Games on graphs

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Mathematicians have a habit of naming concepts and ideas after terms used in colloquial language. One such is the notion of a *clique*. Let's remind ourselves that a *clique* of size n means a group of n people such that any two are related by a common property. Examples include cliques of friends, cliques of enemies, cliques of blondes, cliques of males etc. With that said, let us begin with a question.

Question In a group of 6 people where people are mutually strangers or friends, what are the possible sizes of cliques (of strangers or of friends)?

Now imagine the six people in the party are represented by six vertices. If person A is a friend of person B we connect the corresponding vertices by a red line segment (called an *edge*). If A is a stranger to B then we draw a blue edge between them. Phrased in this language, a clique of strangers of size 3 is equivalent to a blue triangle.

Question How does a clique of friends of size 4 look like in this setting?

We'll get back to the first question and find a definitive answer. For the moment, we leave it aside and switch to playing games, as the title suggests.

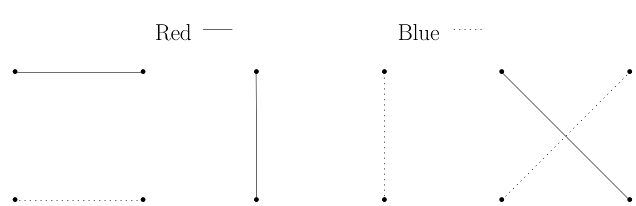
Let us begin by playing a two-player game. The board consists of a plane with n vertices, $n \geq 2$. The game begins by player 1 drawing a red edge joining any two distinct vertices. Player 2 then draws a blue line segment between two distinct vertices. The game goes on where each player draws a line segment between two distinct vertices which is not joined already. The game ends when there are no two vertices left to be joined. We say that the first player *looses immediately* if there is a red triangle and the second player *looses immediately* if there is a blue triangle. The game ends in a *tie* if no further moves can be made and none of the players have lost yet.

Question Why does the game end?

For $n = 2, 3$ this game is boring. Try playing it on four vertices where each player has 3 moves. With perfect play no player looses and the game ends in a tie. One can see this in the following way.

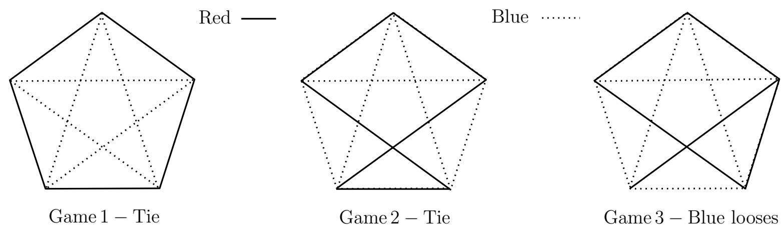
Strategy for player 1 No player looses after having played two moves each. Now the first player has to choose one of the remaining two edges and colour it red. Since only one edge possibly completes a red triangle, he chooses one of the two which doesn't.

Strategy for player 2 In the first move colour that edge blue which is opposite to the red edge. The three possible first two moves are shown below.



It is clear from either of the three configuration above that the new red edge appearing due to player 1's second move must be connected to the already present red edge. Now there is exactly one edge, which when coloured blue, completes a 2-blue and 1-red triangle. Player 2 colours that edge blue. One can easily see that player 2 can now never loose no matter what happens henceforth.

Let us play it on 5 vertices. A few sample finished games are given below.



Play sufficiently many games to see if you can spot some general pattern.

Question Can you deduce if either player has a winning strategy, i.e., if the player plays perfectly then he never looses no matter what the other player plays?

Before playing the game on 6 vertices, let's introduce some useful notation. We shall denote by K_n the object with n vertices and edges joining any two vertices. It is called the *complete graph* on n vertices. Observe that our game consists of player taking turns in colouring the edges red and blue. For $n = 6$ there are 15 edges to be coloured and player 1 has 8 moves while player 2 has 7 moves. Naively, it seems that player 2 has a better chance of winning simply because he has to play one move less. This game is often called *Sim*, named after Gustavus Simmons in 1969. There are online softwares where you can play against a computer.

Exercise Try playing this game with your friends a few times and see what you get. Does the first player win more often or the second player?

If you've played sufficiently many times and played smartly, you'll observe that the second player tends to win more than the first player. However, the game never ends in a tie. This is not an accident - it's a theorem.

Theorem 1. *The game of Sim cannot end in a tie.*

In fact, a completed game of Sim has 8 edges of K_6 coloured red and 7 edges of K_6 coloured blue. These are particular cases of a colouring of the edges of K_6 using two colours. One can imagine the vertices as the people and a red edge means mutual friends while a blue edge stand for mutual strangers. Towards that end, the result above is a simple corollary of the more general result.

Theorem 2. *Any edge colouring of K_6 using red and blue admits a red triangle or a blue triangle. Equivalently, in any group of 6 people there is a clique of size 3, either of friends or of strangers.*

Proof Let the six vertices be placed such that five of it are placed at the fingertips and one at the palm, labelled P. The vertex P is connected to all the other five vertices. At least three of these edges are coloured with the same colours, let's say red. Without loss of generality, lets assume that the thumb, the index finger and the middle finger are the three fingers. Now out of the three edges joining these three fingers, if at least one is red then we have a red triangle. if none of the edges are red then we have a blue triangle. □

Question Suppose that a game of Sim actually lasts for 15 moves. Is it true that the first player actually completes two triangles at once with the last move?

It has been shown by computer search that the second player in Sim has a perfect strategy for winning although it is humanly impossible to memorize it.

It's good to introduce some terminology at this point. The *Ramsey number* $R(m, n)$ is defined to be the least number k such that in any party of k people where any two are mutual friends or mutual strangers, there is a clique of friends of size m or there is a clique of strangers of size n. We have seen that $R(3, 3) \leq 6$. Can you guess what it is?

Exercise Show that $R(m, n) = R(n, m)$.

The study of such numbers and related colouring problems is called *Ramsey theory*, named after F. P. Ramsey. He proved that $R(m, n)$ exists for any positive integers m, n. Although the famous mathematician Paul Erdös didn't discover this basic theorem, he was largely responsible for creating the theory. However, quite surprisingly, only a handful of Ramsey numbers are known explicitly. Moreover, the known ones are for small integers m and n. For instance, $R(3, 10)$ lies between 40 and 43. We know that $R(4, 4) = 18$ although $R(5, 5)$ is not known except that it lies between 43 and 49. Things aren't so bad as it seems as there are explicit upper bounds. In general, the most useful result is the following.

Theorem 3. (Erdös-Szekeres)
The Ramsey number obey the following upper bound for $k, l \geq 3$ and both not equal to 3 :

$$R(k, l) < \binom{k + l - 2}{l - 1}.$$

Without worrying about the proof, let's go ahead and use this for $m = 3, n = 4$ to get $R(3, 4) \leq 9$. To conclude that $R(3, 4) = 9$ we just need to exhibit a two colouring of K_8 such that there are no monochromatic triangles or no four vertices joined by edges of the same colour, i.e., a clique of size 4.

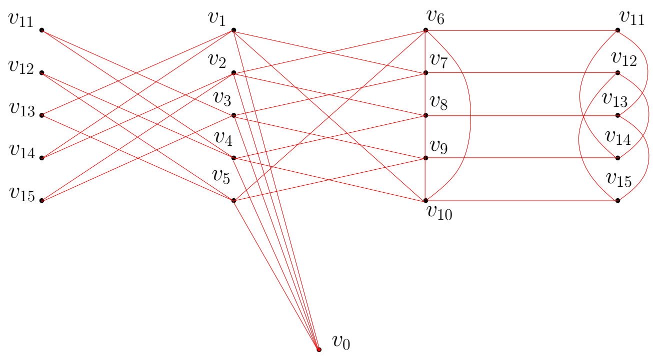
Exercise Find an edge colouring of K_8 (visualize an octagon with all its possible diagonals) with two colours such that the red edges do not have a triangle while the blue edges do not have a K_4.

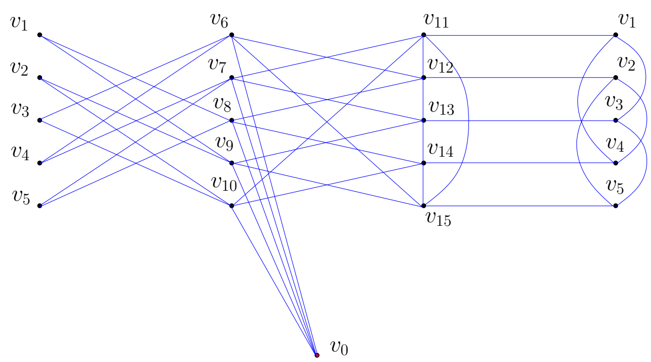
To reiterate the difficulty in knowing Ramsey numbers explicitly here's a story of Erdös. He asks us to *imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5, 5)$ or they will destroy our planet. In that case*, he claims, *we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6, 6)$. In that case*, he believes, *we should attempt to destroy the aliens.* With recent bounds all we can conclude is that $R(6, 6)$ lies between 102 and 165. Even if we search for all possible two colouring of edges in such graphs, the order of complexity grows very fast with n. The rate is fast enough to make computations infeasible even with the modern computational power at our disposal.

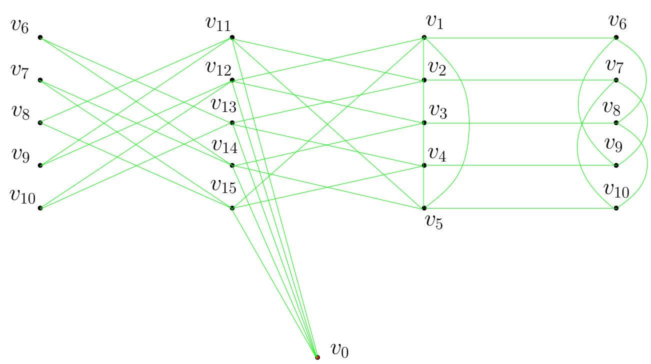
You may have wondered what happens if we play the same game of edge colouring using three colours. In our new language, two players are taking turns by colouring edges of K_n either red, blue or green. The aim of either player is to avoid a monochromatic triangle. Again, this is a particular instance of the more general party problem.

Question What is the last number n such that in a party of n people, where any two are either mutual strangers or mutual friends or mutual enemies, such that there is a clique of size 3 of strangers or of friends or of enemies?

Before we get to the answer let us note we may denote this number by $R(3,3,3)$ pretty much like we used the notation $R(m,n)$.







If we take the three pictures above and superimpose all of them coherently then we shall see a three colouring of the edges of K_{16} without a monochromatic triangle. Thus, $R(3,3,3) \geq 16$ and in fact $R(3,3,3) = 17$. The proof is just part of another long story. However, taking a cue we may define $R(i,j,k)$ to be the least number of people required in a party such that either there is a clique of strangers of size i or there is a clique of friends of size j or there is a clique of enemies of size k. One can define $R(n_1, n_2, \ldots, n_k)$ in general by using $k \geq 3$ colours. These are called *multicolour Ramsey numbers*. It's indeed quite amazing that the only such number explicitly known is $R(3, 3, 3)$!