An expository note on BV algebra and gravity algebra

Let $(A,\cdot)$ be a (graded) commutative, associative algebra.

Definition 0.1. $(A,\cdot)$ is called a *Gerstenhaber algebra* if there is a (graded) Lie bracket $\{,\}$ of degree 1 which is also a graded derivation of $\cdot$ in both variables, i.e.,

$$\{a,b\} = -(-1)^{(|a|+1)(|b|+1)}\{b,a\} \tag{0.1}$$

$$\{a,\{b,c\}\} = \{\{a,b\},c\} + (-1)^{(|a|+1)(|b|+1)}\{b,\{a,c\}\} \tag{0.2}$$

$$\{a,b\cdot c\} = \{a,b\}\cdot c + (-1)^{(|a|+1)|b|} b\cdot\{a,c\} \tag{0.3}$$

$$\{b\cdot c,a\} = (-1)^{(|a|+1)|c|}\{b,a\}\cdot c + b\cdot\{c,a\}. \tag{0.4}$$

Observe that demanding $\{,\}$ to be a derivation in any one variable is equivalent to (0.3),(0.4). For example, if we have (0.3), then using (0.1) we get

$$\begin{aligned}
\{b\cdot c,a\} &= -(-1)^{(|a|+1)(|b|+|c|+1)}\{a,b\cdot c\}\\
&= -(-1)^{(|a|+1)(|b|+|c|+1)}\{a,b\}\cdot c - (-1)^{(|a|+1)|b|+(|a|+1)(|b|+|c|+1)} b\cdot\{a,c\}\\
&= (-1)^{(|a|+1)(|b|+|c|+1)+(|a|+1)(|b|+1)}\{b,a\}\cdot c + b\cdot\{c,a\}\\
&= (-1)^{(|a|+1)|c|}\{b,a\}\cdot c + b\cdot\{c,a\}
\end{aligned}$$

which is (0.4). What is known as a Batalin-Vilkovsky algebra (or BV algebra in short) has a few (equivalent) definitions :

Definition 0.2. A *Batalin-Vilkovsky algebra* is a graded commutative, associative algebra $(A,\cdot)$ equipped with an operator $\Delta : A \to A$ of degree 1 such that $\Delta\circ\Delta = 0$ and the deviation of Δ from being a derivation is a derivation, i.e., the bracket $\{,\}$ defined by

$$\{a,b\} := (-1)^{|a|}\Delta(a\cdot b) - (-1)^{|a|}\Delta a\cdot b - a\cdot\Delta b. \tag{0.5}$$

is a derivation.

Definition 0.3. A *Gerstenhaber algebra* $(A,\cdot,\{,\})$ is called a *Batalin-Vilkovsky algebra* if it is equipped with an operator $\Delta : A \to A$ of degree 1 such that $\Delta\circ\Delta = 0$ and the bracket $\{,\}$ measures the deviation of Δ from being a derivation, i.e.,

$$\{a,b\} = (-1)^{|a|}\Delta(a\cdot b) - (-1)^{|a|}\Delta a\cdot b - a\cdot\Delta b.$$

Definition 0.4. A graded commutative, associative algebra $(A,\cdot)$ is called a *Batalin-Vilkovsky algebra* if it has an operator $\Delta : A \to A$ of degree 1 such that $\Delta\circ\Delta = 0$ and

$$\begin{aligned}
\Delta(a\cdot b\cdot c) &= \Delta(a\cdot b)\cdot c + (-1)^{|a|}a\cdot\Delta(b\cdot c) + (-1)^{(|a|+1)|b|}b\cdot\Delta(a\cdot c)\\
&\quad -\Delta a\cdot b\cdot c - (-1)^{|a|}a\cdot\Delta b\cdot c - (-1)^{|a|+|b|}a\cdot b\cdot\Delta c.
\end{aligned} \tag{0.6}$$

As will follow from the calculations, (0.6) is equivalent to (0.3). We establish the equivalence of these definitions by proving $0.2 \Rightarrow 0.3 \Rightarrow 0.4 \Rightarrow 0.2$

<u>$0.2 \Rightarrow 0.3$</u>

We need to show that $(A, \cdot, \{,\})$ is a Gerstenhaber algebra.

$$\begin{aligned}
\{a,b\} &= (-1)^{|a|}\Delta(a\cdot b) - (-1)^{|a|}\Delta a\cdot b - a\cdot\Delta b \\
&= (-1)^{|a|+|a||b|}\Delta(b\cdot a) - (-1)^{|a|+(|a|+1)|b|} b\cdot\Delta a - (-1)^{(|b|+1)|a|}\Delta b\cdot a \\
&= -(-1)^{(|a|+1)(|b|+1)}\big((-1)^{|b|}\Delta(b\cdot a) - b\cdot\Delta a - (-1)^{|b|}\Delta b\cdot a\big) \\
&= -(-1)^{(|a|+1)(|b|+1)}\{b,a\}.
\end{aligned}$$

Thus, $\{,\}$ is anti-symmetric (satisfies (0.1)). Using $\Delta^2 = 0$:

$$\begin{aligned}
\Delta\{a,b\} &= \Delta\big((-1)^{|a|}\Delta(a\cdot b) - (-1)^{|a|}\Delta a\cdot b - a\cdot\Delta b\big) \\
&= -(-1)^{|a|}\Delta(\Delta a\cdot b) - \Delta(a\cdot\Delta b) \\
&= (-1)^{|a|+1}\Delta(\Delta a\cdot b) - (-1)^{|a|+1}\Delta^2 a\cdot b - \Delta a\cdot\Delta b \\
&\quad -(-1)^{|a|}((-1)^{|a|}\Delta(a\cdot\Delta b) - (-1)^{|a|}\Delta a\cdot\Delta b - a\cdot\Delta^2 b) \\
\Delta\{a,b\} &= \{\Delta a, b\} - (-1)^{|a|}\{a,\Delta b\}. \qquad (0.7)
\end{aligned}$$

Thus, Δ is a (graded) derivation of $\{,\}$. Using this

$$\begin{aligned}
\{\{a,b\},c\} &= (-1)^{|a|+|b|+1}\underbrace{\Delta(\{a,b\}\cdot c)}_{1} - (-1)^{|a|+|b|+1}\underbrace{\{\Delta a, b\}\cdot c}_{2} \qquad (0.8) \\
&\quad -(-1)^{|b|}\underbrace{\{a,\Delta b\}\cdot c}_{3} - \underbrace{\{a,b\}\cdot\Delta c}_{4}
\end{aligned}$$

$$\begin{aligned}
(-1)^{(|a|+1)(|b|+1)}\{b,\{a,c\}\} &= (-1)^{|a||b|+|a|+1}\Delta(b\cdot\{a,c\}) + (-1)^{(|b|+1)|a|}\Delta b\cdot\{a,c\} \\
&\quad -(-1)^{(|a|+1)(|b|+1)} b\cdot\{\Delta a, c\} + (-1)^{|a||b|+|b|+1} b\cdot\{a,\Delta c\} \\
&= (-1)^{|a|+|b|+1}\Delta(\underbrace{\{a,b\cdot c\}}_{5} - \underbrace{\{a,b\}\cdot c}_{1}) \\
&\quad -(-1)^{|b|}(\{a,\Delta b\cdot c\} - \underbrace{\{a,\Delta b\}\cdot c}_{3}) \\
&\quad +(-1)^{|a|+|b|}(\underbrace{\{\Delta a, b\cdot c\}}_{6} - \underbrace{\{\Delta a, b\}\cdot c}_{2}) \\
&\quad -(\{a, b\cdot\Delta c\} - \underbrace{\{a,b\}\cdot\Delta c}_{4}) \\
&= -\{\{a,b\},c\} + (-1)^{|b|}\{a,\Delta(b\cdot c)\} \\
&\quad -(-1)^{|b|}(\{a,\Delta b\cdot c\} - (\{a, b\cdot\Delta c\} \\
&= -\{\{a,b\},c\} + \{a,\{b,c\}\}.
\end{aligned}$$

In the process of proving the Jacobi identity above, we have used (0.7) to cancel out 6 with one of the terms of 5.

<u>$0.3 \Rightarrow 0.4$</u>

We just need to verify (0.6). We claim that it is equivalent to $\{,\}$ being a derivation.

$$\begin{aligned}
\{a, b\cdot c\} &= (-1)^{|a|}\Delta(a\cdot b\cdot c) - (-1)^{|a|}\Delta a\cdot b\cdot c - a\cdot\Delta(b\cdot c) \\
&= (-1)^{|a|}\big(\Delta(a\cdot b\cdot c) - \Delta a\cdot b\cdot c - (-1)^{|a|} a\cdot\Delta(b\cdot c)\big) \qquad (0.9)
\end{aligned}$$

$$
\begin{aligned}
\{a,b\}\cdot c &= (-1)^{|a|}\Delta(a\cdot b)\cdot c-(-1)^{|a|}\Delta a\cdot b\cdot c-a\cdot\Delta b\cdot c\\
&= (-1)^{|a|}\big(\Delta(a\cdot b)\cdot c-\Delta a\cdot b\cdot c-(-1)^{|a|}a\cdot\Delta b\cdot c\big)
\end{aligned} \tag{0.10}
$$

$$
\begin{aligned}
(-1)^{(|a|+1)|b|}b\cdot\{a,c\} &= (-1)^{(|a|+1)|b|}b\cdot\big((-1)^{|a|}\Delta(a\cdot c)-(-1)^{|a|}\Delta a\cdot c-a\Delta c\big)\\
&= (-1)^{|a|}\big((-1)^{(|a|+1)|b|}b\cdot\Delta(a\cdot c)-(-1)^{(|a|+1)|b|}b\cdot\Delta a\cdot c\\
&\quad +(-1)^{(|a|+1)(|b|+1)}b\cdot a\cdot\Delta c\big)\\
&= (-1)^{|a|}\big((-1)^{(|a|+1)|b|}b\cdot\Delta(a\cdot c)-\Delta a\cdot b\cdot c-(-1)^{|a|+|b|}a\cdot b\cdot\Delta c\big)
\end{aligned} \tag{0.11}
$$

Thus, $\{,\}$ is a derivation of $\cdot$ (satisfying (0.3)) if and only if (0.9) equals the sum of (0.10) and (0.11). Cancelling the factor of $(-1)^{|a|}$ from both sides and rearranging we get an equivalent condition which is precisely (0.6).

<u>$0.4 \Rightarrow 0.2$</u>

Define $\{,\}$ on the algebra using (0.5). It follows from the arguments in the previous implication that this is a derivation.

Remarks
(1) Since Δ is of degree 1, this forces $\{,\}$ to be of degree 1 (as in definition 0.4 or 0.2) while in definition 0.3, $\{,\}$ is already of degree 1, which forces Δ to be of degree 1.
(2) The identity (0.6) is equivalent to $\{,\}$ being a derivation of $\cdot$ in each variable (0.3).
(3) The Jacobi identity is equivalent to Δ being a derivation of $\{,\}$, which is itself a derivation of $\cdot$.
(4) A BV algebra (as defined in 0.2 or 0.4) is also a Gerstenhaber algebra.

The concept of a BV algebra is related to that of a gravity algebra.

Definition 0.5. Let V be a chain complex over a field k. A *gravity algebra* on V is a sequence of graded skew-symmetric operators:

$$
c_n : V^{\otimes n} \longrightarrow V
$$

of degree $2-n$, satisfying the following relations: if $k>2$ and $l\geq 0$, and $a_1,\cdots,a_k,b_1,\cdots,b_l\in V$,

$$
\sum_{1\leq i<j\leq k}(-1)^{\epsilon}\{\{a_i,a_j\},a_1,\cdots,\widehat{a_i},\cdots,\widehat{a_j},\cdots,a_k,b_1,\cdots,b_l\} \tag{0.12}
$$
$$
=\begin{cases}\{\{a_1,\cdots,a_k\},b_1,\cdots,b_l\}, & l>0,\\ 0, & l=0,\end{cases}
$$

where we write $c_n(a_1,\cdots,a_n)$ as $\{a_1,\cdots,a_n\}$, and $\epsilon=|a_i|\sum_{k=1}^{i-1}|a_k|+|a_j|\sum_{l=1,l\neq i}^{j-1}|a_l|$.

There is no intrinsic way to get a gravity algebra out of a BV algebra except defining the algebra multiplication to be c_2. Before one can define the other operators, note that (0.6) holds. We shall see where the signs arise from. Assume by induction that

$$
\begin{aligned}
\Delta(a_1\cdot\ldots\cdot a_n) &= \sum_{1\leq i<j\leq n}(-1)^{\epsilon}\Delta(a_i\cdot a_j)\cdot a_1\cdot\ldots\widehat{a_i}\ldots\widehat{a_j}\cdot\ldots\cdot a_n\\
&\quad +(2-n)\sum_{1\leq i\leq n}(-1)^{\delta}a_1\cdot\ldots\cdot\Delta a_i\cdot\ldots\cdot a_n,
\end{aligned} \tag{0.13}
$$

where $\epsilon = |a_i| \sum_{k=1}^{i-1} |a_k| + |a_j| \sum_{l=1, l\neq i}^{j-1} |a_l|$ and $\delta = \sum_{k=1}^{i-1} |a_k|$. Then it follows that

$$\begin{aligned}
\Delta(a_1 \cdot \ldots \cdot a_n \cdot a_{n+1}) &= \sum_{1 \leq i < j < n} (-1)^{\epsilon} \Delta(a_i \cdot a_j) \cdot a_1 \cdot \ldots \widehat{a_i} \ldots \widehat{a_j} \ldots \cdot a_n \cdot a_{n+1} \\
&+ \sum_{i=1}^{n-1} (-1)^{\epsilon} \Delta(a_i \cdot a_n \cdot a_{n+1}) \cdot a_1 \cdot \ldots \widehat{a_i} \ldots \cdot a_{n-1} \\
&+ (2-n) \sum_{1 \leq i < n} (-1)^{\delta} a_1 \cdot \ldots \cdot \Delta a_i \cdot \ldots \cdot a_n \cdot a_{n+1} \\
&+ (2-n)(-1)^{\delta} a_1 \cdot \ldots \cdot \Delta(a_n \cdot a_{n+1}).
\end{aligned}$$

Using (0.6) for $\Delta(a_i \cdot a_n \cdot a_{n+1})$ and rearranging the a_i's we see that the second sum in the above equality reads

$$\begin{aligned}
&\sum_{i<n} (-1)^{\epsilon} \Delta(a_i \cdot a_n) \cdot a_1 \ldots \widehat{a_i} \ldots \widehat{a_n} \cdot a_{n+1} \\
&+ \sum_{i<n} (-1)^{\epsilon} \Delta(a_i \cdot a_{n+1}) \cdot a_1 \ldots \widehat{a_i} \ldots \cdot a_n \\
&- \sum_{i<n} (-1)^{\delta} a_1 \cdot \ldots \cdot \Delta a_i \cdot \ldots \cdot a_n \cdot a_{n+1} + (n-1)(-1)^{\epsilon} \Delta(a_n \cdot a_{n+1}) \cdot a_1 \ldots \cdot a_{n-1} \\
&+ (1-n)(-1)^{\delta} a_1 \cdot \ldots \cdot \Delta a_n \cdot a_{n+1} + (1-n)(-1)^{\delta} a_1 \cdot \ldots \cdot a_n \cdot \Delta a_{n+1},
\end{aligned}$$

with the appropriate ϵ and δ. Combining this with the previous equality we see that (0.13) holds for any $n \geq 3$.

Example (gravity algebra on the equivariant homology of the free loop space) Let M be a compact, oriented manifold and let $H_*(LM)$ denote the homology of its free loop space. It is known that $(H*(LM), \bullet, \Delta)$ is a BV algebra where the BV operator Δ arises from the circle action on elements of LM. Closely associated with the loop homology is the equivariant homology $H_*^{S^1}(LM)$, defined to be $H_*(LM \times_{S^1} ES^1)$. It can be shown that $\Delta = M \circ E$ where E stand for erasing marked points and M stands for marking all possible points. Further, $E : H_*(LM) \to H_*^{S^1}(LM)$ while $M : H_*^{S^1}(LM) \to H_{*+1}(LM)$. This helps us to define a gravity algebra structure on $H_*^{S^1}(LM)$ as follows :