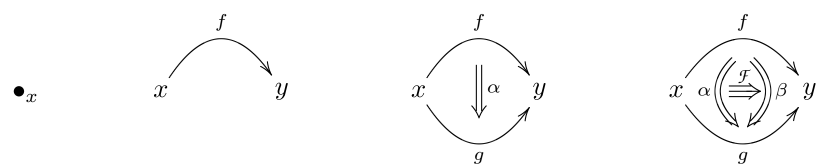
Higher Categories and Topology - 19th May '10

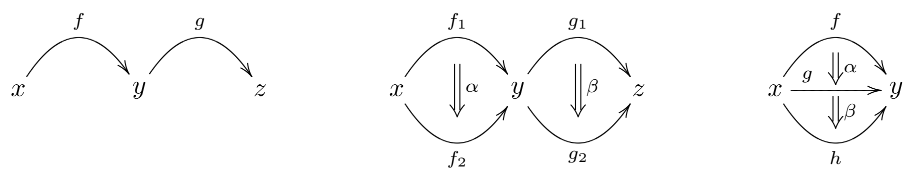
John Baez

There was a discussion about whether the title should read "... algebraic topology" instead of "... topology". Dennis had a reservation againts the word "topology" since it encapsulates a lot of non-algebraic stuff. It was John who then pointed out that knots and links also fall into the scope of higher categories and Dennis grudgingly agreed.

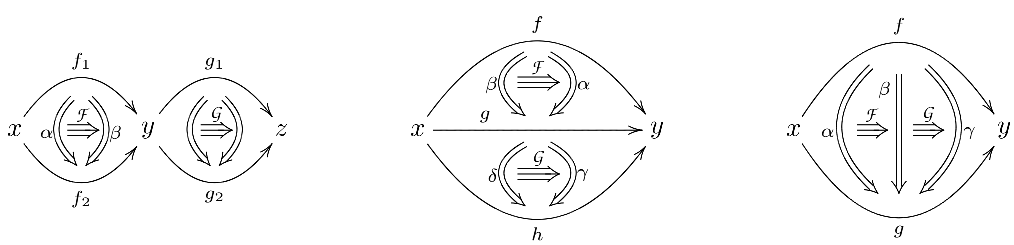
We start with a sketch of what ∞-categories are. The pictures we'll be drawing will be following the *globular approach* by M. Batanin, where objects are usually denoted by dots and 1-morphisms by arrows, 2-morphisms by double arrows and so on, as the following picture suggests :



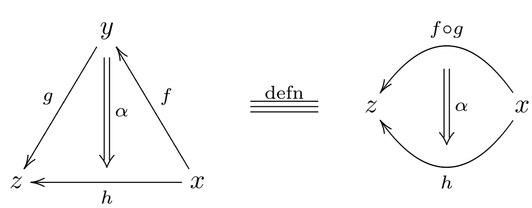
There is a natural way to compose these things :



For composing 3-morphisms the following are the only possible ways :



Before giving an example of such a category, let's note that not all things have to be blobs. As Dennis asked, can one do this for triangulations? The answer is yes and given a triangle with vertices x, y, z and morphisms $f : x \to y, g : y \to z, h : x \to z$ we can associate $\alpha : (f \circ g) \to h$:



Therefore, one can have elaborate pictures.

Example 1. (Chain complexes)[1]
Let's start with a chain complex

$$C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} \cdots$$

[1]John has a habit of writing arrows to the left so that composition of maps, written in the usual way, agrees with the order of arrows.

and define the objects as $\mathcal{O}b := C_0$. Given an 1-chain $f \in C_1$ we have a 0-chain $\partial f \in C_0$ but there is no canonical way of splitting this boundary. The way out is to define[2] $\mathcal{M}or := C_0 \oplus C_1$ such that f sends $x \in C_0$ to $x + \partial f \in C_0$. The composition is given by the usual procedure as depicted previously.

Remark 2. (Vectors in $\mathbb{R}^2$)
Consider the chain complex $C_0 = C_1 = \mathbb{R}^2$ and $C_i = 0, i \geq 2$ with the boundary map $\mathbb{R}^2 \xrightarrow{\partial = Id} \mathbb{R}^2$. The objects are points in $\mathbb{R}^2$ and a morphism $(x, f) \in \mathbb{R}^2 \oplus \mathbb{R}^2$ is realized by a vector from x to $x + f$.

The collection of 2-morphisms is given by $2\mathcal{M}or := C_0 \oplus C_1 \oplus C_2$. Given $(x, f, \alpha) \in 2 - \mathcal{M}or$ we think of $\alpha : f \to f + \partial\alpha, f : x \to x + \partial f$ and

$$f + \partial\alpha : x \to x + \partial(f + \partial\alpha) = x + \partial f.$$

It's nice confirmation of why we need $\partial^2 = 0$.

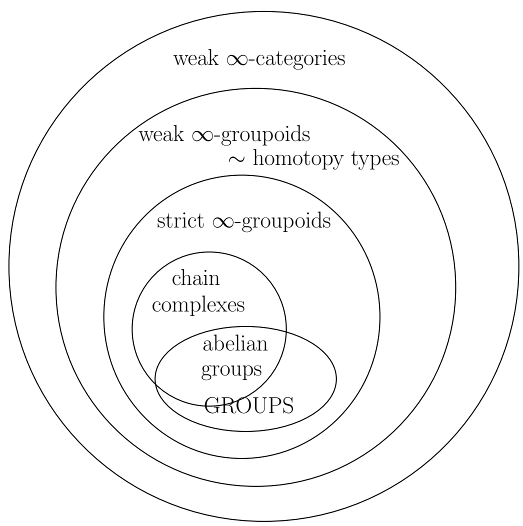
Definition 3. A category *internal* to $\mathcal{A}b\mathcal{G}p$ has an abelian group of objects $\mathcal{O}b$ and an abelian group of morphisms $\mathcal{M}or$ such that

$$\mathcal{M}or \otimes_{\mathcal{O}b} \mathcal{M}or \longrightarrow \mathcal{M}or$$

is a homomorphism of abelian groups and so are the source and target maps.

One can jazz this up to define ∞-categories internal to any category since typically the objects, morphisms, 2-morphisms etc. are sets and saying it is internal to a category $\mathcal{C}$, which admits pullbacks, is demanding that these objects, morphisms etc. are objects of $\mathcal{C}$ satisfying certain coherence conditions.

Theorem 4. *A strict ∞-category internal to $\mathcal{A}b\mathcal{G}p$ is the same as a chain complex. Moreover, these are automatically strict ∞-groupoids.*



Theorem 5. *Any strict ∞-groupoid is a disjoint union (coproduct) of connected ones (i.e., given objects x, y there exists $f : x \to y$). A connected one corresponds to a space that's a bundle over a $K(\pi_1, 1)$ with fibre $\prod_{n\geq 2}^{\infty} K(\pi_n, n)$, with possibly non-trivial action of π_1 on the higher π_i's.*

This is basically saying that the homotopy 2-types are determined by the two stage Postnikov system.

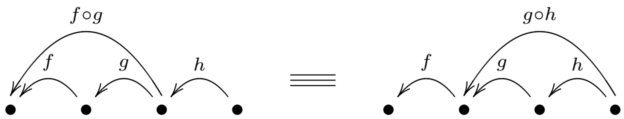
Let me digress and say what a *crossed module* is. In fact, it is a special case of *crossed complexes* which can be taken to be *connected strict ∞-groupoids*, viz., it is like a chain complex

$$G \xleftarrow{\partial} C_1 \xleftarrow{\partial} C_2 \cdots$$

[2]We shall think of *morphisms* as the maps along with the source and target objects.

with the bottom group non-abelian and acting on the rest. A *crossed module*, originally defined by J.H.C. Whitehead, has $C_i = 0$ for $i > 2$.

A strict 2-category has objects, morphisms and morphisms between morphisms



such that associativity and other necessary coherence conditions hold on the nose. A good example is given by

$$\mathcal{CAT} = [\textit{small categories}, \textit{functors}, \textit{natural transformations}]\,.$$

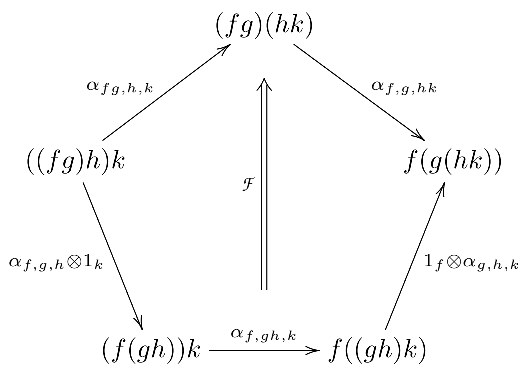
However,

$$\mathcal{D} = [\textit{rings}, \textit{bimodules}, \textit{bimodule homomorphisms}]$$

is not! To backtrack a little, a bimodule is thought of as ${}_RM_S : R \to S$ and ${}_RN_S \otimes {}_SM_T$ is the *composite* of N and M. This composition is not associative. Benebou invented *bicategories* (weak 2-category in the modern language) to handle this. In this setting, the composition of 1-morphisms comes with an *associator* 2-isomorphism

$$\alpha_{fgh} : (fg)h \xrightarrow{\cong} f(gh)$$

obeying the pentagon identity.[3]



The map $\mathcal{F}$ is an isomorphism that identifies the two ways of going from $((fg)h)k$ to $f(g(hk))$. In dimension two, however, the weak and strict notions are essentially the same in the following sense :

Theorem 6. (MacLane)
Every bicategory is equivalent to a strict 2-category.

Remark 7. *This is not true in higher categories!*

An analogous statement is the relation of two stage Postnikov systems and a twisted $K(\pi_2, 2)$ fibration over $K(\pi_1, 1)$. This is modelled by the *fundamental 2-groupoid* of a space. A lot has been worked out in dimension 2 and things work out fairly well.

One can also prove that every weak 2-groupoid is equivalent to a strict 2-groupoid and one doesn't loose any homotopy theoretic information by passing from a weak version to a strong one. This, however, was the point of confusion for a long time and people thought that strict n-categories are going to be enough. It turns out that there are weak 3-categories that are not equivalent to a strict one. One reason why people started looking at abstract ways of studying higher categories instead of writing out all the identities and relations is that it gets out of hand pretty soon. A few things are known when $n = 3$ but a hands-on

[3] Dennis asked who was the first person brave enough to ask such things and wade through the details. John answered that MacLane was certainly one of them and possibly Adams. Dennis agreed by saying that Adams seemed like a guy who could have done it.

approach for defining a *tetracategory*, i.e., a weak 4-category was originally done by Todd Trimble. It was lost for a year till it resurfaced later when various people showed interest in seeing how Todd did it initially. It's definitely worth a look and I have put it on my website :

`http://math.ucr.edu/home/baez/trimble/tetracategories.html`

What are examples of one object bicategories? One could take

$$\mathcal{P} := [\bullet,\ \textit{rings},\ \textit{bimodules}, \textit{bimodule homomorphisms}]\,,$$

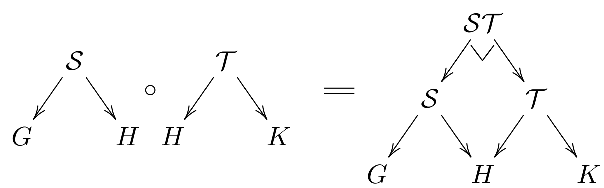
where $R : \bullet \to \bullet$ and the composition is $R \otimes S$. Similarly, one can take

$$\mathcal{A} := [\bullet,\ R\textit{-algebras},\ R\textit{-algebra bimodules}, R\textit{-algebra bimodule homomorphisms}]\,.$$

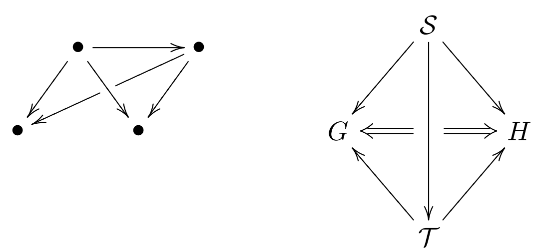
This object is related to three *scary* terms, viz., *Azumaya algebras*, *Morita equivalence* and *Picard group*. In fact, the invertible ones in this bicategory are the Azumaya algebras. There's yet another example given by

$$\mathcal{G} := [\textit{groupoids},\ \textit{spans},\ \textit{maps of spans}, \textit{maps of maps of spans}]\,.$$

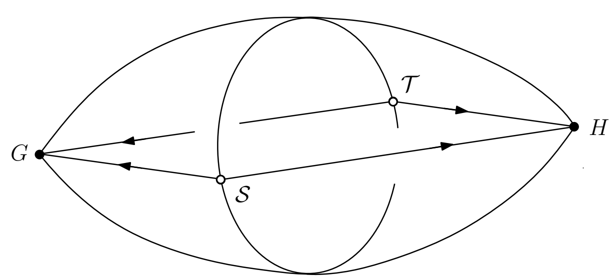
A *span* S of groupoids G and H is nothing but another groupoid S such that there are maps $G \leftarrow S \to H$. The composition of morphisms is given by pullbacks.



Here's a pictorial description of a map of spans.

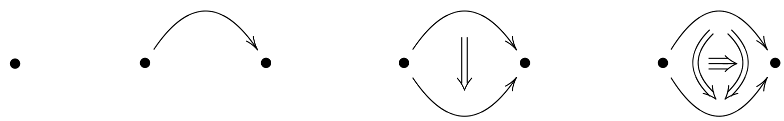


The picture of a map of map of spans is given below. One should think of the triangles in the 2-skeleton represented by suitable 2-morphisms and the interior of the solid represented by a 3-morphism.



We do know a bit about tricategories. For example, given a space X, we can find it's *fundamental 3-groupoid* $\mathbb{T}_3(X)$, loosely thought of as paths between paths between paths in X.

Remark 8. *Batanin and Street defined weak ∞-categories using the globular approach :*



There are also simplicial approaches where we take simplicial weak ∞-groupoids to be Kan complexes.