An Introduction to Equivariant Homology

1 Introduction

For a topological group G acting on a space X, there is a notion of G-equivariant homology of X. For free actions of G, one hopes to get back the usual homology of X/G. In what follows, a G-space would mean a space with a G-action and a G-map between two G-spaces is one that commutes with the G-action. We also assume that G is connected CW complex. The classical notion, pioneered by Borel, was via classifying spaces. More concretely, for any topological group G there is a principal G-bundle with contractible total space. This is unique up to homotopy and we denote it by $G \hookrightarrow EG \to BG$. Given an action of G on X one replaces X by $X \times EG$. This doesn't change the homotopy type of X but the diagonal action of G on $X \times EG$ is free. One then defines the G-equivariant homology of X to the usual homology of $X \times_G EG$. Since EG is the universal object for G-bundles, it is possible to give an axiomatic definition of equivariant homology. The aim of this note is to show the equivalence of the two and give some examples.

2 What is equivariant homology?

We state the two definitions first.

Definition 2.1. The *G-equivariant homology*, denoted H^G_*, is functor from the category of G-spaces and G-maps to the category of abelian groups and homomorphisms satisfying (i) and (ii). More precisely, to each G-space X it assigns abelian groups $H^G_i(X;\mathbb{Z}), i \geq 0$. Given a G-map $f : X \to Y$, we also get homomorphisms $f^G_* : H^G_i(X;\mathbb{Z}) \to H^G_i(Y;\mathbb{Z})$ induced by f. This functor must satisfy :

(i) If G acts freely on X, then $H^G_*(X;\mathbb{Z}) = H_*(X/G;\mathbb{Z})$.

(ii) If $f : X \to Y$ induces an isomorphism $f_* : H_*(X;\mathbb{Z}) \to H_*(Y,\mathbb{Z})$, then it also induces an isomorphism $f^G_* : H^G_*(X;\mathbb{Z}) \to H^G_*(Y,\mathbb{Z})$.

For any given G-action on X, one can tweak it to make it free, i.e., replace X by the principal G-bundle $X \times EG$ and the old action by the diagonal action. Recall that EG is contractible and it is the universal G-bundle over BG, which is the classifying space for principal G-bundles. Clearly, the action on $X \times EG$ is free. The quotient $(X \times EG)/G$ will be denoted by $X \times_G EG$. Define

$$\pi : X \times EG \to X, \;\; (x, e) \mapsto x.$$

This a G-map and induces an isomorphism

$$\pi_* : H_*(X \times EG;\mathbb{Z}) \xrightarrow{\cong} H_*(X;\mathbb{Z})$$

since EG is contractible. By (ii), this induces an isomorphism

$$\pi^G_* : H^G_*(X \times EG;\mathbb{Z}) \xrightarrow{\cong} H^G_*(X;\mathbb{Z}). \tag{2.1}$$

Applying (i) to the free G-action on $X \times EG$ we have

$$H_*^G(X \times EG; \mathbb{Z}) = H_*(X \times_G EG; \mathbb{Z}). \tag{2.2}$$

Putting together (2.1) and (2.2) we get

$$H_*^G(X; \mathbb{Z}) \cong H_*(X \times_G EG; \mathbb{Z}). \tag{2.3}$$

This may lead to (although historically it was the other way round) :

Definition 2.2. (Borel) For a G-space X, the *G-equivariant homology* of X, denoted $H_*^G(X; \mathbb{Z})$, is defined to be $H_*(X \times_G EG; \mathbb{Z})$.

To prove the equivalence of the two definitions, we need only show that the latter implies the former. First, let G act freely on X and fix $e_0 \in EG$. For $(x, e) \in X \times EG$ there is a unique element $g \in G$ such that $ge_0 = e$. Hence, $[(x, e)] = [(g^{-1}x, e_0)]$. Observe that for the same reason $[(g_1 x, g_1 e)] = [(g^{-1}x, e_0)]$ for any $g_1 \in G$. By the freeness of the action on X, $[(x, e)]$ can be identified bijectively with $g^{-1}x$, i.e., there is a homeomorphism

$$\phi : X \times_G EG \to X/G, \ [(x, e)] \mapsto g^{-1}x,$$

whence (i) holds. To prove (ii), let $f : X \to Y$ be a G-map inducing isomorphisms on ordinary homology. This induces a G-map

$$f \times \mathrm{id} : X \times EG \to Y \times EG, \ \ (x, e) \mapsto (f(x), e) \tag{2.4}$$

which turns out to be a map of principal G-bundles. Consequently, there is a map of the base spaces

$$f^G : X \times_G EG \to Y \times_G EG, \ \ [(x, e)] \mapsto [(f(x), e)] \tag{2.5}$$

and we need to show that f_*^G is an isomorphism. Observe that the contractibility of EG implies that $f \times \mathrm{id}$ induces an isomorphism on homology. For any fibration, the fundamental group of the base acts on the homology of the fibre. Assuming the action is trivial in our case, the following proposition then finishes off the proof.

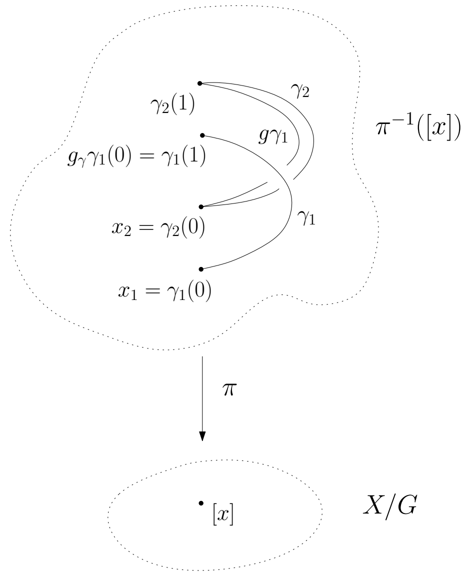
Proposition 2.3. *Suppose we have a map between fibrations such that the base acts trivially on the homology of the fibre. Then if any two of the three maps $F \to F'$, $X \to X'$ and $B \to B'$ induce isomorphisms on ordinary homology, then so does the third.*

To prove the proposition we need to use the machinery of Serre spectral sequences which calculates $H_*(X; \mathbb{Z})$ in terms of $H_*(B; \mathbb{Z})$ and $H_*(F; \mathbb{Z})$ where $F \hookrightarrow X \xrightarrow{\pi} B$ is a fibre bundle. The existence of such a gadget depends on the action of $\pi_1(B)$ being trivial on $H_*(F; \mathbb{Z})$. For our case, we need :

Lemma 2.4. *For a principal G-bundle X, $\pi_1(X/G)$ acts trivially on $H_*(G; \mathbb{Z})$.*

Proof Let $\gamma \in \pi_1(X/G)$ with $\gamma(0) = [x] \in X/G$. Choose $x_1, x_2 \in X$ to be pre-images of $[x]$. There is a unique $g \in G$ such that $gx_1 = x_2$. Now by the path lifting property, γ lifts to paths $\gamma_i, i = 1, 2$ in X with $\gamma_i(0) = x_i$. Then $g\gamma_1$ is a path in X starting at x_2 and is a lift of γ. By the uniqueness of the endpoints, $g\gamma_1(1) = \gamma_2(1)$. Consequently, if $g_\gamma \in G$ is the unique element such that $g_\gamma \gamma_1(0) = \gamma_1(1)$ then

$$\gamma_2(1) = g\gamma_1(1) = gg_\gamma\gamma_1(0) = gg_\gamma g^{-1} g\gamma_1(0) = gg_\gamma g^{-1}\gamma_2(0).$$



Fixing x_1 induces a twisted conjugation map on $G = \pi^{-1}([x])$ via γ and defined by

$$\gamma : G \to G, \;\; x_2 \mapsto g g_\gamma g^{-1} x_2, \tag{2.6}$$

where g is the unique element such that $gx_1 = x$. Since G is connected, g_γ can be joined by a path to the identity element. Thus, γ as in (2.6) can be homotoped to the identity map on G. This implies $\gamma_* : H_*(G;\mathbb{Z}) \to H_*(G;\mathbb{Z})$ is the identity map. □

We invoke a result about maps between fibrations. For further reference look up any text on algebraic topology containing spectral sequences.

Proof of the propisition Let us label the maps

$$\begin{array}{ccccc} F & \xrightarrow{\iota} & X & \xrightarrow{\pi} & B \\ \downarrow{\scriptstyle \mathcal{F}} & & \downarrow{\scriptstyle \mathcal{X}} & & \downarrow{\scriptstyle \mathcal{B}} \\ F' & \xrightarrow{\iota'} & X' & \xrightarrow{\pi'} & B' \end{array}$$

In what follows, homology with $\mathbb{Z}$-coefficients is understood unless mentioned otherwise. The proof is divided into three cases :

Case 1 *$\mathcal{F}_*$ and $\mathcal{B}_*$ are isomorphisms*

By the universal coeffieicient theorem for homology

$$\begin{array}{ccccccccc} 0 & \to & H_p(B) \otimes H_q(F) & \to & H_p(B; H_q(F)) & \to & \mathrm{Tor}(H_{p-1}(B), H_q(F)) & \to & 0 \\ & & \cong \downarrow {\scriptstyle \mathcal{B}_* \otimes \mathcal{F}_*} & & \downarrow {\scriptstyle H_p(\mathcal{B};\mathcal{F}_*)} & & \cong \downarrow {\scriptstyle \mathrm{Tor}(\mathcal{B}_*, \mathcal{F}_*)} & & \\ 0 & \to & H_p(B') \otimes H_q(F') & \to & H_p(B'; H_q(F')) & \to & \mathrm{Tor}(H_{p-1}(B'), H_q(F')) & \to & 0 \end{array}$$

and applying five-lemma to the above we get an isomorphism

$$H_p(B; H_q(F)) \cong H_p(B'; H_q(F')) \tag{2.7}$$

which is natural which follows from the naturality of the exact sequences above. Thus, the map $f_2 : E^2_{p,q} \to E'^2_{p,q}$ is an isomorphism. Since f_2 induces f_3 and so on, all the maps f_r are isomorphisms and so is f_∞. The map $\mathcal{X}_* : H_n(X) \to H_n(X')$ preserves filtrations and induces the isomorphism f_∞ between successive quotients in the filtrations. It follows by induction and the five-lemma that it restricts to an isomorphism between each term in the filtration F_n^p of $H_n(X)$ and $F_n'^p$ of $H_n(X')$. In particular, $H_n(X)$ is isomorphic to $H_n(X')$.

Case 2 *$\mathcal{F}_*$ and $\mathcal{X}_*$ are isomorphisms*
The pullback fibration for $\mathcal{B} : B \to B'$ fits into a commutative diagram :

$$\begin{array}{ccccc}
F & \xrightarrow{\mathcal{F}} & \{b\} \times F' & \xrightarrow{\text{id}} & F' \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle i_2 \circ \iota'} & & \downarrow{\scriptstyle \iota'} \\
X & \xrightarrow{(i_1 \circ \pi, \mathcal{X})} & \mathcal{B}^*(X') & \xrightarrow{\pi_2} & X' \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi'} \\
B & \xrightarrow{\text{id}} & B & \xrightarrow{\mathcal{B}} & B'
\end{array}$$

By the first case, the map

$$(i_1 \circ \pi, \mathcal{X}) : X \to \mathcal{B}_*(X')$$

induces an isomorphism in homology. Thus, it suffices to deal with the second and the third fibrations where the fibres are equal. One can reduce to the case where $\mathcal{B}$ is an inclusion by the use of the mapping cylinder $M_\mathcal{B}$ which is just $B \times [0,1]$ glued to B' along $B \times \{1\}$ using $\mathcal{B}$. By construction,

$$\phi_t : M_\mathcal{B} \times [0,1] \to M_\mathcal{B}$$

$$\phi_t(b,s) = (b, s(1-t)), \ \phi_t|_{B'} \equiv \text{id},$$

is a deformation retraction of $M_\mathcal{B}$ to B' and therefore ϕ_{1*} is an isomorphism. There is also

$$i : B \hookrightarrow M_\mathcal{B}, \ \ b \mapsto (b, 0)$$

and $\phi_1 \circ i : B \to B'$ is just $\mathcal{B}$. Thus we get

$$\begin{array}{ccccc}
F & \xrightarrow{\text{id}} & F & \xrightarrow{\text{id}} & F \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{B}^*(X') & \longrightarrow & \phi_1^*(X') & \xrightarrow{\pi_2} & X' \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi'} \\
B & \xrightarrow{i} & M_\mathcal{B} & \xrightarrow{\phi_1} & B'
\end{array}$$

and by the first case, $\pi_2 : \phi_1^*(X') \to X'$ induces an isomorphism. Consequently, $\mathcal{B}^*(X') \to \phi_1^*(X')$ is an isomorhism on homology too. Therefore it is enough to show that for $i : B \hookrightarrow M_\mathcal{B}$ induces an isomorphism. Look at the relative fibration

$$F \to (\phi_1^*(X'), \mathcal{B}^*(X')) \to (M_\mathcal{B}, B).$$

For the corresponding spectral sequence

$$E^2_{p,q} = H_p(M_\mathcal{B}, B; H_q(F))$$

converges to $H_{p+q}(\phi_1^*(X'), \mathcal{B}^*(X'))$ which is zero. If there exists $n \in \mathbb{N}$ such that $H_i(M_{\mathcal{B}}, B) = 0$ for $i < n$ and $H_n(M_{\mathcal{B}}, B)$ is non-zero, then in E^2-page the first $n-1$ columns are zero. This would imply that $E^2_{n,0} = H_n(M_{\mathcal{B}}, B; H_0(F))$ is non-zero and survives to E^∞, a contradiction.

Case 3 $\quad$ *$\mathcal{X}_*$ and $\mathcal{B}_*$ are isomorphisms*
First, the pullback fibration for $\mathcal{B} : B \to B'$ fits into the commutative diagram as in the proof of case 2. By case 1, it suffices to consider the first and second fibrations, i.e., given

$$\begin{array}{ccccc} F & \xrightarrow{\iota} & X & \xrightarrow{\pi} & B \\ \downarrow{\scriptstyle \mathcal{F}} & & \downarrow{\scriptstyle \mathcal{X}} & & \downarrow{\scriptstyle \mathrm{id}} \\ F' & \xrightarrow{\iota'} & X' & \xrightarrow{\pi'} & B \end{array}$$

such that $\mathcal{X}_*$ is an isomorphism, we need to show the same for $\mathcal{F}_*$. Let $M_{\mathcal{F}}$ denote the mapping cylinder for $\mathcal{F} : F \to F'$. Fix a deformation retraction ϕ_t of $M_{\mathcal{F}}$ to F' and let $i : F \to M_{\mathcal{F}}$ be the natural inclusion map. Observe that $i \circ \phi_1 : F \to F'$ is just $\mathcal{F}$. Then we have the following commutative diagram :

$$\begin{array}{ccccc} F & \xrightarrow{i} & M_{\mathcal{F}} & \xrightarrow{\phi_1} & F' \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota' \circ \phi_1} & & \downarrow{\scriptstyle \iota'} \\ X & \xrightarrow{\mathcal{X}} & X' & \xrightarrow{\mathrm{id}} & X' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi'} \\ B & \xrightarrow{\mathrm{id}} & B & \xrightarrow{\mathrm{id}} & B \end{array}$$

Consequently, it reduces to showing that i_* is an isomorphism, since ϕ_1 is a homotopy equivalence. One may alternatively consider the induced fibration

$$M_{\mathcal{F}}/F \to X'/\mathcal{X}(X) \to B/B = \{\mathrm{pt}\}.$$

Since the homology of $\{\mathrm{pt}\}$ is concentrated in dimension 0, the only possible non-zero terms in the E^2-page are

$$E^2_{0,q} = H_0(\{\mathrm{pt}\}; H_q(M_{\mathcal{F}}/F)) = H_q(M_{\mathcal{F}}/F),$$

which survive to E^∞. For good pairs (all pairs of CW complexes certainly are) (X, A) it can be shown that

$$H_i(X, A) \cong H_i(X/A),\ i > 0.$$

Thus

$$H_i(X'/\mathcal{X}(X)) \cong H_i(X', \mathcal{X}(X)) = 0 \ \text{ for } i > 0.$$

Consequently, $E^2_{0,0} = H_0(M_{\mathcal{F}}/F) = \mathbb{Z}$ and is zero otherwise. By the goodness of the pair $(M_{\mathcal{F}}, F)$, this means

$$H_q(M_{\mathcal{F}}, F) \cong H_q(M_{\mathcal{F}}/F) = 0 \ \text{ for } q > 0.$$

It follows from the long exact sequence in homology of the couple $(M_{\mathcal{F}}, F)$ that

$$i_* : H_*(F) \to H_*(M_{\mathcal{F}})$$

is an isomorphism. $\square$

Remark 2.5. *The above proof works essentially verbatim in proving that if $f : X \to Y$ is a G-map between free G-spaces such that $f_* : H_*(X) \xrightarrow{\cong} H_*(Y)$ then there is an isomorphism $f_*^G : H_*(X/G) \to H_*(Y/G)$.*

Remark 2.6. *If G acts trivially on X then $X \times_G EG = X \times BG$ and $H_*^G(X) = H_*(X \times BG)$. In particular, if X is a point (or contractible, in general) then we get back the homology of BG. The significance of BG is that the cohomology ring of BG lies at the heart of the theory of characteristic classes.*

Remark 2.7. *The equivariant cohomology $H_G^*(X)$ can be given the structure of a $H^*(BG)$-module. Let $p : E \times_G EG \to BG$ be the G-equivariant projection map. Given $\alpha \in H^*(BG), \beta \in H_G^*(X)$, we define $\alpha \cdot \beta := p^*(\alpha) \cup \beta$.*

3 A few examples

We discuss a few diverse examples to convey the flavour of the topic, starting with simple ones.

Example 3.1. The S^1-equivariant homology of a point is just the homology of BS^1. One can take S^∞ to be a model for ES^1 with the free action of the unit complex numbers. The quotient BS^1 is then $\mathbb{CP}^\infty$. It can be shown the homology ring

$$H_*^{S^1}(\bullet) = H_*(\mathbb{CP}^\infty; \mathbb{Z}) \cong \mathbb{Z}[u],$$

where u is an element of degree 2 corresponding to the 2-cycle $\mathbb{CP}^1 \subset \mathbb{CP}^\infty$.

Example 3.2. The equivariant homology may not detect triviality of G-bundles. For instance, $S^1 \times S^2$ and S^3 are both principal S^1-bundles over S^2. Since the action of S^1 on S^3 is free, $H_*^{S^1}(S^3) = H_*(S^2)$. On the other hand, it's clear that $H_*^{S^1}(S^1 \times S^2) = H_*(S^2)$ as well.

Example 3.3. (From the 1970 MIT notes by D. Sullivan) Let X be a locally compact polyhedron with an involution T. Let F denote the subcomplex of fixed points of T. Let $X_T := X \times_{\mathbb{Z}_2} S^\infty$ denote the space arising from the Borel construction. Then $H^*(X; \mathbb{Z}_2)$ is an R-module, where

$$R = \mathbb{Z}_2[x] = H^*(\mathbb{RP}^\infty; \mathbb{Z}_2).$$

Let S denote the multiplicative set generated by x and let R_x denote the the localization $S^{-1}R$. Then it can be shown that

$$H^*(F; R_x) \cong H^*(X_T; \mathbb{Z}_2) \otimes_R R_x.$$

Example 3.4. (String topology) Let $LM = \mathrm{Map}(S^1, M)$ be the free loop space on a manifold M. The circle action on itself defines an S^1-action on LM. This, however, is not free since constant loops are fixed by the circle. The S^1-equivariant homology of LM is denoted by $\mathcal{H}_*(LM)$ and called the *string topology* of M. It can be shown (part of my thesis) that for $M = S^{2k+1}, k > 0$, the equivariant cohomology is given by

$$\mathcal{H}^*(LS^{2k+1}; \mathbb{Z}) = \mathbb{Z}[u] \oplus \mathbb{Z}[y],$$

where u is of degree 2 and y is of degree $2k$. For $M = S^{2k}$ we have

$$\mathcal{H}^*(LS^{2k}; \mathbb{Q}) = \mathbb{Q}[u] \oplus (\Lambda(y_1) \otimes \mathbb{Q}[y_2]),$$

where u, y_1, y_2 has degree $2, 2k-1$ and $4k-2$ respectively.

Example 3.5. One can generalize the previous example. Let $L^d M = \mathrm{Map}(S^d, M)$ denote the space of maps of d-spheres into M. Then there is a natural $SO(d+1)$-action on $L^d M$ and one can speak of $H^*_{SO(d+1)}(L^d M)$. However, $SO(d+1)$ has non-trivial cohomology and if one wants to work with less material then there are alternatives. Let G_d be the group of all degree 1 maps from S^d to itself. We can then speak of $H^*_{G_d}(L^d M)$ or alternatively, it is possible to work over $\mathrm{Diff}^+(S^d)$. For $d = 1$, this is topologically the same as $SO(2)$.

Example 3.6. We think of $X = \mathbb{C} \times S^1$ as the open solid torus. Define an action of the torus $S^1 \times S^1$ on X by setting

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot (z, e^{i\theta}) := (ze^{i\theta_1}, e^{i(\theta+\theta_2)}).$$

This is not a free action since $S^1 \times \{1\}$ fixes $\{0\} \times S^1$ pointwise. However, the natural inclusion $S^1 \hookrightarrow \mathbb{C} \times S^1$ into the core circle is an equivariant homotopy equivalence. Therefore,

$$H_*^{S^1 \times S^1}(\mathbb{C} \times S^1) \cong H_*^{S^1 \times S^1}(S^1),$$

which can be seen to be $H_*(\mathbb{CP}^\infty)$ since the first copy of S^1 in the torus acts trivially on the core circle and the second copy acts freely.