Floer Homology in Gauge Theory

Somnath Basu

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Abstract

This is an expository account of the Chern-Simons functional, leading to various invariants on homology 3-spheres. It is based mainly on a review article by K. Fukaya [F], D. Freed's [DF1] on classical Chern-Simons theory, A. Floer's instanton invariant [AF] and K. Walker's [W] book on Casson's invariant. Certain parts are taken verbatim from these sources with most of the basic details filled in. This essay is also inspired by the seminal paper of S. S. Chern and J. Simons [CS] **???**.

1 The Chern-Simons Functional

A principal G-bundle E over B is a fibration with fibre a Lie group G and there is a free right (resp. left) action on the total space by elements of G such that this is free and transitive on each fibre. The map $\pi : E \to B$ induces $d\pi : TE \to TB$, where T_eE locally splits into $T_{\pi(e)}B \oplus T_1G$ by local triviality. Thus, $\ker d\pi$ is T_1G which can be canonically identified with $\mathfrak{g}$. For such a bundle, a connection is a 1-form θ on E taking values in $\mathfrak{g}$ and satisfying

$$R_g^*\theta = (\mathrm{ad}_g)^{-1} \circ \theta \tag{1.1}$$

$$\theta(v) = v_{\mathfrak{g}},\ v \in \ker d\pi. \tag{1.2}$$

Here the right action R_g acts by g on the right and $v_{\mathfrak{g}}$ refers to the identification of v with an element $v_{\mathfrak{g}}$ of $\mathfrak{g}$ as mentioned before. In particular, this means that if θ_x denote the Maurer-Cartan form on $E_x = G$ and $i_x : E_x \hookrightarrow E$, then $i_x^*(\theta) = \theta_x$. Also recall the structure equation

$$d\theta_x + \frac{1}{2}[\theta_x, \theta_x] = 0. \tag{1.3}$$

Proposition 1.1. *Let $E \to B$ be as above such that G is connected and simply connected. Let B be a manifold of dimension at most 3. Then the bundle is trivial.*

Proof It suffices to prove that a section $s : B \to E$ exists; then $\phi : B \times G \to E$ defined by $\phi(b, g) = s(b)g$ gives an isomorphism of bundles. The obstructions to finding a section $s : B \to E$ are the cohomology classes $\theta_i \in H^i(B, \pi_{i-1}(G))$. It is well known that $\pi_2(G) = 0$. Therefore, the primary obstruction is at least of degree 4 and there is a section, whence E is trivial. $\square$

In what follows, let M be a closed, connected and oriented 3-manifold and let E be a principal $SU(2)$-bundle on M and we have chosen a trivialization for E. Let

$$\Omega^i(M, su_2) = \Gamma(M, \wedge^i T^*M \otimes (M \times su_2))$$

denote the space of forms on M with values in su_2. There are two obvious ways to extend $a \in \Omega^1(M, su_2)$ to a 1-form on $E = M \times SU(2)$:

(1) using (1.1) and declaring it to be the Maurer-Cartan form on each fibre,

(2) just extend to each copy of $M \times \{g\}$ by (1.1).

Thus, the space of connections on any (principal) $SU(2)$-bundle over M (making use of (1) above) is just $\mathcal{A}(M) = \Omega^1(M, su_2)$ - the space of su_2-valued 1-forms on M. $\mathcal{A}(M)$ is an affine space but when thought of as $\Omega^1(M, su_2)$, it can be treated as a vector space. The curvature of a connection $a \in \mathcal{A}(M)$ is defined to be

$$F_a = da + a \wedge a \in \Omega^2(M, su_2). \tag{1.4}$$

We observe here that

$$(a \wedge a)(X, Y) = \frac{1}{2}\big(a(X)a(Y) - a(Y)a(X)\big) = \frac{1}{2}[a(X), a(Y)]$$

and

$$R_g^* F_a = \mathrm{ad}_{g^{-1}} F_a \tag{1.5}$$

$$i_x^* F_a = 0. \tag{1.6}$$

The last equation follows from (1.3). The *Bianchi identity* follows by differentiating F_a :

$$dF_a + [a, F_a] = 0. \tag{1.7}$$

We introduce the covariant derivative

$$d_a = d + [a, \;] \tag{1.8}$$

which induces the complex

$$\Omega^0(M, su_2) \xrightarrow{d_a} \Omega^1(M, su_2) \xrightarrow{d_a} \Omega^2(M, su_2) \xrightarrow{d_a} \Omega^3(M, su_2). \tag{1.9}$$

The failure of the above to be a chain complex, i.e., $d_a^2 = 0$, is measured by the flatness of a since $d_a^2 = F_a$. In this form, the Bianchi identity is equivalent to $d_a^3 = 0$. We also observe that the adjoint bundle $(su_2)_E = E \times_{SU(2)} su_2$ is just $M \times su_2$. We will encounter (1.9) again in §??? regarding flat connections. In that setting d_a turns out to be an elliptic operator ???. **Add stuff from [DF1]**

Definition 1.2. For $a \in \mathcal{A}(M)$, choose an oriented 4-manifold W which bounds M and collar neighbourhood of the boundary

$$\partial W \subset U = M \times [0, 1].$$

Take any $\tilde{a} \in \mathcal{A}(W)$ such that it is the pullback of a near the boundary. The **Chern-Simons functional** is then defined to be

$$CS(a) = \frac{1}{8\pi^2} \int_W tr(F_{\tilde{a}} \wedge F_{\tilde{a}}). \tag{1.10}$$

This is well defined for two reasons :

(1) Since any closed oriented 3-manifold is cobordant to a point, such a W (as above) always exist. This is a result by R. Thom who originally proved it by calculating the stable homotopy groups of Thom spaces up to 3 using the cohomology structure and homotopy theory. Later Likorish gave another proof in the 60's using Heegard decompositions and the fact that homeomorphisms of surfaces are generated by Dehn twists.

(2) The Chern-Simons functional is well defined because if we take two such 4-manifolds W_1, W_2 with collar neighbourhoods U_1, U_2 and connections a_1, a_2 respectively, then gluing these two give a closed oriented 4-manifold $\tilde{W}$ with a connection $\tilde{a}$. Since $F_{\tilde{a}} \in \Omega^2(M, su_2)$, let

$$F_{\tilde{a}} = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}, \; w_1 + w_4 = 0, \; w_2 + \overline{w_3} = 0,$$

where w_i's are complex valued 2-forms on M. The equivalence class of the coefficient of t^j in the expansion of $\det(I + \frac{ti}{2\pi} F_{\tilde{a}})$ is the Chern class c_j. The expansion is

$$1 + \frac{it}{2\pi}(w_1 + w_4) + \frac{t^2}{4\pi^2}(w_2 \wedge w_3 - w_1 \wedge w_4),$$

whence $c_2 = [(w_2 \wedge w_3 - w_1 \wedge w_4)/4\pi^2]$. On the other hand,

$$\begin{aligned} \frac{1}{8\pi^2} tr(F_{\tilde{a}} \wedge F_{\tilde{a}}) &= \frac{1}{8\pi^2} tr \begin{pmatrix} w_1 \wedge w_1 + w_2 \wedge w_3 & w_1 \wedge w_2 + w_2 \wedge w_4 \\ w_3 \wedge w_1 + w_4 \wedge w_3 & w_3 \wedge w_2 + w_4 \wedge w_4 \end{pmatrix} \\ &= \frac{w_1 \wedge w_1 + w_2 \wedge w_3 + w_3 \wedge w_2 + w_4 \wedge w_4}{8\pi^2} \\ &= \frac{w_2 \wedge w_3 - w_1 \wedge w_4}{4\pi^2}. \end{aligned}$$

Since the $SU(2)$-bundle is trivial c_2 is exact, i.e., $(w_2 \wedge w_3 - w_1 \wedge w_4)/4\pi^2 = \delta\alpha$. Therefore

$$CS(a_1) - CS(a_2) = \int_{\tilde{W}} c_2 = \int_{\partial \tilde{W}} \alpha = 0.$$

Actually, there is an explicit expression for the Chern-Simons functional :

Lemma 1.3.

$$CS(a) = \frac{1}{8\pi^2} \int_M tr(a \wedge da + \frac{2}{3} a \wedge a \wedge a). \tag{1.11}$$

Proof Let $(a_t)_{t \in [0,1]}$ be a path of connections; it determines a connection $a \in \mathcal{A}(M \times [0,1])$. It follows easily from definition that

$$CS(a_1) - CS(a_0) = \frac{1}{8\pi^2} \int_{M \times [0,1]} tr(F_a \wedge F_a).$$

Now take $a_t = ta$. Then

$$F_a = dt \wedge a + t\, da + t^2 a \wedge a$$

and

$$
\begin{aligned}
tr(F_a \wedge F_a) &= tr(t\,dt \wedge a \wedge da + t^2\,dt \wedge a \wedge a \wedge a + t\,dt \wedge da \wedge a + t^2\,da \wedge da) \\
&\quad + tr(t^3\,da \wedge a \wedge a + t^2\,dt\,a \wedge a \wedge a + t^3\,a \wedge a \wedge da) \\
&= 2t\,dt \wedge tr(a \wedge da) + t^2\,tr(da \wedge da) + 2t^2 dt \wedge tr(a \wedge a \wedge a) \\
&\quad + 2t^3 tr(da \wedge a \wedge a) \\
&= d\big(tr(t^2 a \wedge da + \frac{2}{3}t^3 a \wedge a \wedge a)\big).
\end{aligned}
$$

The proof is complete by using Stokes' theorem. □

Reverting back to thinking of a as a connection on the trivial $SU(2)$-bundle on M, we shall see that $\frac{1}{8\pi^2}tr(a\,da + \frac{2}{3}a \wedge a \wedge a)$ is a 3-form on the total space which restricts to the volume form on each fibre. This will be called the Chern-Simons form and denoted by $CS_3(a)$. It is also true (as seen by simply differentiating) that this form is the anti derivative of $\frac{1}{8\pi^2}tr(F_a \wedge F_a)$. We remark in passing that the trace appears since

$$tr : su_2 \otimes su_2 \to \mathbb{R}, \quad A \otimes B \mapsto tr(AB)$$

is the (non-degenerate) Killing form on $SU(2)$.

The differential of the Chern-Simons functional is

$$
\begin{aligned}
d\,CS_a(b) &= \frac{d}{dt}\Big|_{t=0} CS(a+tb) \\
&= \lim_{t\to 0} \frac{1}{8\pi^2 t} \int_M tr\Big(\frac{2t}{3}(a \wedge b \wedge a + a \wedge a \wedge b + b \wedge a \wedge a)\Big) \\
&\quad + \lim_{t\to 0} \frac{1}{8\pi^2 t} \int_M tr\Big(\frac{2t^2}{3}(a \wedge b \wedge b + b \wedge a \wedge b + b \wedge b \wedge a)\Big) \\
&\quad + \lim_{t\to 0} \frac{1}{8\pi^2 t} \int_M tr\Big(\frac{2t^3}{3} b \wedge b \wedge b\Big) + \lim_{t\to 0} \frac{1}{8\pi^2 t} \int_M tr\big(t(a \wedge db + b \wedge da)\big) \\
&= \frac{1}{8\pi^2} \int_M tr(b \wedge da + a \wedge db + 2b \wedge a \wedge a)
\end{aligned}
$$

For any two connections a, b, it follows from definition that $tr(a \wedge b) = 0$; then

$$0 = d\,tr(a \wedge b) = tr(da \wedge b - a \wedge db),$$

which implies $tr(a \wedge db) = tr(da \wedge b) = tr(b \wedge da)$. Thus

$$d\,CS_a(b) = \frac{1}{4\pi^2} \int_M tr(b \wedge F_a). \tag{1.12}$$

In particular, the critical points of the Chern-Simons functional $\mathrm{Crit}(CS) = \mathcal{A}^{\mathrm{flat}}(M)$ is the space of flat connections, i.e., those with curvature $F_a = 0$.

Remark *One can think of curvature as a 1-form on $\mathcal{A} = \mathcal{A}(M)$ by just defining $F : \mathcal{A} \to T^*\mathcal{A}$ such that $F(a)(b)$ is defined to be the right hand side of* (1.12)*. Then it can be verified, independent of* (1.12)*, that F is a closed form. And the strange form of the CS 3-form is just what makes F exact!*

The Hessian is

$$\begin{aligned}
\mathrm{Hess}(CS)_a(b,c) &= \frac{\partial^2\, CS_a}{\partial b\,\partial c} \\
&= \frac{\partial}{\partial b}\Big(\frac{1}{4\pi^2}\int_M tr(c\wedge F_a)\Big) \\
&= \lim_{t\to 0}\frac{1}{4\pi^2 t}\int_M tr\big(c\wedge(F_{a+tb}-F_a)\big) \\
&= \lim_{t\to 0}\frac{1}{4\pi^2 t}\int_M tr\big(c\wedge(t\,db+t\,b\wedge a+t\,a\wedge b+t^2 b\wedge b)\big) \\
&= \frac{1}{4\pi^2}\int_M tr(c\wedge db+c\wedge b\wedge a+c\wedge a\wedge b),
\end{aligned}$$

Since $d_a b = db + b\wedge a + a\wedge b$ (as defined in (1.8)), we get

$$\mathrm{Hess}(CS)_a(b,c) = \frac{1}{4\pi^2}\int_M tr(c\wedge d_a b). \tag{1.13}$$

It is easily verified that $\mathrm{Hess}(CS)_a$ is symmetric.

When a is flat, $\operatorname{im} d_a$ is contained in $\ker d_a$. We shall see later that there is an infinite dimensional symmetry group $\mathcal{G}$, called the group of gauge transformations (see §2). Moreover, $\operatorname{im} d_a$ is the tangent space, at a, of the $\mathcal{G}$-orbit of a. This means that $\ker d_a$ is infinite dimensional, whence the Hessian is degenerate and CS is not Morse.

???Sobolev completion??? from Floer's paper

2 Gauge Symmetry

The group of *gauge transformations* $\mathcal{G}(M)$ is the space of smooth maps from M to $SU(2)$, with point wise multiplication as the group action. An element $g \in \mathcal{G}(M)$ of the gauge group defines a diffeomorphism of the trivial $SU(2)$-bundle E over M by the right action on each fibre; we shall denote this diffeomorphism also by g. Thus g acts on $a \in \mathcal{A}(M)$ by $g \cdot a = g^*a$. Actually, there is an explicit formula for the pullback. Working locally, let $v = (v_1, v_2) \in T_{(m,h)}E$. Then

$$dg : T_{(m,h)}E \to T_{(m,hg(m)^{-1})}E$$

and

$$\begin{aligned} dg(v) &= dg(\gamma_1'(0), \gamma_2'(0)) \\ &= \frac{d}{dt}\Big|_{t=0} \big(\gamma_1(t), \gamma_2(t) g(\gamma_1(t))^{-1}\big) \\ &= \big(v_1, v_2 g^{-1} + h\, d(g^{-1})_m v_1\big). \end{aligned}$$

Thus

$$\begin{aligned} g^*a(v) &= a(dg(v)) \\ &= a(v_1 + v_2 g^{-1} + h\, d(g^{-1})_m v_1) \\ &= a\Big(v_1 + (R_{hg^{-1}})_*(v_2 h^{-1}) + (R_{hg^{-1}})_*\big(h\, d(g^{-1})(v_1) g h^{-1}\big)\Big) \\ &= a\big((R_{hg^{-1}})_*(v_1 + v_2 h^{-1} + h\, d(g^{-1})(v_1) g h^{-1})\big) \\ &= R^*_{hg^{-1}} a\big(v_1 + v_2 h^{-1} + h\, d(g^{-1})(v_1) g h^{-1}\big) \\ &= g h^{-1} a\big(v_1 + v_2 h^{-1} + h\, d(g^{-1})(v_1) g h^{-1}\big) h g^{-1} \\ &= g h^{-1} \big(a(v_1) + v_2 h^{-1} + h\, d(g^{-1})(v_1) g h^{-1}\big) h g^{-1} \\ &= g h^{-1} a(v_1) h g^{-1} + g h^{-1} v_2 g^{-1} + g\, d(g^{-1}) v_1 \\ &= g h^{-1} a(v_1) h g^{-1} + g h^{-1} v_2 g^{-1} - dg(v_1) g^{-1}, \end{aligned}$$

where the last equality follows from differentiating $gg^{-1} = 1$; $dg\, g^{-1} + g\, d(g^{-1}) = 0$.

On the other hand

$$\begin{aligned} gag^{-1}(v) &= ga(v_1 + v_2) g^{-1} \\ &= ga(v_1) g^{-1} + ga(v_2) g^{-1} \\ &= g h^{-1} a(v_1) h g^{-1} + g (R_h)^* a(v_2 h^{-1}) g^{-1} \\ &= g h^{-1} a(v_1) h g^{-1} + g h^{-1} v_2 g^{-1} \end{aligned}$$

and we also have $dg\, g^{-1}(v) := dg(d\pi(v)) g^{1-} = dg(v_1) g^{-1}$. This implies that

$$g \cdot a = gag^{-1} - dg\, g^{-1}. \tag{2.1}$$

It is clear that g^* is natural, i.e., commutes with wedge products and d. In particular, the pullback of the CS 3-form arising from a is just the CS 3-form generated by $g \cdot a$.

Two elements $g_1, g_2 \in \mathcal{G}(M)$ are (path) connected if and only if $\deg g_1 = \deg g_2$ since the target space for both maps $g_i : M \to SU(2)$ is S^3. Consequently, the connected component of $\mathcal{G}(M)$ is

$\mathcal{G}_0(M)$, consisting of elements of degree 0. The cosets of $\mathcal{G}_0(M)$ are the elements of $\pi_0(\mathcal{G}(M))$ and there is a natural injective homomorphism $\deg : \pi_0(\mathcal{G}(M)) \to \mathbb{Z}$ such that

$$[g] \to \deg g.$$

This map is an isomorphism since there exist maps of any given degree.

Lemma 2.1. *For $g_1, g_2 \in g\mathcal{G}_0(M)$, $CS(g_1 \cdot a) = CS(g_2 \cdot a)$.*

Proof Since $\deg g_1 = \deg g_2$, choose a path $\gamma(t)$ connecting the two gauge elements such that $\gamma = g_1$ near $t = 0$ and $\gamma = g_2$ near $t = 1$. This defines an element $\tilde{g} \in \mathcal{G}(M \times [0,1])$ satisfying $\tilde{g} = g_1$ near $M \times \{0\}$ and $\tilde{g} = g_2$ near $M \times \{1\}$. Let $\tilde{a} := \pi_1^*(a) \in \mathcal{A}(M \times [0,1])$. Then

$$\tilde{g}\tilde{a} = \gamma(t)a\gamma(t)^{-1} - d(\gamma(t))\gamma(t)^{-1}$$

and

$$\begin{aligned} F_{\tilde{g}\tilde{a}} &= d(\gamma(t) \cdot a) + (\gamma(t) \cdot a) \wedge (\gamma(t) \cdot a) \\ &= d(\gamma(t)) \wedge a\gamma(t)^{-1} + \gamma(t)da\gamma(t)^{-1} - \gamma(t)a \wedge d(\gamma(t)^{-1}) + d(\gamma(t)) \wedge d(\gamma(t)^{-1}) \\ &\quad + \gamma(t)a \wedge a\gamma(t)^{-1} - \gamma(t)a\gamma(t)^{-1}d(\gamma(t))\gamma(t)^{-1} - d(\gamma(t)) \wedge a\gamma(t)^{-1} \\ &\quad + d(\gamma(t))\gamma(t)^{-1}d(\gamma(t))\gamma(t)^{-1} \\ &= \gamma(t)(da + a \wedge a)\gamma(t)^{-1} \\ &= \tilde{g}F_{\tilde{a}}\tilde{g}^{-1}, \end{aligned}$$

since $d(\gamma(t))\gamma(t)^{-1} + \gamma(t)d(\gamma(t)^{-1}) = 0$. Therefore

$$tr(F_{\tilde{g}\tilde{a}} \wedge F_{\tilde{g}\tilde{a}}) = tr\big(\tilde{g}(F_{\tilde{a}} \wedge F_{\tilde{a}})\tilde{g}^{-1}\big) = tr(F_{\tilde{a}} \wedge F_{\tilde{a}}).$$

Hence

$$CS(g_2 \cdot a) - CS(g_1 \cdot a) = \int_{M\times[0,1]} tr(F_{\tilde{g}\tilde{a}}^2) = \int_{M\times[0,1]} tr(F_{\tilde{a}}^2) = 0$$

because $F_{\tilde{a}}^2 = F_{\pi^*(a)}^2 = \pi^*(F_a^2) = 0$. □

In particular, for $g \in \mathcal{G}_0(M)$

$$CS(g \cdot a) = CS(a). \tag{2.2}$$

We actually have

Lemma 2.2. *For $g \in \mathcal{G}(M), a \in \mathcal{A}(M)$, $CS(g \cdot a) - CS(g \cdot 0) = CS(a)$.*

Proof Let $a_t = ta$ denote the linear path joining a and the trivial connection. This can viewed as a connection on $M \times [0,1]$. Then $g \cdot a_t$ is a path joining $gag^{-1} - dg\,g^{-1}$ and $-dg\,g^{-1}$. Since $g \cdot a_t = tgag^{-1} - dg\,g^{-1}$, the respective curvature

$$\begin{aligned} F &= d(tgag^{-1}) - d(dg\,g^{-1}) + (tgag^{-1} - dg\,g^{-1})^2 \\ &= dt\,gag^{-1} + t^2ga^2g^{-1} + tg\,da\,g^{-1} - 2tgag^{-1}dg\,g^{-1}. \end{aligned}$$

While calculating $F \wedge F$ we observe that any 4-form not involving dt is zero. Thus

$$F \wedge F = 2t^2dt\,ga^3g^{-1} + 2tdt\,ga\,da\,g^{-1} - tdt\,ga^2g^{-1}dg\,g^{-1} - tdt\,gag^{-1}dg\,g^{-1}gag^{-1}.$$

Integrating $tr(F \wedge F)$ on $M \times [0,1]$, we can integrate out the t variable first leading to

$$\begin{aligned}
\frac{1}{8\pi^2}\int_{M\times[0,1]} tr(F\wedge F) &= \frac{1}{8\pi^2}\int_M tr\big(ga\,da\,g^{-1} + \frac{2}{3}ga^3g^{-1}\big) \\
&\quad -\frac{1}{16\pi^2}\int_M tr\big(gag^{-1}(gag^{-1}dg\,g^{-1} + dg\,g^{-1}gag^{-1})\big) \\
&= \frac{1}{8\pi^2}\int_M tr\big(g(a\,da + \frac{2}{3}a^3)g^{-1}\big) \\
&= CS(a).
\end{aligned}$$

Since the integral on the LHS is $CS(g\cdot a) - CS(g\cdot 0)$, this completes the proof. □

We shall see in theorem 2.3 that $CS(g\cdot 0) = \deg g$ and therefore we get the identity

$$CS(g\cdot a) = CS(a) + \deg g. \tag{2.3}$$

There is a more geometric way to see this - one can think of M as the identity section, i.e., $M\times\{1\}$ in the total space E. The CS form coming from a is integrated over this section to give $CS(a)$. The map $g : E \to E$ maps the homology class $[M]$ corresponding to the identity section to $g_*[M]$. Since $CS_3(g\cdot a) = g^*(CS_3(a))$,

$$CS(g\cdot a) = \int_M CS_3(g\cdot a) = \int_{g_*(M)} CS_3(a).$$

We can perturb g (degree remaining fixed) such that $g_*(M) \pitchfork M$. Since $\dim E = 6$, $S = g_*(M)\cap M$ is of dimension 0 and consists of finitely many points. Let $S = \{x_1,\ldots,x_k\}$ and for ε sufficiently small, fix mutually disjoint balls $B_\varepsilon(x_i)$ in M. Then the section given by g on M on the complement of these neighbourhoods can be homotoped to $M\times\{h\}, h\neq 1$ since $SU(2)\setminus\{1\}$ is path connected. Homotope the g-section on $B_\varepsilon(x_i)\setminus B_{\varepsilon/2}(x_i)$ to a section which is given by g near $B_{\varepsilon/2}(x_i)$ and is given by $M\times\{g\}$ near $B_\varepsilon(x_i)$. Denote this new section by $\tilde g$. Then $\tilde g_* M \pitchfork M$ and

$$\int_{g_*M} CS_3(a) = \int_{\tilde g_* M} CS_3(a).$$

But as $\varepsilon\to 0$ this process splits the (smooth) integrating space g_*M into the union of the trivial section $M\times\{h\}$ (which is homotopic to $M\times\{1\}$) and $\{x_i\}\times E_{x_i}$, in the limit. Thus

$$\begin{aligned}
CS(g\cdot a) &= \int_M CS_3(a) + \sum_{i=1}^k \int_{\pm E_{x_i}} CS_3(a) \\
&= CS(a) + \sum_{i=1}^k \pm 1 \\
&= CS(a) + \deg g.
\end{aligned}$$

A few explanations are in order. Here $\pm E_{x_i}$ means that the CS 3-form, which is the volume form when restricted to a fibre, is integrated on $E_{x_i} = G$ with the orientation preserved/reversed respectively. Hence the integral contributes ± 1 accordingly. Thus, changing the section by a map into the group $SU(2)$ changes the integral by the degree of the map of M into $SU(2)$.

Theorem 2.3. *For $g \in \mathcal{G}(M)$, $CS(g \cdot 0) = deg\, g$.*

Proof Let $G = SU(2)$ in the proof, which is divided into three steps :

Step 1

Let $g : M \to G$ be a smooth map. Since $dg : TM \to TG$, one views $dg\, g^{-1}$ as a 1-form with values in the Lie algebra su_2. More explicitly, for $v \in T_m M$

$$(dg\, g^{-1})(v) = (R_{g(m)^{-1}})_*(dg(v)).$$

Working locally

$$dg\, g^{-1} = Adv_1 + Bdv_2 + Cdv_3,$$

where dv_i's form a basis around $m \in M$ for the cotangent space and $A, B, C \in su_2$. Consequently we write

$$dg\, g^{-1} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

where α, β are 1-forms satisfying $\alpha + \overline{\alpha} = 0$. Using the fact that $w \wedge w = 0$ for any 1-form w, we get

$$(dg\, g^{-1})^2 = \begin{pmatrix} -\beta\overline{\beta} & (\alpha - \overline{\alpha})\beta \\ (\alpha - \overline{\alpha})\overline{\beta} & -\beta\overline{\beta} \end{pmatrix},$$

whence

$$(dg\, g^{-1})^3 = \begin{pmatrix} (\overline{\alpha} - 2\alpha)\beta\overline{\beta} & -\alpha\beta\overline{\alpha} \\ \alpha\overline{\beta}\overline{\alpha} & (2\overline{\alpha} - \alpha)\beta\overline{\beta} \end{pmatrix}.$$

Taking trace we get

$$tr\big((dg\, g^{-1})^3\big) = 3(\overline{\alpha} - \alpha)\beta\overline{\beta} = -6\alpha\beta\overline{\beta}.$$

Now let $A = (a_{ij}), B = (b_{ij}), C = (c_{ij})$. Calculating $-6\alpha\beta\overline{\beta}$ in terms of these co-ordinates we get

$$\begin{aligned} tr\big((dg\, g^{-1})^3\big) &= -6(a_{12}dv_1 + b_{12}dv_2 + c_{12}dv_3)(\overline{a}_{12}dv_1 + \overline{b}_{12}dv_2 + \overline{c}_{12}dv_3)(a_{11}dv_1 \\ &\quad + b_{11}dv_2 + c_{11}dv_3) \\ &= -6(a_{12}\overline{b}_{12}dv_1dv_2 + a_{12}\overline{c}_{12}dv_1dv_3 + b_{12}\overline{a}_{12}dv_2dv_1 + b_{12}\overline{c}_{12}dv_2dv_3 \\ &\quad + c_{12}\overline{a}_{12}dv_3dv_1 + c_{12}\overline{b}_{12}dv_3dv_2)(a_{11}dv_1 + b_{11}dv_2 + c_{11}dv_3) \\ &= -6((a_{12}\overline{b}_{12} - \overline{a}_{12}b_{12})c_{11} + (b_{12}\overline{c}_{12} - \overline{b}_{12}c_{12})a_{11} \\ &\quad + (c_{12}\overline{a}_{12} - \overline{c}_{12}a_{12})b_{11})dv_1dv_2dv_3\, (=: -6\Phi). \end{aligned}$$

We calculate

$$CS_3(g \cdot 0) = tr\big((d(-dg\, g^{-1})(-dg\, g^{-1}) - \frac{2}{3}(dg\, g^{-1})^3\big) = \frac{1}{3}tr\big((dg\, g^{-1})^3\big) = -2\Phi. \tag{2.4}$$

The action of an element g of the gauge group on the trivial connection gives us $-dg\, g^{-1}$. Thus, in view of (2.4)

$$CS(g \cdot 0) = -\frac{1}{4\pi^2}\int_M \Phi. \tag{2.5}$$

Step 2

For the group G, the identity element $e = \mathrm{Id}_{2\times 2}$ is represented by $(1,0,0,0) \in S^3$. Let $\omega_e = dx_1 \wedge dx_2 \wedge dx_3$ be a generator of $\bigwedge^3 \mathfrak{g}^*$. Right translation allows us to define a global 3-form ω on G by defining

$$\omega_{g^{-1}}(w_1, w_2, w_n) = \omega_e((R_g)_* w_1, (R_g)_* w_2, (R_g)_* w_3)$$

for w_i's in $T_{g^{-1}}G$. We claim that

Proposition 2.4. *The* 3*-form on* S^3

$$\tilde{\omega} = x_0 dx_1 dx_2 dx_3 - x_1 dx_0 dx_2 dx_3 + x_2 dx_0 dx_1 dx_3 - x_3 dx_0 dx_1 dx_2 \tag{2.6}$$

that arises from it's embedding in $\mathbb{R}^4$ *and the global* 3*-form* ω *(defined by right-translating* $\omega_e = dx_1 dx_2 dx_3$*) are the same.*

Proof of proposition Recall that an element $y = (y_0, y_1, y_2, y_3) \in S^3$ is identified with the matrix

$$\begin{pmatrix} y_0 + iy_1 & y_2 - iy_3 \\ -y_2 - iy_3 & y_0 - iy_1 \end{pmatrix} \in G.$$

Then if $x = (x_0, -x_1, -x_2, -x_3)$, the inverse is $x^{-1} = (x_0, x_1, x_2, x_3)$. Let $\overline{a} = (a_0, a_1, a_2, a_3), \overline{b}, \overline{c} \in T_{x^{-1}}S^3$. This implies that $\sum_{j=0}^{3} a_j x_j = 0$ etc. We have

$$\tilde{\omega}(\overline{a}, \overline{b}, \overline{c}) = \frac{x_0}{6}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} - \frac{x_1}{6}\begin{vmatrix} a_0 & b_0 & c_0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \frac{x_2}{6}\begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} - \frac{x_3}{6}\begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix},$$

which is just one-sixth of the determinant of

$$\tilde{A} = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \end{pmatrix}.$$

Now let us calculate $(R_x)_*\overline{a}$, which by our observation before equals the action of x on the right of $\overline{a}$, thought of as a matrix, i.e.,

$$\begin{aligned}(R_x)_*\overline{a} &= \begin{pmatrix} a_0 + ia_1 & a_2 - ia_3 \\ -a_2 - ia_3 & a_0 - ia_1 \end{pmatrix}\begin{pmatrix} x_0 - ix_1 & -x_2 + ix_3 \\ x_2 + ix_3 & x_0 + ix_1 \end{pmatrix} \\ &= \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}\alpha &= \sum_{j=0}^{3} a_j x_j + i(-a_0 x_1 + a_1 x_0 + a_2 x_3 - a_3 x_2) \\ \beta &= (-a_0 x_2 - a_1 x_3 + a_2 x_0 + a_3 x_1) - i(-a_0 x_3 + a_1 x_2 - a_2 x_1 + a_3 x_0).\end{aligned}$$

Translating this to vector form we have :

$$(R_x)_*\overline{a} = (0, -a_0x_1 + a_1x_0 + a_2x_3 - a_3x_2, -a_0x_2 - a_1x_3 + a_2x_0 + a_3x_1, -a_0x_3 + a_1x_2 - a_2x_1 + a_3x_0)$$

and similarly for $\overline{b}, \overline{c}$. By definition,

$$\begin{aligned}
\omega(\overline{a}, \overline{b}, \overline{c}) &= \omega_e((R_x)_*\overline{a}, (R_x)_*\overline{b}, (R_x)_*\overline{c}) \\
&= \frac{1}{6}\begin{vmatrix} dx_1((R_x)_*\overline{a}) & dx_1((R_x)_*\overline{b}) & dx_1((R_x)_*\overline{c}) \\ dx_2((R_x)_*\overline{a}) & dx_2((R_x)_*\overline{b}) & dx_2((R_x)_*\overline{c}) \\ dx_3((R_x)_*\overline{a}) & dx_3((R_x)_*\overline{b}) & dx_3((R_x)_*\overline{c}) \end{vmatrix} \\
&= \frac{1}{6}\begin{vmatrix} -a_0x_1 + a_1x_0 + a_2x_3 - a_3x_2 & \cdots & -c_0x_1 + c_1x_0 + c_2x_3 - c_3x_2 \\ -a_0x_2 - a_1x_3 + a_2x_0 + a_3x_1 & \cdots & -c_0x_2 - c_1x_3 + c_2x_0 + c_3x_1 \\ -a_0x_3 + a_1x_2 - a_2x_1 + a_3x_0 & \cdots & -c_0x_3 + c_1x_2 - c_2x_1 + c_3x_0 \end{vmatrix} \\
&= \frac{1}{6}\left|\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \end{pmatrix}\begin{pmatrix} -x_1 & -x_2 & -x_3 \\ x_0 & -x_3 & x_2 \\ x_3 & x_0 & -x_1 \\ -x_2 & x_1 & x_0 \end{pmatrix}\right| \\
&= \frac{1}{6}\text{det}AB.
\end{aligned}$$

If we augment A, B with x such that

$$\tilde{A} = \left(\frac{x}{A}\right),\ \tilde{B} = \left(\ B \mid x^t\ \right).$$

It is easy to check that

$$\tilde{A}\tilde{B} = \left(\begin{array}{c|c} 0 & 1 \\ \hline AB & 0 \end{array}\right)$$

and hence $\text{det}\tilde{A}\tilde{B} = \text{det}AB$. We claim that $\tilde{B} \in SL_4(\mathbb{R})$. We use a cyclic permutation to put the fourth column as the first one and

$$\begin{aligned}
\text{det}\tilde{B} &= \begin{vmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{vmatrix} \\
&= x_0\begin{vmatrix} x_0 & -x_3 & x_2 \\ x_3 & x_0 & -x_1 \\ -x_2 & x_1 & x_0 \end{vmatrix} + x_1\begin{vmatrix} x_1 & -x_3 & x_2 \\ x_2 & x_0 & -x_1 \\ x_3 & x_1 & x_0 \end{vmatrix} - x_2\begin{vmatrix} x_1 & x_0 & x_2 \\ x_2 & x_3 & -x_1 \\ x_3 & -x_2 & x_0 \end{vmatrix} \\
&\quad + x_3\begin{vmatrix} x_1 & x_0 & -x_3 \\ x_2 & x_3 & x_0 \\ x_3 & -x_2 & x_1 \end{vmatrix} \\
&= x_0^2(\sum_{i=0}^{3} x_i^2) + x_1^2(\sum_{i=0}^{3} x_i^2) + x_2^2(\sum_{i=0}^{3} x_i^2) + x_3^2(\sum_{i=0}^{3} x_i^2) \\
&= 1.
\end{aligned}$$

This completes our claim. Consequently, $\text{det}AB = \text{det}\tilde{A}$ and the forms are the same. $\square$

Finally we observe that

$$\int_G \omega = \int_G \tilde{\omega} = \int_{\partial B_4} \tilde{\omega} = \int_{B_4} d\tilde{\omega} = 4\int_{B_4} dx_0dx_1dx_2dx_3 = 4\text{vol}(B_4) = 2\pi^2. \tag{2.7}$$

Step 3

Since $g: M \to G$ is a smooth map, it follows from (2.7) that

$$2\pi^2 \deg(g) = \deg(g) \int_G \omega := \int_M g^*(\omega).$$

As before, we work locally and write

$$dg\, g^{-1} = Adv_1 + Bdv_2 + Cdv_3,$$

where dv_i's form a basis around $m \in M$ for the cotangent space and $A, B, C \in su_2$. Thus

$$\begin{aligned}
g^*(\omega)(v_1, v_2, v_3) &= \omega_e(dg\, g^{-1}(v_1), dg\, g^{-1}(v_2), dg\, g^{-1}(v_3)) \\
&= \det(dx_j(dg\, g^{-1}(v_k))_{jk} \\
&= \det \begin{pmatrix} a_{11}/i & b_{11}/i & c_{11}/i \\ (a_{12} + \overline{a}_1 2)/2 & (b_{12} + \overline{b}_1 2)/2 & (c_{12} + \overline{c}_1 2)/2 \\ -(a_{12} - \overline{a}_{12})/2i & -(b_{12} - \overline{b}_{12})/2i & -(c_{12} - \overline{c}_{12})/2i \end{pmatrix} \\
&= \frac{1}{4} \det \begin{pmatrix} a_{11} & b_{11} & c_{11} \\ a_{12} + \overline{a}_{12} & b_{12} + \overline{b}_{12} & c_{12} + \overline{c}_{12} \\ a_{12} - \overline{a}_{12} & b_{12} - \overline{b}_{12} & c_{12} - \overline{c}_{12} \end{pmatrix} \\
&= -\frac{1}{2} \big((a_{12}\overline{b}_{12} - \overline{a}_{12} b_{12}) c_{11} + (b_{12}\overline{c}_{12} - \overline{b}_{12} c_{12}) a_{11} + (c_{12}\overline{a}_{12} - \overline{c}_{12} a_{12}) b_{11} \big).
\end{aligned}$$

This implies that $g^*(\omega) = -\frac{1}{2}\Phi$ and hence (using (2.5))

$$\deg(g) = \frac{1}{2\pi^2} \int_M g^*(\omega) = -\frac{1}{4\pi^2} \int_M \Phi = CS(g \cdot 0), \tag{2.8}$$

concluding the proof. □

Comments about the $<,>$ being integral such that ... and its relation to $H^4(BG, /Z?)$

3 Gauge etc.

It follows from (2.3) that CS is not well-defined on $\mathcal{B}(M) := \mathcal{A}(M)/\mathcal{G}(M)$ but it is on its infinite cyclic cover $\tilde{\mathcal{B}}(M) := \mathcal{A}(M)/\mathcal{G}^0(M)$. The action of $\mathcal{G}(M)$ on $\mathcal{A}(M)$ is not free but has very small stabilizers :

Lemma 3.1. *For $a \in \mathcal{A}(M)$, the stabilizer $\mathcal{G}^a = \{g \in \mathcal{G}(M) |\, g{\cdot}a = a\}$ is one of the groups $\{\pm 1\}, U(1)$ or $SU(2)$.*

Proof First notice that $\{\pm 1\} \subset \mathcal{G}^a$ for any $a \in \mathcal{A}(M)$. If $g \in \mathcal{G}^a$ then

$$dg = ga - ag. \tag{3.1}$$

Fix $m \in M$ and define a map :

$$\phi_m : \mathcal{G}^a \to SU(2),\ g \mapsto g(m). \tag{3.2}$$

Observe that this map is a homomorphism and $\operatorname{Im} \phi_m$ is a closed subgroup of $SU(2)$ since $\mathcal{G}^a$ is closed in $\mathcal{G}(M)$. ϕ_m is injective because if $g_1, g_2 \in \mathcal{G}^a$ satisfies $g_1(m) = g_2(m)$, then set $g = g_2 g_1^{-1} \in \mathcal{G}^a$. Then g satisfies (3.1) and $g(m) = 1$. Thus $dg_m = 0$ and hence g is locally constant around m. Since g exists globally, by connectedness of M, $g \equiv 1$, whence $g_1 \equiv g_2$. Thus an element of the stabilizer is uniuely determined by its value at one point. Changing m to another point simply changes the image of $\mathcal{G}^a$ by an isomorphism. Consequently, one may think of $\mathcal{G}^a$ as a closed subgroup of $SU(2)$. By a simple dimension argument the possibilities (upto isomorphism) are $SU(2), U(1)$ or finite subgroups of $SU(2)$.

Let $a \in \mathcal{A}(M)$ such that $\mathcal{G}^a$ is finite. This rules out the zero connection which has the full stabilizer. We claim that each $g \in \mathcal{G}^a$ is a constant map. Let $\tilde{g} : I \to \mathcal{G}(M)$ with $g \in \mathcal{G}^a$; write $g_t = \tilde{g}(t)$ and observe that if it satisfies the equation

$$\frac{dg_t}{dt} = g_t a - a g_t, \tag{3.3}$$

then by discreteness of $\mathcal{G}^a$, $g_t \equiv g$. This implies that

$$ga - ag = 0 \tag{3.4}$$

and $dg = ga - ag = 0$; hence g is a constant map (which we also denote by g by abuse of notation).

Set $V \subset su_2$ to be the vector space generated by the values taken by a over TM and choose a basis; (3.4) then translates to :

$$V \subseteq \ker(\mathrm{ad}_g).$$

There are three cases :

(i) dim $V = 1$: Exponentiating V we get a connected 1-parameter abelian subgroup of $SU(2)$ (denoted by T); it must be a maximal tori since the rank of $SU(2) = 1$; hence isomorphic to $U(1)$. It is known that the centralizer of a maximal torus is the torus itself. Therefore, any element of T would suffice as g. Thus $\mathcal{G}^a \cong U(1)$, a contradiction.

(ii) dim $V = 2$: Exponentiating each linear subspace spanned by the basis vectors we get two maximal tori T_1 and T_2 and g lies in the centralizer of both. Thus $g \in T_1 \cap T_2$. It is also known that the intersection of any two maximal tori (of $SU(2)$) is $\{\pm 1\}$. Hence $\mathcal{G}^a = \{\pm 1\}$.

(iii) dim $V = 3$: $g \in \{\pm 1\}$; consequently $\mathcal{G}^a = \{\pm 1\}$.

This completes the proof. □

Definition 3.2. If $\mathcal{G}^a = \{\pm 1\}$, a is called *irreducible.* All other connections are *reducible.*

Principal Orbit Type Theorem - Bourbaki ??? Palais
Stabilizers - Fukaya pg 28
elliptic complex
Bott-Morse ?

Let $\exp : su_2 \to SU(2)$ denote the exponential map. Given $f : M \to su_2$, define a family of maps $\{g_t\}$ such that

$$g_t := \exp(tf) : M \to SU(2).$$

Clearly, $g_0 \equiv 1$ and $g_0' = f$. Then $g_t \cdot a \in \mathcal{G}a$ and the derivative lies in $T_a\mathcal{G}a$, i.e.,

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} g_t \cdot a &= g_0' a g_0^{-1} + g_0 a (g_0^{-1})' - \frac{d}{dt}\Big|_{t=0} dg_t g_0^{-1} + dg_0 \frac{d}{dt}\Big|_{t=0} g_t^{-1} \\ &= fa + af + df \ (= d_a f). \end{aligned} \tag{3.5}$$

This implies that $\operatorname{im} d_a \subseteq T_a\mathcal{G}a$ (ref (1.9)). Conversely, if $\gamma(t) := g_t \cdot a$ is curve on $\mathcal{G}(M)$ satisfying $g_0 = 1$ (consequently $\gamma(0) = a$), then $f = g_0' \in \Omega^0(M, su_2)$ and (3.5) shows that

$$d_a f = \frac{d}{dt}\Big|_{t=0} g_t \cdot a.$$

As a consequence,

$$\operatorname{im} d_a = T_a\mathcal{G}a. \tag{3.6}$$

Invoking (1.9) again for a flat connection $a \in \Omega^1(M, su_2)$, we see that $d_a^2 = 0$, whence $T_a\mathcal{G}a$ lies in the kernel of d_a. This proves that $\ker d_a$ is infinite dimensional.

Add M to be Riemannian somewhere for L^2-adjoint and Hodge star

4 ASD connections and Floer Homology

For any oriented Riemannian manifold X, let $*_X$ denote the Hodge star operator on X. To interpret the gradient trajectories for the Chern-Simons functional, we need *anti-self-dual* connections on $M \times \mathbb{R}$:

Definition 4.1. Let W be an oriented Riemannian 4-manifold. An $a \in \mathcal{A}(W)$ is called anti-self-dual (ASD) if

$$F_a + *_W F_a = 0. \tag{4.1}$$

Given a family $a_t \in \mathcal{A}(M), t \in \mathbb{R}$, one can get $\tilde{a} \in \mathcal{A}(M \times \mathbb{R})$. If $\tilde{a}$ has a vanishing dt-term then it is in *temporal gauge*. Let $E \to M$ be the trivial $SU(2)$-bundle. Then $E \times [0,\infty) \to M \times [0,\infty)$ is the pullback of $E \to M$. Choose a connection $\tilde{a}$ on $E \times [0,\infty)$, written as

$$\tilde{a} = \eta_t + \xi_t\, dt,\ \ t \in [0,\infty),\ \eta_t \in \Omega^1(M, su_2),\ \xi_t \in \Omega^0(M, su_2).$$

Lemma 4.2. *There is a unique gauge transformation $\tilde{g}$ of $E \times [0,\infty)$ which satisfies*

$$\tilde{g}\big|_{E\times\{0\}} = g \in \mathcal{G}(M),\ \ \tilde{g}^*(\tilde{a}) = \tilde{\eta}_t \in \Omega^1(M, su_2). \tag{4.2}$$

In other words, the transformed connection has no dt-component. Thus if we have $a_t \in \mathcal{A}(M)$ as before (and hence $\tilde{a}$), we split it into $\tilde{a}_+$ and $\tilde{a}_-$, thought of as connections as $E \times [0,\infty)$ and $E \times (-\infty, 0]$ respectively. Thus, there exists gauge transformation $\tilde{g}_+$ (resp. $\tilde{g}_-$) on $E \times [0,\infty)$ (resp. $E \times (-\infty, 0]$) satisfying (4.2). Gluing the two gauge transformations give a gauge transformation $\tilde{g}$ on $E \times \mathbb{R}$ such that $\tilde{g} \cdot \tilde{a} := \tilde{g}^*(\tilde{a})$ has no dt-component. We conclude that for any $\tilde{a} \in \mathcal{A}(M \times \mathbb{R})$ there is a $\tilde{g} \in \mathcal{G}(M \times \mathbb{R})$ such that $\tilde{g} \cdot \tilde{a}$ is in temporal gauge and it is unique up to $\mathcal{G}(M)$.

Proof Let $\tilde{g}_t : M \to SU(2)$ denote the map associated to $\tilde{g}\big|_{E\times\{t\}}$ and $\phi_t = \tilde{g_t}^*(\theta)$. Then the first condition asserts $\tilde{g}_0 = g$. The second condition is equivalent to saying $\tilde{g}^*(\tilde{a})(\frac{\partial}{\partial t}) = 0$. Since

$$\tilde{g}^*(\tilde{a})\big|_{E\times\{t\}} = \mathrm{ad}_{\tilde{g}_t^{-1}}\tilde{a} + \phi_t,$$

we see that

$$\mathrm{ad}_{\mathrm{g}_t{}^{-1}}\xi_t + \phi_t(\frac{\partial}{\partial t}) = 0.$$

Written as a differential equation (omitting the t-subscripts), this looks like

$$\tilde{g}^{-1}\xi\, \tilde{g} + \tilde{g}^{-1}\frac{\partial \tilde{g}}{\partial t} = 0. \tag{4.3}$$

This is a first order PDE having a unique solution satisfying $\tilde{g}_0 = g$. □

The link between the Chern-Simons functional and ASD connections is the following :

Lemma 4.3. *Let $\tilde{a} \in \mathcal{A}(M \times \mathbb{R})$ be in temporal gauge. Then $\tilde{a}$ is ASD if and only if for $a_t = \tilde{a}|_{M\times\{t\}}$,*

$$\frac{da_t}{dt} = *_M F_{a_t}. \tag{4.4}$$

Proof Let $\tilde{a} = \eta_t \in \Omega^1(M, su_2)$ by the temporal gauge assumption. Since it suffices to prove the result locally, let

$$\eta_t = \eta_1 dx_1 + \eta_2 dx_2 + \eta_3 dx_3$$

where η_i's are functions on $M \times \mathbb{R}$ taking values in su_2. Since

$$d\tilde{a} = \sum_{i \neq j} (\partial_i \eta_j - \partial_j \eta_i) dx_i \wedge dx_j - \sum_{i=1}^{3} \partial_t \eta_i dx_i \wedge dt,$$

we get

$$\frac{da_t}{dt} = d\tilde{a}(\partial/\partial t) = \sum_{i=1}^{3} \partial_t \eta_i dx_i. \tag{4.5}$$

Let $S_1 = \{(1,2),(2,3),(3,1)\}$ and $S_2 = \{(1,2,3),(2,3,1),(3,1,2)\}$ to make equations look neat. Since

$$F_{a_t} = \sum_{(i,j) \in S_1} (\partial_i \eta_j - \partial_j \eta_i) dx_i \wedge dx_j + \sum_{(i,j) \in S_1} [\eta_i, \eta_j] dx_i \wedge dx_j$$

applying the Hodge star we have

$$*_M F_{a_t} = \sum_{(i,j,k) \in S_2} (\partial_i \eta_j - \partial_j \eta_i + [\eta_i, \eta_j]) dx_k. \tag{4.6}$$

Consequently, $\tilde{a}$ satisfies (4.4) if and only if the following holds :

$$\partial_t \eta_1 = \partial_2 \eta_3 - \partial_3 \eta_2 + [\eta_2, \eta_3] \tag{4.7}$$
$$\partial_t \eta_2 = \partial_3 \eta_1 - \partial_1 \eta_3 + [\eta_3, \eta_1] \tag{4.8}$$
$$\partial_t \eta_3 = \partial_1 \eta_2 - \partial_2 \eta_1 + [\eta_1, \eta_2]. \tag{4.9}$$

Now we calculate :

$$F_{\tilde{a}} = \sum_{(i,j) \in S_1} (\partial_i \eta_j - \partial_j \eta_i + [\eta_i, \eta_j]) dx_i \wedge dx_j - \sum_{i=1}^{3} \partial_t \eta_i dx_i \wedge dt \tag{4.10}$$

and

$$*_W F_{\tilde{a}} = \sum_{(i,j,k) \in S_2} (\partial_i \eta_j - \partial_j \eta_i + [\eta_i, \eta_j]) dx_k \wedge dt - \sum_{(i,j,k) \in S_2} \partial_t \eta_i dx_j \wedge dx_k. \tag{4.11}$$

Since $\tilde{a}$ is ASD if and only if $F_{\tilde{a}} + *_W F_{\tilde{a}} = 0$, using (4.10) and (4.11) we get the same conditions as (4.7),(4.8) and (4.9). $\square$

5 The Moduli Space of Flat Connections

As before, let E denote the trivial $SU(2)$-bundle over M. Let $\mathcal{A}^p_k(M) = L^p_k(\Omega^1(M, su_2))$ denote the Sobolev type space of 1-forms on M with values in su_2 such that the zero element is the product trivial connection on E. The gauge group can be identified with $\mathcal{G}^p_k(M) = L^p_{k+1}(M, SU(2))$ acting on $\mathcal{A}^p_k(M)$ by the usual rule (2.1). By the Sobolev embedding theorem applied to manifolds ??? if $k = 1, p = 4$ then $\mathcal{A}(M) = \mathcal{A}^4_1(M)$ consists of continuous 1-forms and $\mathcal{G}(M) = \mathcal{G}^4_2(M)$ consists of C^1-maps. Since spaces which are quotient by infinite dimensional groups can behave badly, we take completions of smooth connections and smooth gauge transformations inside suitable Sobolev spaces.

Define $\mathcal{B}(M) = \mathcal{A}(M)/\mathcal{G}(M)$ and $\widetilde{\mathcal{B}}(M) = \mathcal{A}(M)/\mathcal{G}_0(M)$. These spaces are well-defined since $k+1 > 3/p$ with our choice of $k = 1, p = 4$. ??? As seen before, the CS functional is not well defined on $\mathcal{B}(M)$ but it is on the infinite cyclic cover $\widetilde{\mathcal{B}}(M)$. Let $\mathcal{A}^{\text{flat}}(M) \subset \mathcal{A}(M)$ denote the space of flat connections on P. The curvature transforms as a tensor; hence $\mathcal{A}^{\text{flat}}(M)$ is gauge invariant.

Definition 5.1. The space $\mathcal{A}^{\text{flat}}(M)/\mathcal{G}(M)$ of equivalence classes of flat connections is defined to be the moduli space of flat connections and denoted $\mathcal{M}_M$.

We have an identification :

Theorem 5.2.

$$\mathcal{M}_M \cong \mathcal{R}(M) := \text{Hom}(\pi_1(M), SU(2))/SU(2). \tag{5.1}$$

This let's us interpret the moduli space more geometrically. In fact, we analyze $\mathcal{R}(M)$ in §6 and it turns out to be a real algebraic set. For the identification at hand, we make use of holonomy groups.

Proof Let $\pi_1(M) = \pi_1(M, m)$ and $e \in E_m$. If we change the point $e \in E_m$ to $e' \in E_m$, then $e' = e \cdot h$ for some unique $h \in SU(2)$. Then

$$\text{Hol}_{e'}(a) = h^{-1}\text{Hol}_e(a)\, h.$$

Thus, the holonomy group $\text{Hol}(a) \subset SU(2)$ is well defined up to conjugation. Observe that $\text{Hol}(a)$ is a discrete subgroup of $SU(2)$ since $\text{Hol}^0(a) = \{1\}$. This also implies that the holonomy along a loop depends only on the homotopy class of the loop. The existence of a natural surjective homomorphism $\pi_1(M) \to \text{Hol}_p(a)$ induces a map

$$\phi : \mathcal{A}^{\text{flat}}(M) \to \mathcal{R}(M).$$

If we change a to $g \cdot a$ for some $g \in \mathcal{G}(M)$, then the horizontal lifts $\tilde{\gamma}, \tilde{\gamma}_g$ of any loop γ for $a, g \cdot a$ respectively are related by the conjugation by g. Thus,

$$\phi(g \cdot a) = g(m)\phi(a)g(m)^{-1};$$

consequently ϕ descends to an injective map

$$\overline{\phi} : \mathcal{M}_M \to \mathcal{R}(M).$$

To prove surjectivity, let $\rho : \pi_1(M) \to SU(2)$ be a representation. Consider the universal cover $\widetilde{M}$ as a principal $\pi_1(M)$-bundle over M. One should think of an element $\gamma \in \pi_1(M)$ acting on $\widetilde{M}$ by Deck (gauge) transformations. Extend this to an action of $\pi_1(M)$ on $\widetilde{M} \times SU(2)$ by ρ, i.e., $\gamma \cdot (\tilde{m}, h) = (\gamma \cdot \tilde{m}, \rho(\gamma^{-1})h)$. The quotient $E_\rho := (\widetilde{M} \times SU(2))/\pi_1(M)$ is a principal $SU(2)$-bundle over M (and hence trivial). The trivial connection $\o$ on $\widetilde{M} \times SU(2)$ descends to a 1-form a on E_ρ with values in su_2 such that $da + a \wedge a = 0$. The (free) right action R_g on $\tilde{M} \times SU(2)$ commutes with the action of $\pi_1(M)$. Thus $R_g^* a = \mathrm{ad}_{g^{-1}} \circ a$. Since $\o$ was the Maurer-Cartan form on $SU(2)$, so is a. Therefore, a is a flat connection on E_ρ. If we change ρ to $g_0 \rho g_0^{-1}$ for some $g_0 \in SU(2)$, then one defines a bundle map

$$g : E_\rho \to E_{g_0 \rho g_0^{-1}},\ [\tilde{m}, h] \mapsto [\tilde{m}, gh].$$

$\o$ induces flat connections on E_ρ and $E_{g_0 \rho g_0^{-1}}$ and the former is the pullback of the latter by g. Therefore, the connections are gauge equivalent. □

ASD stuff from Fukaya; may need Floer's transversality theorem (?).

6 Invariants for Homology 3-Spheres

In lectures at MSRI in 1985, Andrew Casson described an integer valued invariant λ of oriented homology 3-spheres (see also [AM]). This can be thought of as counting the number of conjugacy classes of representations $\pi_1(M) \to SU(2)$, in the same sense that the Lefschetz number of a map counts the number of fixed points. According to Casson's original definition, $\lambda(M)$ is actually half this number. K. Walker (see [W]) extended this invariant to rational homology 3-sphere. In another direction, Floer defined a $\mathbb{Z}_8$-graded homology group associated to any oriented, closed homology 3-sphere such that the Euler characteristic is $\lambda(M)$. This section gives an overview of these constructions, following [W] and [AF] closely for Casson's invariant and instanton-invariant for 3-manifolds, respectively.

6.1 Topology of Representation Spaces

For any finitely presented group H, let $\mathcal{R}(H)^\sharp$ denote $\mathrm{Hom}(H, SU(2))$, topologized with the compact open topology with H having the discrete topology and $SU(2)$ having the usual topology. Notice that a homomorphism $\phi : H \to H'$ induces $\phi^\sharp : \mathcal{R}(H)^\sharp \to \mathcal{R}(H')^\sharp$. The natural (faithful) action of $(SO(3))$ $SU(2)$ on $SU(2)$ induces a natural action on $\mathcal{R}(H)^\sharp$ and $\mathcal{R}(H')^\sharp$ such that it commutes with any $\phi^\sharp$. (This rather odd appearance of $\sharp$ becomes clear only later in the section and the convenience is purely notational.)

Let

$$\begin{aligned} S_0(H) &= \{\rho \,|\, \rho : H \to \{\pm \mathrm{I}\}\} \\ S_1(H) &= \{\rho \,|\, \rho : H \to S^1\} \,\text{(diagonal repn.)} \\ S(H) &= \{\rho \,|\, \rho : H \to SU(2) \,\text{is reducible}\}. \end{aligned}$$

It follows from definition that

$$S_0(H) \subset S_1(H) \subset S(H) \subset \mathcal{R}(H)^\sharp. \tag{6.1}$$

Since any compact Lie group is the union of the conjugates of any maximal torus,

$$SU(2) = \bigcup_g g\, S^1 g^{-1}. \tag{6.2}$$

Using this one can show that any reducible representation is conjugate to a diagonal representation. In our case, $H = \pi_1(M)$ is the fundamental group of a homology 3-sphere. Since any reducible representation ρ factors through $\overline{\rho} : \pi_1(M) \to S^1$, it corresponds to an element of $H^1(M, S^1)$, which is zero since $H_1(M, \mathbb{Z}) = 0$. Thus, in the quotient space of representations $\mathcal{R}(M) := \mathcal{R}(H)^\sharp / SU(2)$, all the reducible and diagonal elements vanish. Thus the equivalence classes of irreducible representations is just $\mathcal{R}(M)^* = \mathcal{R}(M) \setminus \{1\}$.

Choose a finite set of generators $S = \{g_1, \ldots, g_n\}$ for $H = \pi_1(M)$ and define the map

$$F_S : \mathcal{R}(H)^\sharp \to (SU(2))^n, \;\; \rho \mapsto (\rho(g_1), \ldots, \rho(g_n)).$$

Let $R = \{r_\alpha\}$ be a set of relations. Then

$$V(S) = \{A_1, \ldots, A_n) \in (SU(2))^n \,|\, r(A_1, \ldots, A_n) = 1\,, r \in R\}$$

is a real algebraic variety, homeomorphic to $\mathcal{R}(H)^\sharp$. Consequently

$$\mathcal{R}(M) = \mathcal{R}(H)^\sharp/SU(2) \subset (SU(2))^n/SU(2) = \underbrace{S^1 \times \cdots \times S^1}_{n}.$$

We **claim** that $\mathcal{R}(M)$ is a real algebraic set.

Definition 6.1. For any space X, let

$$\mathcal{R}(X) := \mathrm{Hom}(\pi_1(X), SU(2))/SU(2) \tag{6.3}$$

be the set of equivalence classes of representations where $SU(2)$ acts by conjugation.

Let M be an oriented homology 3-sphere with a genus g *Heegaard splitting*, i.e., $M = M_1 \cup_{\Sigma_g} M_2$ is the union of two genus g handlebodies glued along $\partial M_1 = \partial M_2 = \Sigma_g$, a surface of genus g. The minimum such g is called the *Heegaard genus* of M. Since the generators of $\pi_1(M_i), i = 1, 2$ (free group on g generators) can be chosen to be lying on ∂M_i resp., it follows that $\iota_i : \Sigma_g \hookrightarrow M_i$ induces a surjection of fundamental groups. Using this observation and Van Kampen's theorem, we conclude that the inclusions $M_i \hookrightarrow M$ induces surjection at the level of π_1. Let Σ_g^* denote $\Sigma_g \setminus D_2$. We thus have

$$\begin{array}{ccccc} & & \pi_1(M_2) & & \\ & \nearrow & & \searrow & \\ \pi_1(\Sigma_g^*) \longrightarrow \pi_1(\Sigma_g) & & & & \pi_1(M). \\ & \searrow & & \nearrow & \\ & & \pi_1(M_2) & & \end{array}$$

where all maps are surjective. If we apply the functor $\mathrm{Hom}(., SU(2))$ we get the following diagram of injections

$$\begin{array}{ccccc} & & \mathcal{R}(M_2)^\sharp & & \\ & \swarrow & & \nwarrow & \\ \mathcal{R}(\Sigma_g^*)^\sharp \longleftarrow \mathcal{R}(\Sigma_g)^\sharp & & & & \mathcal{R}(M)^\sharp \\ & \nwarrow & & \swarrow & \\ & & \mathcal{R}(M_2)^\sharp & & \end{array}$$

Here $\mathcal{R}(X)^\sharp$ denotes the space $\mathrm{Hom}(\pi_1(X), SU(2))$. Again, by Van Kampen we see that $\mathcal{R}(M)^\sharp$ is just the intersection of $\mathcal{R}(M_1)^\sharp$ and $\mathcal{R}(M_2)^\sharp$ inside $\mathcal{R}(\Sigma_g)^\sharp$. Clearly $SU(2)$ acts on these spaces by the adjoint action. Taking quotients, we have

$$\begin{array}{ccccc} & & \mathcal{R}(M_2) & & \\ & \swarrow & & \nwarrow & \\ \mathcal{R}(\Sigma_g^*) \longleftarrow \mathcal{R}(\Sigma_g) & & & & \mathcal{R}(M) = \mathcal{R}(M_1) \cap \mathcal{R}(M_2). \\ & \nwarrow & & \swarrow & \\ & & \mathcal{R}(M_2) & & \end{array}$$

Later on in this section, the superscript $\sharp$ will be used to denote the following - if X has been defined as $Y/SU(2)$, then $X^\sharp$ will denote Y.

Roughly speaking, $\mathcal{R}(M_1)$ and $\mathcal{R}(M_2)$ have complementary dimension in $\mathcal{R}(\Sigma_g)$ and we may define $\lambda(M)$ to be the intersection number

$$\lambda(M) = \langle \mathcal{R}(M_1), \mathcal{R}(M_2) \rangle.$$

There is an obvious difficulty in defining the above intersection number as $\mathcal{R}(M_i), i = 1, 2$ and $\mathcal{R}(\Sigma_g)$ are typically real algebraic sets. The condition $H_1(M, \mathbb{Z}) = 0$ guarantees that after removing the trivial representation, $\mathcal{R}(M_1) \cap \mathcal{R}(M_2)$ is a compact subset of the non-singular part of $\mathcal{R}(\Sigma_g)$. Thus, the intersection will consist of a well-defined (signed) number of non-singular point and $\lambda(M)$ is defined to be this number. The reason we require $H_1(M, \mathbb{Z}) = 0$ is that the equivalence class of irreducible representations of $\pi_1(M)$ then becomes $\mathcal{R}(M) \setminus \{1\}$ since all reducible representations are trivial. **Put Fukaya's interpretation pg 28 and Floer's review pg 216** We observe in passing that $\pi_1(M)$ is countable and has a finite presentation with at most g generators for any genus g Heegaard splitting.

For simplicity of notation let $\pi = \pi_1(M)$ and $G = SU(2)$. In general, G could be any Lie group and π could be any finitely presented group and the arguments that follow still hold. Let $\rho_t : \pi \to G$ be a differentiable 1-parameter family of representations such that $\rho_0 = \rho$ is a given representation. Expanding ρ_t for t sufficiently close to 0 and $x \in \pi$,

$$\rho_t(x) = \exp(tu(x) + O(t^2))\rho(x)$$

and therefore, by differentiating, $\rho_0^{'}(x) = u(x)\rho(x)$ (here $u : \pi \to \mathfrak{g}$). Since ρ_t is a homomorphism, differentiating this condition we get

$$u(xy)\rho(xy) = \rho_0^{'}(xy) = \rho_0^{'}(x)\rho(y) + \rho(x)\rho_0^{'}(y) = u(x)\rho(x)\rho(y) + \rho(x)u(y)\rho(y).$$

Consequently,

$$u(xy) = u(x) + \mathrm{Ad}_{\rho(x)}u(y). \tag{6.4}$$

Here $\mathfrak{g}$ is a π module by the action of $\mathrm{Ad}_{\rho(x)}$ and we henceforth denote $\mathfrak{g}$ by $\mathfrak{g}_{\mathrm{Ad}\rho}$ to indicate this. Then it follows from the definition of *group cohomology* that u is a 1-cocycle with coefficients in $\mathfrak{g}_{\mathrm{Ad}\rho}$. Conversely, solutions of u being a cocycle lead to maps $\rho_t : \pi \to G$ which satisfy the homomorphism condition to first order in t. Thus, the Zariski tangent space of $\mathrm{Hom}(\pi, G)$ at ρ can be identified with $Z^1(\pi, \mathfrak{g}_{\mathrm{Ad}\rho})$.

To compute the tangent space of the Ad-orbit containing ρ, let $\mathfrak{g}_t$ be a path with $g_0 = 1$. Let

$$\rho_t(x) = g_t^{-1}\rho(x)g_t.$$

Writing $g_t = \exp(tu_0 + O(t^2))$, then

$$\begin{aligned}\rho_t(x) &= \exp(-tu_0 + O(t^2))\rho(x)\exp(tu_0 + O(t^2)) \\ &= \exp(-tu_0 + O(t^2))\mathrm{Ad}_{\rho(x)}(\exp(tu_0 + O(t^2)))\rho(x) \\ &= \exp(t\mathrm{Ad}_{\rho(x)}u_0 - tu_0 + O(t^2))\rho(x).\end{aligned}$$

Therefore, the cocyle corresponding to ρ_t,

$$u(x) = \mathrm{Ad}_{\rho(x)}u_0 - u_0 \tag{6.5}$$

is a coboundary δu_0. Consequently, the tangent space of the Ad-orbit through ρ is $B^1(\pi, \mathfrak{g}_{\mathrm{Ad}\rho})$. Based on the above results we define

Definition 6.2. The *Zariski tangent space* of $\mathrm{Hom}(\pi, G)/G$ is defined to be $H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\rho})$.

Now specialize to $\pi = \pi_1(M)$ and G being any reductive Lie group with the Cartan-Killing form $\langle \cdot, \cdot \rangle$. This means that the centre $Z(G)$ is discrete. For e.g., we would later use $G = SU(2)$ with $\langle A, B \rangle := tr(AB)$. In what follows, we collect together various results of Goldman (see [G1] and [G2] for more details) without proofs.

Proposition 6.3. (Goldman) *The dimension of* $Z^1(\pi, \mathfrak{g}_{Ad\rho})$ *is*

$$(2g-1)\, dim\, G + dim\, Z_G(\rho(\pi)). \tag{6.6}$$

The dimension of $H^1(\pi, \mathfrak{g}_{Ad\rho})$ *is*

$$(2g-2)\, dim\, G + dim\, Z_G(\rho(\pi)). \tag{6.7}$$

Note that dim $Z^1(\pi, \mathfrak{g}_{\mathrm{Ad}\rho})$ is minimal, and hence ρ is non-singular point of $\mathrm{Hom}(\pi, G)$, if and only if $\mathrm{dim} Z(\rho)/Z(G) = 0$. We denote the set of all such points as $\mathrm{Hom}(\pi, G)^{-}$.

7 unknown

Donaldson invariants and relations with the above

References

[AM] S. Akbulut, J. McCarthy, *Casson's invariant for oriented homology* 3*-spheres - an exposition*, Princeton Math. Notes **36**, Princeton Univ. Press.

[CS] S. S. Chern, J. Simons, *Characteristic forms and geometric invariants*, Ann. Math. **99** (1974) 48-69.

[AF] A. Floer, *An instanton-invariant for* 3*-manifolds*, Comm. Math. Phys. **118** (1988) 215-240.

[DF1] D. S. Freed, *Classical Chern-Simons Theory, Part* 1, Adv. Math. **113** (1995) 237-303.

[DF2] D. S. Freed, *Classical Chern-Simons Theory, Part* 2, Houston J. Math., Vol **76** No 1.

[F] K. Fukaya, *Floer homology,* A_∞*-categories and topological field theory*, Geom. and Phys., Aarhus 1995.

[G1] W. Goldman, *The symplectic nature of the fundamental groups of surfaces*, Adv. Math. **54** (1984) 220-225.

[G2] W. Goldman, *Representations of fundamental groups of surfaces*, LNM **1167**, Springer-Verlag, 1985.

[W] K. Walker, *An extension of Casson's invariant*, Ann. Math. Studies **126**, Princeton Univ. Press.