MAT 131, Calculus I Fall 2005
Midterm I Solutions

(1) (**10pts**) *For each of the questions below, indicate if the statement is true* (**T**) *or false* (**F**).

a	**F**	f	**T**
b	**F**	g	**F**
c	**T**	h	**F**
d	**T**	i	**T**
e	**F**	j	**F**

(a) *If* $\lim_{x\to a} f(x) = L$, *then* $f(a) = L$.
F: f may not be even defined at a.

(b) $\lim_{x\to a}[f(x)g(x)] = [\lim_{x\to a} f(x)]\cdot[\lim_{x\to a} g(x)]$ *for any two functions* f *and* g.
F: LHS may be defined even if RHS is not; e.g. $f(x)=x$, $g(x)=1/x$, $a=0$.

(c) *The equation* $3^x + x^3 = 3$ *has a solution in the interval* $(0,1)$.
T: If $f(x)=3^x+x^3$, then $f(0)=1$ and $f(1)=4$. Since f is continuous on $[0,1]$ and $f(0)\le 3\le f(1)$, $f(x)=3$ for some $x\in[0,1]$ by the Intermediate Value Theorem. Since $f(0), f(1)\neq 3$, it follows that $f(x)=3$ for some $x\in(0,1)$.

(d) *The line* $y=0$ *is a horizontal asymptote for the function* $f(x)=\frac{\cos x}{(x+1)^6}$.
T: $\lim\limits_{x\longrightarrow\infty}\frac{\cos x}{(x+1)^6}=0$, since $-1\le\cos x\le 1$ for all x, while $(x+1)^6\longrightarrow\infty$ as $x\longrightarrow\infty$.

(e) *If* f *is continuous, then* f *must be differentiable.*
F: $f(x)=|x|$ is continuous everywhere, but not differentiable at $x=0$.

(f) $\lim_{x\to 0} f\left(e^{2x}+xf(x)\right) = f(1)$, *provided that* f *is continuous.*
T: As $x\longrightarrow 0$, $e^{2x}+xf(x)\longrightarrow e^{2\cdot 0}+0\cdot f(0)=1$. Thus, $f(e^{2x}+xf(x))\longrightarrow f(1)$, since f is continuous.

(g) *The function* $f(x)=\begin{cases} x^3 & \text{if } x\le 0,\\ x & \text{if } x\ge 0\end{cases}$ *is differentiable everywhere.*
F: The derivatives of $g(x)=x^3$ and $h(x)=x$ at $x=0$ are 0 and 1, respectively. Thus, the function f defined piecewise is not differentiable at the junction point 0.

(h) *There exists an even function* f *such that* $f'<0$ *at every point.*
F: Since $f(-x)=f(x)$ for all x, $f'(-x)=-f'(x)$ for all x. Thus, $f'(x)$ cannot be always negative.

(i) *If the graph of* f *is always concave upward, then the tangent line at every point must be below the graph.*
T: The assumption means the graph rises faster (or descends slower) than any of its tangent lines.

(j) f *attains a local maximum or a local minimum value whenever* $f'=0$.
F: If $f(x)=x^3$, then $f'(0)=0$, but $f(x)$ achieves neither a local maximum nor a local minimum at $x=0$.

(2) (**10 pts**) *Consider the function*

$$f(x) = \frac{\ln|x+2|}{x^2-9}.$$

Determine the points at which f is discontinuous. Explain your reasoning briefly.

The function $y = \ln x$ is defined and continuous for all $x > 0$. Thus, numerator of f, $\ln|x+2|$, is defined and continuous for all x such that $|x+2| > 0$, i.e. for all $x \neq -2$. It is not defined, and thus not continuous, for $x = -2$.

The denominator of f,

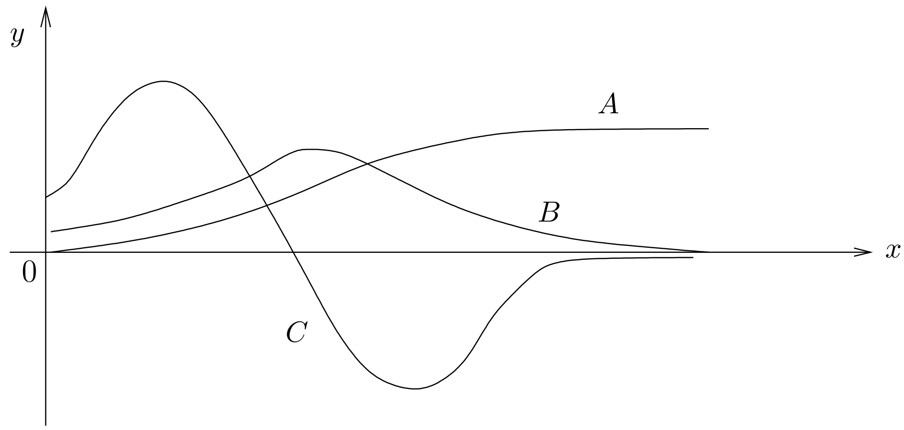
$$x^2 - 9 = (x-3)(x+3),$$

is a polynomial. Thus, it is defined and continuous for all x. Its zeros are $x = -3, 3$. Since we cannot divide by 0, f is discontinuous at $x = -3, 3$.

Thus, the points at which f is discontinuous are $\boxed{x = -3, -2, 3}$

(3) (**20 pts**) *Answer the following questions.*

(a) (**15 pts**) *The figure shows the graphs of f, f', and f''. Identify each curve and write your answers in the table below.*



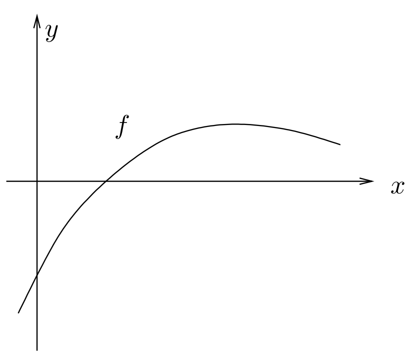
Answer:

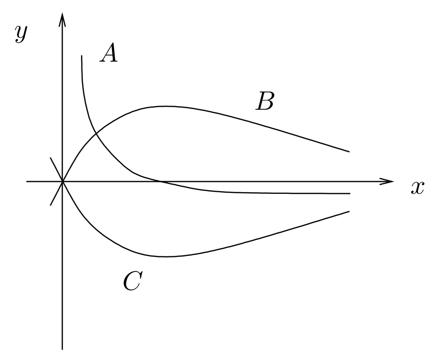
A	B	C
f	f'	f''

Grading: Each correct answer is 5 pts. Among the wrong answers: f'', f, f' is 7 pts. Otherwise give 0 pts.

Note that C has two points (the max and the min) at which the tangent line to C is horizontal, i.e. of slope 0. Since the derivative of a function is the slope of the tangent line and neither A nor B crosses the x-axis where C has a max or min, neither A nor B is the graph of the derivative of the function represented by C. Thus, C must correspond to f''. Since the x-intercept of C corresponds to the maximum of B, C must be the derivative of B. Thus, B corresponds to f' and A to f.

(b) (**5 points**) *The first figure shows the graph of f. In the second figure, which of the graphs A, B and C is the graph of an antiderivative of f? Cross the box corresponding to your answer.* (*An antiderivative of f is a function whose derivative is f.*)





Answer:

A	B	C
		X

Grading: 0 pts otherwise

Since f is negative at first and then positive, its antiderivative must first decrease and then increase.

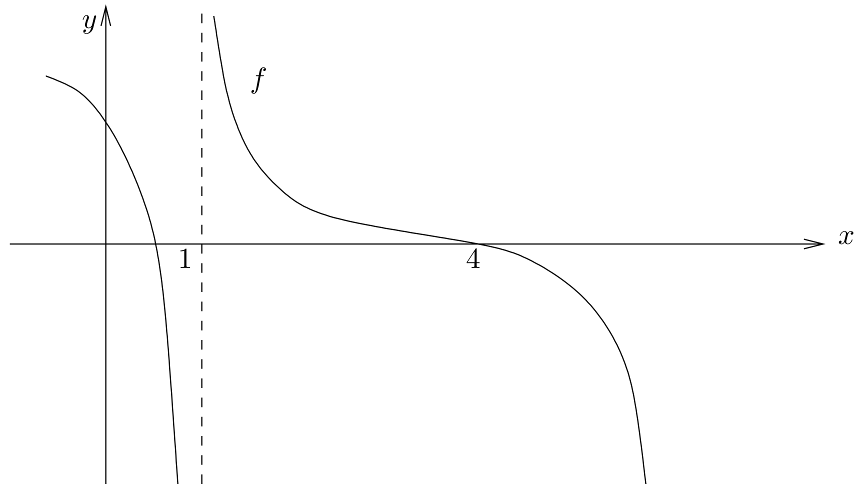
(4) (**20 pts**) *Sketch the graph of a function that satisfies all of the given conditions.*

(a) (**10 pts**)

i. $f'(x) < 0$ *for all* $x \neq 1$, *vertical asymptote* $x = 1$;

ii. $f''(x) < 0$ if $x < 1$ or $x > 4$, $f''(x) > 0$ if $1 < x < 4$;

iii. $f(4) = 0$.



We start by labeling the relevant x-values on the x-axis, i.e. 1 and 4, and drawing the vertical asymptote, $x=1$. Since $f'(x)<0$ for $x\neq 1$, f is always decreasing, except at $x=1$. Thus, f drops toward $-\infty$ as x approaches 1 from the left and ascends toward ∞ as x approaches 1 from the right.

On $(-\infty, 1)$, the graph of f is always concave down, since $f''(x)<0$ if $x<1$. The graph of f crosses the x-axis at $x=4$, since $f(4)=0$. It is concave up on $(1,4)$ and down on $(4,\infty)$, since $f''(x)>0$ on $(1,4)$ and $f''(x)<0$ on $(4,\infty)$. So, $(4,0)$ is an inflection point.

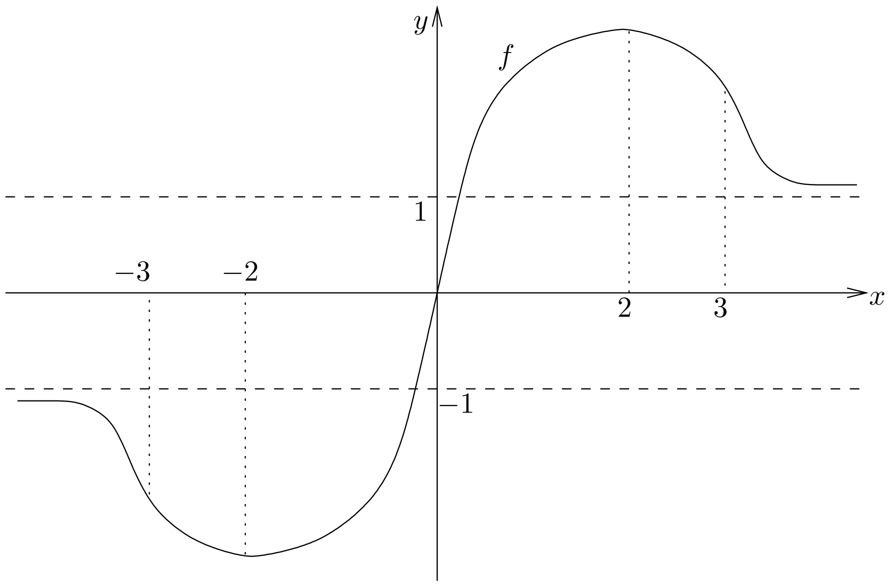
(b) (**10 pts**)

i. $f'(x) > 0$ if $-2 < x < 2$, $f'(x) < 0$ if $x < -2$ or $x > 2$, $f'(2) = 0$;

ii. $\lim_{x\to\infty} f(x) = 1$;

iii. f is odd, i.e., $f(-x) = -f(x)$;

iv. $f''(x) < 0$ if $0 < x < 3$, $f''(x) > 0$ if $x > 3$.



We start by labeling the relevant x-values on the x-axis, i.e. -2, 2, and 3. Since f is odd, the x-value of -3 must also be relevant, so we mark it as well. Since

$$\lim_{x\to\infty} f(x) = 1,$$

we have a horizontal asymptote, $y=1$, which we sketch next. Since f is odd,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(-x) = -1.$$

So, we have another horizontal asymptote, $y=-1$, which we also sketch.

Since $f'(x)<0$ for all $x>2$, f is decreasing for all $x>2$. Thus, the graph of f must approach the horizontal asymptote $y=1$ from above. Since $f'(x)>0$ on $(0,2)$, f is increasing on $(0,2)$. Also, note that there is a local max at 2 since sign of f' changes from positive to negative. Since f is odd, $f(0)=0$. Thus, starting at $(0,0)$, the graph of f rises for x-values in $(0,2)$ to a y-value above $y=1$, which is also the local max value, and then descends on $(2,\infty)$, approaching $y=1$ from above. Furthermore, the graph of f is concave down over $(0,3)$ and up over $(3,\infty)$, since $f''(x)<0$ on $(0,3)$ and $f''(x)>0$ on $(3,\infty)$. Thus, the graph has an inflection point over $x=3$.

Since the function f is odd, the part of the graph to the left of y-axis, i.e. for negative x, is obtained by reflecting the part already constructed about $(0,0)$. In particular, the lowest point of the graph corresponds to $x=-1$ and lies below $y=-2$, which is is the local min value. The graph is concave up on $(-3,0)$ and down on $(-\infty,-3)$. It approaches the asymptote $y=-1$ from below.

(5) (**30 points**) *Evaluate the following limits.*

(a) (**7 points**)

$$\lim_{x\to 0} \frac{x^6 + \ln(2x^2+3)}{8e^x + \sin x}$$

Since $\ln x$, e^x, $\sin x$, and x^n are continuous everywhere they are defined and the entire fraction is defined at $x=0$, we can simply plug in $x=0$:

$$\lim_{x\to 0} \frac{x^6 + \ln(2x^2+3)}{8e^x + \sin x} = \frac{0^6 + \ln(2\cdot 0^2+3)}{8e^0 + \sin 0} = \frac{0+\ln 3}{8\cdot 1 + 0} = \boxed{\frac{\ln 3}{8}}$$

(b) (**8 points**)

$$\lim_{t\to 0} \frac{\sqrt{t^3+25}-5}{t^3}$$

If we simply plug in $t=0$ in this case, we'll get $0/0$, which is undefined. So, we need to do something else. We first multiply and divide by the conjugate of the numerator:

$$\lim_{t\to 0} \frac{\sqrt{t^3+25}-5}{t^3} = \lim_{t\to 0} \frac{\left(\sqrt{t^3+25}-5\right)\left(\sqrt{t^3+25}+5\right)}{t^3\left(\sqrt{t^3+25}+5\right)} = \lim_{t\to 0} \frac{\left(\sqrt{t^3+25}\right)^2 - 5^2}{t^3\left(\sqrt{t^3+25}+5\right)}$$

$$= \lim_{t\to 0} \frac{(t^3+25)-25}{t^3\left(\sqrt{t^3+25}+5\right)} = \lim_{t\to 0} \frac{t^3}{t^3\left(\sqrt{t^3+25}+5\right)} = \lim_{t\to 0} \frac{1}{\sqrt{t^3+25}+5}$$

$$= \frac{1}{\sqrt{0^3+25}+5} = \frac{1}{5+5} = \boxed{\frac{1}{10}}$$

(c) (**7 pts**)

$$\lim_{u\to\infty} \frac{7+12u^5}{1+2u+u^4-3u^5}$$

If we simply plug in ∞ for u in this case, we would get $\infty - \infty$ in the denominator, which is undefined. Instead we first divide the numerator and the denominator by u^5:

$$\lim_{u\to\infty} \frac{7+12u^5}{1+2u+u^4-3u^5} = \lim_{u\to\infty} \frac{(7+12u^5)/u^5}{(1+2u+u^4-3u^5)/u^5}$$

$$= \lim_{u\to\infty} \frac{7/u^5+12}{1/u^5+2/u^4+1/u-3} = \frac{0+12}{0+0+0-3} = \boxed{-4}$$

(d) (**8 pts**)

$$\lim_{x\to\infty} \left(3x - \sqrt{9x^2+5x}\,\right)$$

If we plug in ∞ for x, once again we would get $\infty - \infty$, which is undefined. Instead, we first multiply and divide by the conjugate:

$$\lim_{x\to\infty} \left(3x - \sqrt{9x^2+5x}\right) = \lim_{x\to\infty} \frac{\left(3x - \sqrt{9x^2+5x}\right)\left(3x + \sqrt{9x^2+5x}\right)}{3x + \sqrt{9x^2+5x}}$$

$$= \lim_{x\to\infty} \frac{(3x)^2 - \left(\sqrt{9x^2+5x}\right)^2}{3x+\sqrt{9x^2+5x}} = \lim_{x\to\infty} \frac{9x^2 - (9x^2+5x)}{3x+\sqrt{9x^2+5x}}$$

$$= \lim_{x\to\infty} \frac{-5x}{3x+\sqrt{9x^2+5x}}$$

If we plug in ∞ for x now, we would obtain ∞/∞, which is again undefined. Instead, we next divide the numerator and the denominator by x:

$$\lim_{x\to\infty} \left(3x - \sqrt{9x^2+5x}\right) = \lim_{x\to\infty} \frac{-5x}{3x+\sqrt{9x^2+5x}} = \lim_{x\to\infty} \frac{-5x/x}{\left(3x+\sqrt{9x^2+5x}\right)/x}$$

$$= \lim_{x\to\infty} \frac{-5}{3+\left(\sqrt{9x^2+5x}\right)/x}$$

Note that

$$\frac{\sqrt{9x^2+5x}}{x} = \sqrt{\left(\frac{\sqrt{9x^2+5x}}{x}\right)^2} = \sqrt{\frac{(\sqrt{9x^2+5x})^2}{x^2}} = \sqrt{\frac{9x^2+5x}{x^2}} = \sqrt{9+5/x}$$

Finally, we conclude that

$$\lim_{x\to\infty} \left(3x - \sqrt{9x^2+5x}\right) = \lim_{x\to\infty} \frac{-5}{3+\left(\sqrt{9x^2+5x}\right)/x} = \lim_{x\to\infty} \frac{-5}{3+\sqrt{9+5/x}}$$

$$= \frac{-5}{3+\sqrt{9+0}} = \frac{-5}{3+3} = \boxed{-\frac{5}{6}}$$

(6) (**10 pts**) *Let*

$$f(x) = \sqrt{5x+1}$$

be defined on $(-1/5, \infty)$*. Use the definition of the derivative to compute* $f'(x)$*.*

$$f'(x) \equiv \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{5(x+h)+1} - \sqrt{5x+1}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{5(x+h)+1} - \sqrt{5x+1}\right)\left(\sqrt{5(x+h)+1} + \sqrt{5x+1}\right)}{h\left(\sqrt{5(x+h)+1} + \sqrt{5x+1}\right)}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{5(x+h)+1}\right)^2 - \left(\sqrt{5x+1}\right)^2}{h\left(\sqrt{5(x+h)+1} + \sqrt{5x+1}\right)} = \lim_{h\to 0} \frac{\left(5(x+h)+1\right) - \left(5x+1\right)}{h\left(\sqrt{5(x+h)+1} + \sqrt{5x+1}\right)}$$

$$= \lim_{h\to 0} \frac{\left(5x+5h+1\right) - \left(5x+1\right)}{h\left(\sqrt{5(x+h)+1} + \sqrt{5x+1}\right)} = \lim_{h\to 0} \frac{5h}{h\left(\sqrt{5(x+h)+1} + \sqrt{5x+1}\right)}$$

$$= \lim_{h\to 0} \frac{5}{\sqrt{5(x+h)+1} + \sqrt{5x+1}} = \frac{5}{\sqrt{5(x+0)+1} + \sqrt{5x+1}}$$

$$= \frac{5}{\sqrt{5x+1} + \sqrt{5x+1}} = \boxed{\frac{5}{2\sqrt{5x+1}}}$$