Transversal String Topology & Invariants of Manifolds

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Outline

What is string topology?

Fix a closed, oriented manifold M.

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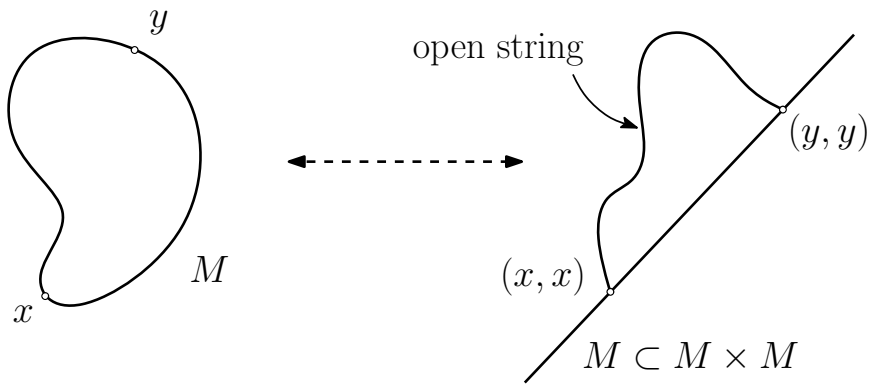
These are known to be *homotopy invariants*.

An alternate viewpoint

Loops in $M \Longleftrightarrow$ Arcs in $M \times M$ with end points in M :



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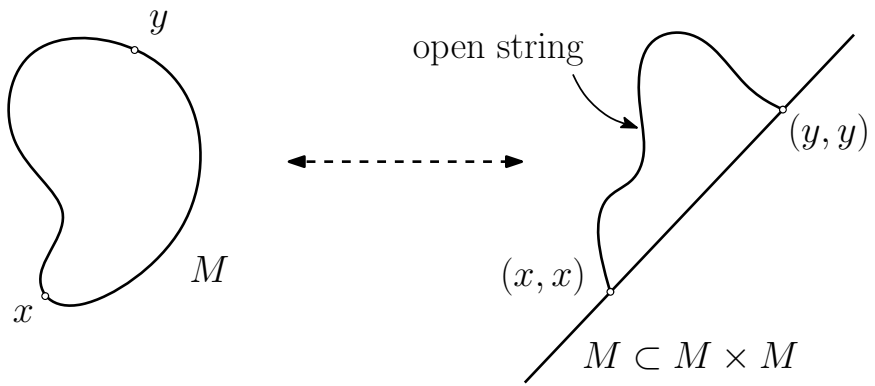
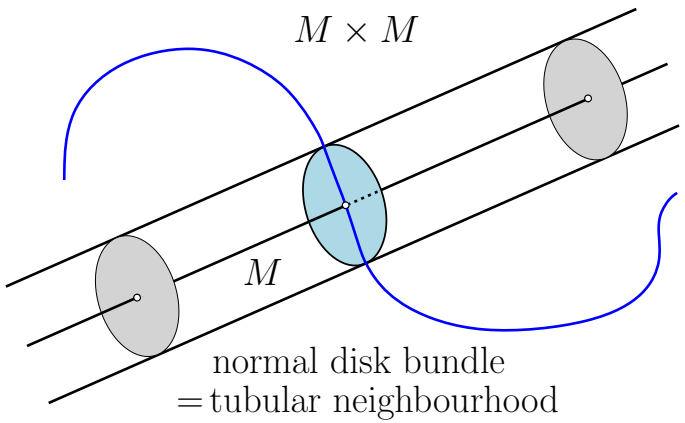


We restrict ourselves to *transversal open strings* in $M \times M$:



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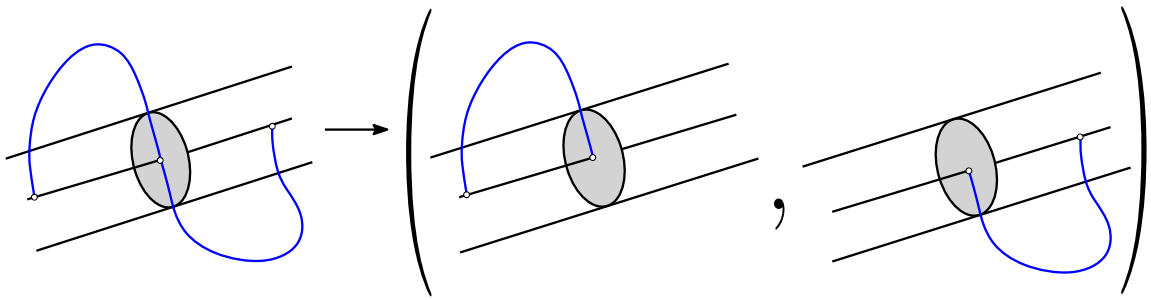

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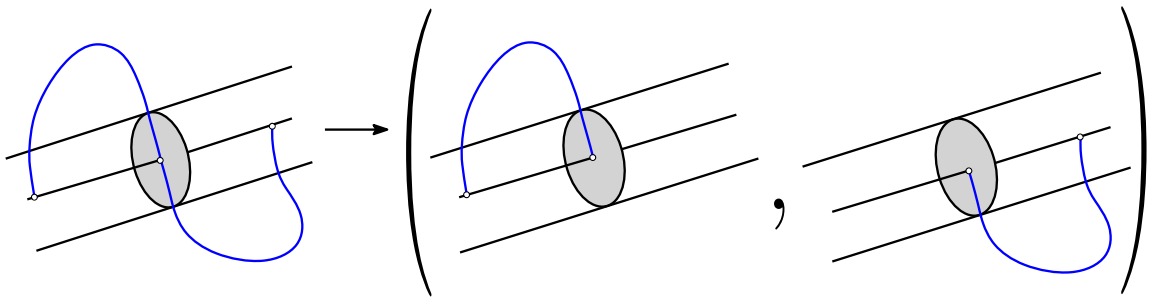
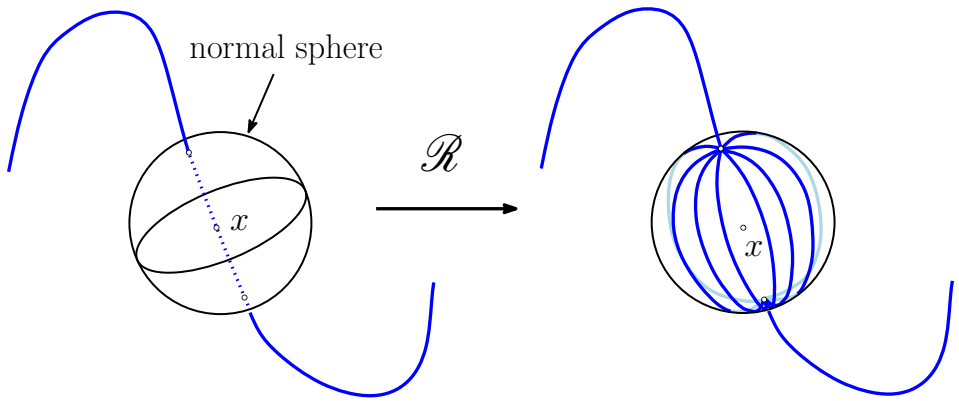


We also have a *resolve* operation $\mathscr{R}$:



Extend $\prec$ and $\mathscr{R}$ appropriately for strings with more intersections.

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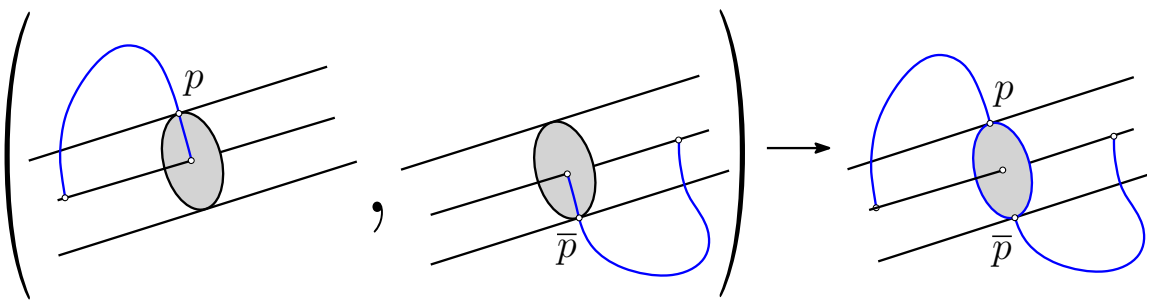

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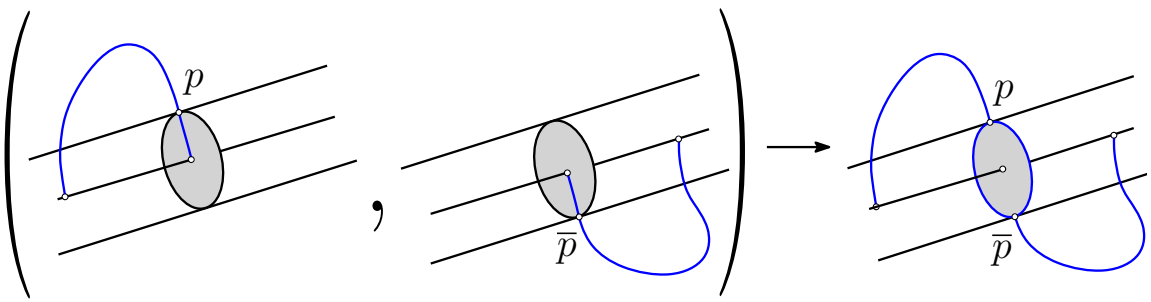

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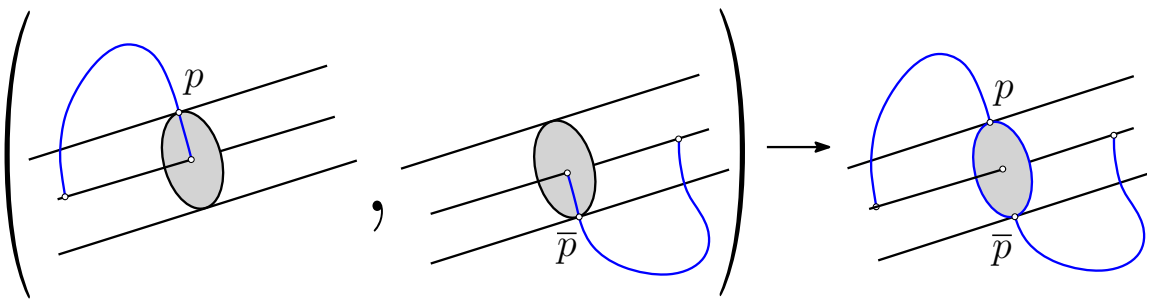

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This is fully defined when we work over a subset of pairs of cords.

Question

What is the appropriate structure that turns $\mathscr{A}$ into an algebra?

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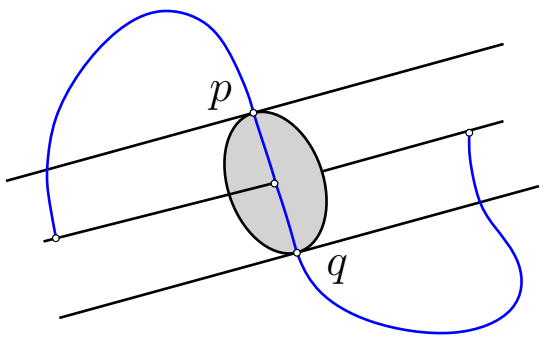
The splitting produces open strings which are *connected* :



Here $\bar{p} = q$ where $\bar{p}$ is the antipode of p.

Bicomodules and the cobar construction

There exists natural maps

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Theorem

We have an isomorphism $\left(\Omega\,\mathfrak{C}(M)\right)_{\text{aug}} \cong \mathscr{A}$ where Ω is the cobar construction and $\mathscr{A}$ is the cord algebra.

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The Pontrjagin rings $H_*(\Omega F_2(L_{7;j}))$ for $j = 1, 2$ are not isomorphic.

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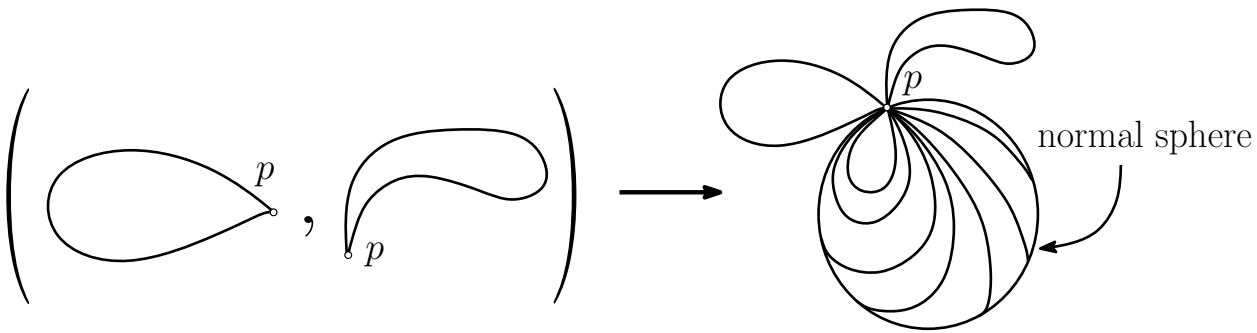

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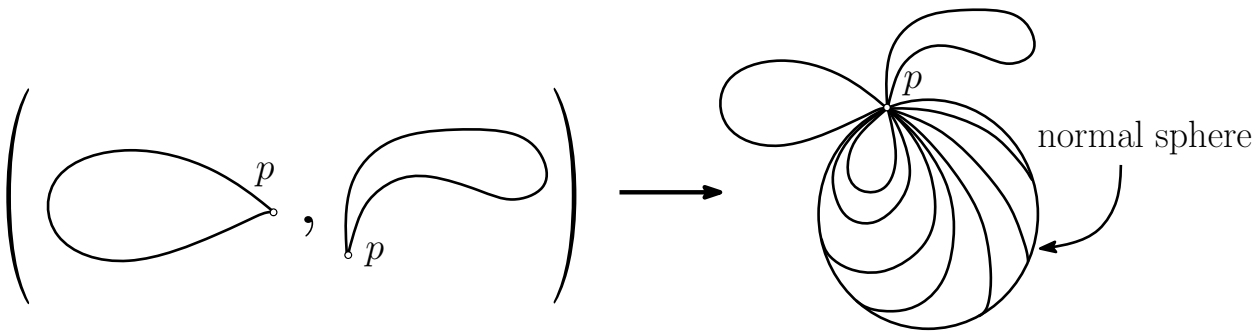

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Theorem

The dg coalgebra $\mathfrak{C}(M)$ encodes non homotopy invariants.