$ (since $U(n) \hookrightarrow SO(2n)$), i.e. T is the set of matrices having 2×2 minors along the diagonal

$$D = \begin{pmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_n \end{pmatrix}$$

where the D_j's are given by

$$\begin{pmatrix} \cos 2\pi x_j & -\sin 2\pi x_j \\ \sin 2\pi x_j & \cos 2\pi x_j \end{pmatrix}.$$

The Lie algebra $\mathfrak{so}_{2n}$ splits into the following summands :

1. $\mathfrak{t}$ - consisting of

$$\begin{pmatrix} \begin{array}{cc|} 0 & -d_1 \\ d_1 & 0 \\ \hline \end{array} & & \\ & \ddots & \\ & & \begin{array}{|cc} \hline 0 & -d_n \\ d_n & 0 \end{array} \end{pmatrix}.$$

2. $M_{rs}, r < s$ - the image of corresponding matrices in $\mathfrak{u}_n$ with a 2×2 matrix W in the (r, s)th entry, $-W^T$ in the (s, r)th entry and zero elsewhere with

$$W = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

3. $E_s M_{rs} E_s^{-1}, r < s$ - the matrix E_s is given by

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \boxed{\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}.$$

The matrix $E_s M_{rs} E_s^{-1}$ looks like M_{rs} with W replaced by

$$V = \begin{pmatrix} x & -y \\ -y & -x \end{pmatrix}.$$

One can verify that T acts on M_{rs} by $\theta_{rs} = x_r - x_s$ and acts on matrices of the form $E_s M_{rs} E_s^{-1}$ by $\theta_{rs} = x_r + x_s$. Thus $V_0 = \mathfrak{t}$, T is maximal and the roots are $(x_r - x_s)$, $\pm(x_r + x_s)$ for $r \neq s$.

3 The Lefschetz fixed point theorem

The (topological) proof of Cartan's theorem relies on the smooth version of the fixed point theorem. For the sake of clarity we shall state both the versions. For a simplicial complex X whose homology groups are finitely generated and vanish in higher dimensions, we define the **Lefschetz number**

$$L(f) = \sum_{i=1}^{n} (-1)^i \text{tr}(f_* : H_i(X) \to H_i(X)).$$

Since $H_i(X)$ (with $\mathbb{Z}$ coefficients) might have torsion, we may also define this number in terms of the homology groups with $\mathbb{Q}$ coefficients instead. Then by the universal coefficient theorem, the split exact sequence

$$0 \to \text{Ext}(H_{n-1}(X), \mathbb{Q}) \to H^n(X; \mathbb{Q}) \to \text{Hom}(H_n(X), \mathbb{Q}) \to 0$$

has the first term zero. The isomorphism between $H^n(X;\mathbb{Q})$ and $\text{Hom}(H_n(X), \mathbb{Q})$ is natural. If f_* acts on $H_n(X;\mathbb{Q})$ by a matrix A then f^* acts on $H^n(X;\mathbb{Q})$ by A^t. Since this doesn't affect the trace, we have

$$L(f) = \sum_{i=1}^{n} (-1)^i \text{tr}(f^* : H^i(X; \mathbb{Q}) \to H^i(X; \mathbb{Q})).$$

Now one can state the

Lefschetz fixed point theroem Let $f : X \to X$ be a continuous map of a finite simplicial complex. If $L(f) \neq 0$ then f has a fixed point.

This holds more generally for any retract of a finite simplicial complex since the trace remains unaltered under the retract and so does the fixed point set. It is part of the theory that the homology groups, the cohomology groups and the Lefschetz number are invariants of subdivisions and $L(f)$ is homotopy invariant. For a quick proof of this theorem, suppose that f has no fixed point. Then by sufficiently small subdivisions one can assume that the image of the simplices land outside itself. One can homotope f to a map g such that for cellular homology the traces are all zero. By the equivalence of cellular and simplicial homology, the traces are zero and so is $L(f)$.
For the version we shall use, let $f : X \to X$ be a smooth map of a compact oriented (smooth) manifold. Let Δ denote the diagonal of $X \times X$ and Γ_f denote the grapf of f, i.e., the submanifold $\{(x, f(x)) | x \in X\} \subset X \times X$. It is easily verified that $\Gamma_f \pitchfork \Delta$ if and only if $\det(I - df)_x \neq 0, x \in \text{Fix}(f)$. We call such a map **transversal** and define

$$\text{ind}(x) = \begin{cases} 1 & \text{if det } (I - df)_x > 0 \\ -1 & \text{if det } (I - df)_x < 0 \end{cases}$$

for each fixed point of f. The index of each fixed point just measures the degree of the induced map of the surrounding sphere. The fixed point set is discrete since it is an embedded 0 dimensional submanifold of $X \times X$. It is finite as X is compact. We now define

Definition The index of f, $\Lambda(f) := \sum_{\text{Fix}(f)} \text{ind}(x)$.

We can now state the revised

C^∞ **Lefschetz fixed point theorem** Let $f : X \to X$ be a transversal map. Then $\Lambda(f) = L(f)$.

It's obvious that this trivially implies the previous theorem. For the sake of completeness we provide an outline of the proof.

Proof We shall work with de Rham cohomology $H^*(X,\mathbb{R})$ instead of $\mathbb{Q}$ coefficients. Let $\{\omega_i\}$ be a basis of the vector space $H^*(X)$ and $\{\tau_j\}$ be the dual basis under Poincaré duality. Let π_1, π_2 denote the two projections of $X \times X$ to X. By the Kunneth formula, $\{\pi_1^*\omega_i \wedge \pi_2^*\tau_j\}$ is a basis for $H^*(X \times X) = H^*(X) \otimes H^*(X)$. The Poincaré dual η_Δ of the diagonal Δ in $X \times X$ is some linear combination of $\pi_1^*\omega_i \wedge \pi_2^*\tau_j$'s. By computing in two ways (we shall see an exactly similar calculation for η_Γ) one can show that

$$\eta_\Delta = \sum (-1)^{|\omega_i|} \pi_1^*\omega_i \wedge \pi_2^*\tau_i.$$

Similarly one can write the Poincaré dual η_Γ of Γ_f as

$$\eta_\Gamma = \sum_{i,j} c_{ij} \pi_1^*\omega_i \wedge \pi_2^*\tau_j.$$

We integrate $\pi_1^*\tau_k \wedge \pi_2^*\omega_l$ over Γ_f in two ways. Let $\alpha : X \to \Gamma_f$ be the map $x \mapsto (x, f(x))$. Then $\pi_1 \circ \alpha : X \to X$ is the the identity map and the pullback on forms is the same. Also $\pi_2 \circ \alpha : X \to X$ is f and the pullback on forms is just f^*. Let $f^*(\omega_i) = \sum_{i,j} a_{ji}\omega_j$ where the sum runs over ω_j's such that $|\omega_i| = |\omega_j|$. Then

$$\int_{\Gamma_f} \pi_1^*\tau_k \wedge \pi_2^*\omega_l = \int_X \alpha^*\pi_1^*\tau_k \wedge \alpha^*\pi_2^*\omega_l = \int_X \tau_k \wedge f^*(\omega_l) = (-1)^{|\omega_l||\tau_k|} a_{kl}.$$

On the other hand, by the definition of the Poincaré dual of a closed oriented submanifold,

$$\begin{aligned}\int_{\Gamma_f} \pi_1^*\tau_k \wedge \pi_2^*\omega_l &= \int_{X\times X} \pi_1^*\tau_k \wedge \pi_2^*\omega_l \wedge \eta_\Gamma \\ &= \sum_{i,j} c_{ij} \int_{X\times X} \pi_1^*\tau_k \wedge \pi_2^*\omega_l \wedge \pi_1^*\omega_i \wedge \pi_2^*\tau_j \\ &= \sum_{i,j} (-1)^{|\omega_i|(|\omega_l|+|\tau_k|)} c_{ij} \int_{X\times X} \pi_1^*(\omega_i \wedge \tau_k) \wedge \pi_2^*(\omega_l \wedge \tau_j) \\ &= (-1)^{|\omega_l|(|\omega_l|+|\tau_k|)} c_{kl}.\end{aligned}$$

Thus $c_{kl} = (-1)^{|\omega_l|} a_{kl}$. Therefore, using $\iota : X \to \Delta$ and $\pi_1 \circ \iota = \pi_2 \circ \iota = \mathrm{Id}$, we have

$$\int_\Delta \eta_\Gamma = \sum (-1)^{|\omega_j|} a_{ij} \int_X \iota^*\pi_1^*\omega_i \wedge \iota^*\pi_2^*\tau_j = \sum (-1)^{|\omega_i|} a_{ii} = L(f).$$

In the event that f is transversal, the integral of η_Γ over the diagonal is exactly the integral of $\eta_\Gamma \wedge \eta_\Delta$ over $X \times X$, which is just the homology intersection of Γ_f and Δ. This is exactly the index $\Lambda(f)$. □

As we shall see, the proof of the main result of this essay uses this version where one shows that the index at each fixed point of G/T is the same and equals 1. Thus $\Lambda(f)$ = number of fixed points $= |N(T)/T|$.

4 É. Cartan's theorem

From basic linear algebra we know that any unitary matrix A can be diagonalized by another unitary matrix U such that UAU^{-1} is diagonal with the entries being the eigenvalues of U. The choices of how one diagonalizes leads to different permutations of the eigenvalues along the diagonal. One should think of the main result as analogous.

Theorem *For T and G as before, any element of G is conjugate to an element of T.*

Proof 1 (Topological) Given $g \in G$ we need to find $x \in G$ such that $x^{-1}gx \in T$. One can reformulate the problem as follows : to find x such that $gxT = xT$, i.e., a fixed point under the left action of g on G/T. Denote this continuous map by f_g. Since G is connected one can homotope f_g to f_e. By homotopy invariance

$$L(f_g) = L(f_e) = L(\mathrm{Id}) = \sum_q (-1)^q \dim_{\mathbb{Q}} H^q(G/T;\mathbb{Q}) = \chi(G/T),$$

Thus, for the existence of a fixed point, it suffices to prove that the Euler characteristic is non-zero. Alternatively, let g_0 be a (topological) generator of T. One can also homotope f_g to f_{g_0}. The fixed points of f_{g_0} are exactly the cosets nT for $n \in N(T)$. By observations before, $N(T)$ is closed (and hence compact) and $T = N(T)_e$ has finitely many cosets. Thus f_{g_0} has only finitely many fixed points and hence isolated. In this setting, once we show that f_{g_0} is transversal, it will follow that

$$\Lambda(f_{g_0}) = L(f_{g_0}) = L(f_g).$$

We observe that

$$f_{g_0}(xT) = g_0 xT = g_0 x g_0^{-1} T.$$

In other words, one can think of f_{g_0} as a quotient of the map $G \to G$ given by $x \mapsto g_0 x g_0^{-1}$. This also maps e to e.

We claim that it suffices to consider any one of these fixed points, say T. Let nT be another fixed point. Then define

$$r_n : G/T \to G/T$$

$$r_n(gT) = gTn.$$

This is well-defined diffeomorphism, maps T to nT and

$$f_{g_0}(r_n(gT)) = f_{g_0}(gTn) = g_0 gTn = r_n(f_{g_0}(gT)).$$

Thus the multiplicity at nT is the same as T.

To obtain a basis of $(G/T)_T$, take a basis of $\mathfrak{t}$, extend it to a basis of $\mathfrak{g}$. Then by the discussion in the first section

$$I - df_{g_0} = \left(\begin{array}{cc|c} 1-\cos 2\pi\theta_1(g_0) & \sin 2\pi\theta_1(g_0) & 0 \\ -\sin 2\pi\theta_1(g_0) & 1-\cos 2\pi\theta_1(g_0) & \\ \hline \multicolumn{2}{c|}{0} & \ddots \end{array}\right)$$

just has 2×2 block of minors along the diagonal. Therefore the determinant

$$\det(I - df_{g_0}) = \prod_1^m \det \begin{pmatrix} 1-\cos 2\pi\theta_i(g_0) & \sin 2\pi\theta_i(g_0) \\ -\sin 2\pi\theta_i(g_0) & 1-\cos 2\pi\theta_i(g_0) \end{pmatrix}$$

which is greater than 0 unless $\cos 2\pi\theta_r(g_0) = 1$ for some r. Since g_0 is a generator for T, $\cos 2\pi\theta_r(g_0) = 1$ implies that $\theta_r(g_0) \equiv 0 \bmod 1$ which contradicts the non-triviality of θ_r. Thus the index of f_{g_0} at T is 1 and

$$\Lambda(f_{g_0}) = |N(T)/T|.$$

This implies that $L(f_g) \neq 0$ and thus f_g has a fixed point. □

We provide another proof of this which is more geometric.

Proof 2 Recall that

(i) T acts non-trivially on V_j ($\mathfrak{g} = \mathfrak{t} \oplus (\oplus_1^m V_j)$) via $\phi : T \to SO(2)$, and

(ii) $M = \cap ker(\phi_j)^c$ is an open dense submanifold of T containing all the generators of T.

Let $t_0 \in M$. Since each $\phi_j(t_0)$ is a non-trivial rotation, it acts trivially only on $\mathfrak{t}$. Let $exp(sX)$ denote a 1 parameter subgroup of $Z_G(t_0)$. Then

$$exp(sX) = t_0 exp(sX) t_0^{-1} = exp(sAd(t_0)X) \ \forall s \in \mathbb{R}$$

implies that $X \in \mathfrak{t}$. In other words, the connected component of identity of $Z_G(t_0)$ is T and

$$\dim G(t_0) = \dim G - \dim Z_G(t_0) = \dim G - \dim T$$

where $G(t_0)$ is the orbit of t_0 under the conjugacy action. $G(t_0)$ and T are (compact) submanifolds of complementary dimensions. Since l_{t_0} commutes with the conjugation of any $t \in T$, it induces an equivariant linear map

$$dl_{t_0} : \mathfrak{g} \to T_{t_0} G$$

with respect to the induced T-actions. It is clear that $dl_{t_0}(\mathfrak{t})$ is the tangent space of T at t_0. The induced T-action on $G(t_0)$ is the induced (left) T-action on $G/Z_g(t_0)$. But this is exactly equivalent to the restriction of the adjoint T-action on $\mathfrak{g}$ to the (orthogonal) complement of $\mathfrak{t}$. Thus we have shown that $G(t_0)$ and T intersects perpendicularly and transversally at t_0. We also note that T is a connected component of $F(T, G)$, the fixed point set of G under the adjoint action of T, which acts by isometries. Thus T is a *totally geodesic* submanifold of G. Now we complete the proof as follows. If $G(y)$ is any other G-orbit then $G(t_0)$ and $G(y)$ are two compact submanifolds in a complete Riemannian manifold G. By Hopf-Rinow, there exists a geodesic interval $\overline{x_1 y_1}$ which realizes the shortest distance between them. It must necessarily be perpendicular to both $G(t_0)$ and $G(y)$. Choose $g \in G$ such that $g x_1 g^{-1} = t_0$. Then $g(\overline{x_1 y_1})g^{-1} = \overline{t_0(g y_1 g^{-1})}$ is again a geodesic interval which is perpendicular to both. By the two preceding observations it follows that $\overline{t_0(g y_1 g^{-1})} \subset T$, which means $g y_1 g^{-1} \in T$ and therefore $G(y) \cap T \neq \phi$. □

This proof allows us to think of a maximal torus as a complete, totally geodesic submanifold normal to $G(t_0)$. As an upshot of either of the proofs we have

Corollary 1 Every element of G lies in a maximal torus. Moreover, any two maximal tori are conjugate.

Proof The first assertion is clear because the conjugate of a maximal torus is a maximal torus. For the second one, let $t \in T$ be a generator of the torus. Then $t \in xT'x^{-1}$ for some $x \in G$. Therefore $T \subset xT'x^{-1}$. The maximality of T forces $T = xT'x^{-1}$. □

As a consequence, any construction that depends on T is independent of the choice of the maximal

torus upto an inner automorphism of G. It also follows that any two maximal tori have the same dimensions and this is called the **rank** of G.
Let H be an abelian subgroup of G. Then $\overline{H}$ is a compact abelian Lie group and $\overline{H}_e$ is a torus. Since $\overline{H}/\overline{H}_e$ is finite and thus isomorphic to $\mathbb{Z}_n$ for some n. One can check that $\overline{H}$ is generated by one element g, viz choose $u \in G$ to project to a generator of $\mathbb{Z}_n$, choose $s \in \overline{H}_e$ such that $t = s^n u^n$ where t generates $\overline{H}$ and set $g = us$. By the theorem g lies in some maximal torus and hence so does $\overline{H}$. This also shows that $Z(T) = T$ for a maximal torus. Therefore $W(G,T) \cong N(T)/T$ and we deduce from the (topological) proof that

$$\chi(G/T) = |W(G,T)|.$$

More generally, for any torus S in G, the centralizer $Z_G(S)$ is the union of all maximal tori containing S (and hence it is connected).
We also see that for two representations ϕ, ψ of G, the respective characters χ_ϕ, χ_ψ are equal if and only if their restriction to T is. Since $\phi \sim \psi$ (similarly $\phi|_T \sim \psi|_T$) holds if and only if $\chi_\phi = \chi_\psi$ (similarly $\chi_\phi|T = \chi_\psi|T$), we have

Corollary 2 Two representations of G are equivalent if and only if their restrictions to T are equivalent.

5 Cell structure of G/T

One can say more about G/T for a simply connected, compact and semisimple Lie group G. It follows from [2] that the homotopy type of G/T is that of a CW-complex having only even dimensional cells. Consequently, the odd cohomology classes vanish and $\chi(G/T)$ is actually the sum of the number of even dimensional cells. We shall revisit a few examples now in this light. But first we need :

Definition The **diagram** of G is the family of hyperplanes in $\mathfrak{t}$ where some root is integral. A hyperplane in $\mathfrak{t}$ that is the zero set of a root will be called a **root plane**.

Definition A point $g \in G$ is **regular** if $\dim N(g) = \operatorname{rank} G$. It will be called singular if $\dim N(g) > \operatorname{rank} G$.

Since any element g lies in some maximal torus T, $\dim N(g) \geq \dim T = \operatorname{rank} G$. Analogously, a point $B \in \mathfrak{t}$ is regular if it's normalizer has minimal possible dimension, or equivalently, if the normalizer is T. It is also well known that B is regular if and only if it does not lie on any of the root planes. If B is regular then the stabilizer under the adjoint action is T and the orbit is G/T. For example, with $G = SU(2)$ and T as in example 3, (refer figure below) 0 is the root plane and any point lying between the nodes is regular.

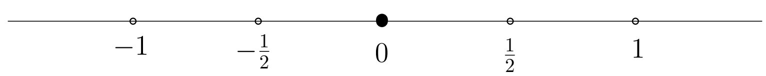

Figure 1 The diagram of $SU(2)$

If $G = SU(3)$ and T is as in example 2 then the diagram of $SU(3)$ consists of (refer figure below) the collection of dashed lines in the plane $x_1 + x_2 + x_3 = 0$. The thickened lines are the root planes.

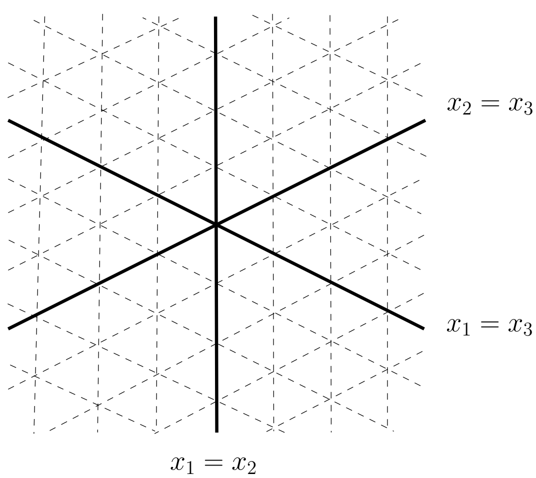

Figure 2 The diagram of $SU(3)$.

Another way to think of the diagram of T is by looking at the universal covering map $p : \tilde{T} \to T$. Let S denote the set of singular elements of T. Then $p^{-1}(S)$ is the diagram of G and provides a simplex structure on $\tilde{T} = \mathfrak{t}$.

It is known that the Killing form (defined by $K(x, y) = tr(\mathrm{ad}x\,\mathrm{ad}y)$ for $x, y \in \mathfrak{g}$) is non-degenerate for semisimple Lie groups. Moreover for compact groups it is negative definite. For example, the

Killing form on $\mathfrak{su}_2$ is just the negative of the usual Euclidean inner product. In general let G be a compact, semisimple, connected and simply connected Lie group and let T be a maximal torus. Let K denote the Killing form. Choose two distinct regular points A and B in $\mathfrak{t}$. Notice that the orbit of B is G/T. Since the restriction of K to $\mathfrak{t}$ is still non-degenarate, we define a function

$$f : G/T \to \mathbb{R},\ f(*) = K(A, *).$$

Let $\{B_j\}$ be all the points in $\mathfrak{t}$ obtained from B by reflecting about the root planes. Then Bott's theorem (refer [2]) tells us that f is a Morse function with critical points precisely the B_j's. Moreover the index of B_j is twice the number of times that the line segment joining A to B_j intersects the root planes.

Example 7 Let $G = SU(2)$ and T as in example 3. The Lie algebra $\mathfrak{t} = \mathbb{R}$ and the roots are $\pm 2x$. Choose regular points A and $B = B_1$ as given below. Then the Morse function (as described before)

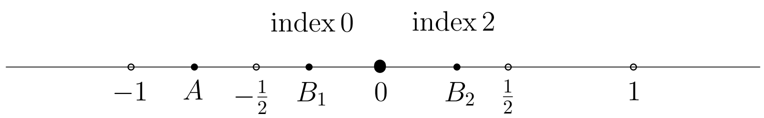

Figure 3 The flag manifold $Fl(1,2)$

has two critical points - B_1 of index 0 and B_2 of index 2. Thus the homotopy type of $SU(2)/S^1$ is just that of the 2-sphere.

Example 8 Let $G = SU(3)$ and T as in example 2. The Lie algebra $\mathfrak{t}$ is the plane given by $x_1 + x_2 + x_3 = 0$ in $\mathbb{R}^3$. Choosing A and B as per the recipe, we see that

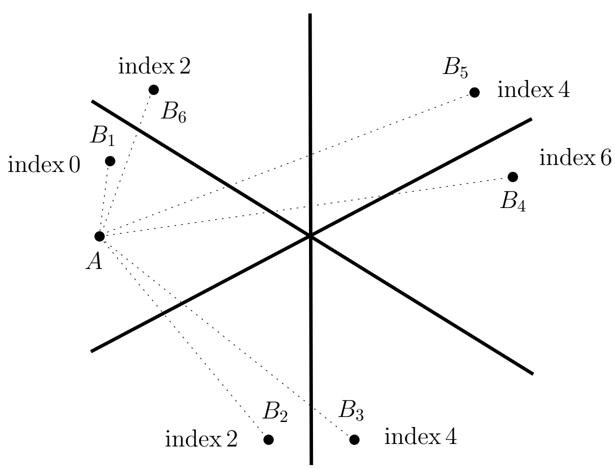

Figure 4 The flag manifold $Fl(1,2,3)$.

the homotopy type of $SU(3)/T = Fl(1,2,3)$ is a CW-complex made up of one 0 cell, two 2 cells, two 4 cells and one 6 cell and $\chi(Fl(1,2,3)) = 6$.

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