MAT 131, Calculus I   Fall 2005

Final Exam Solutions

1. **(10pts)** *For each of the questions below, indicate if the statement is true (**T**) or false (**F**).*

a	**T**
b	**T**
c	**T**
d	**F**
e	**F**

f	**F**
g	**T**
h	**F**
i	**F**
j	**T**

(a) *The equation* $\cos x = 2x$ *has a solution in the interval* $(0,1)$.
T: Let $f(x)=2x-\cos x$. Since

$$-1 = f(0) < 0 < f(1) = 2 - \cos 1,$$

by the Intermediate Value Theorem $f(c)=0$ for some c in the interval $[0,1]$. Since $f(0), f(1) \neq 0$, $f(c)=0$ for some c in the interval $(0,1)$.

(b) *Suppose that* $f'(x) = g'(x)$ *and* $f(2) = g(2)$. *Then necessarily* $f(x) = g(x)$.
T: By the Fundamental Theorem of Calculus,

$$f(x) = f(2) + \int_2^x f'(t)\,dt = g(2) + \int_2^x g'(t)\,dt = g(x).$$

(c) $\displaystyle\int_1^5 e^x\,dx > \int_1^5 \ln x\,dx.$
T: since $e^x > \ln x$ for all $x > 0$.

(d) *If* $f(-1) = 13$, *then* $\lim_{x\to -1} f(x) = 13$ *provided that the limit exists.*
F: $\lim_{x\to -1} f(x)$ says nothing about $f(-1)$ unless f is continuous at -1.

(e) *There exists a function* f *such that* $f(0) = 1$, $f(3) = 2$, *and* $f'(x) \geq 1$ *for all* x.
F: By the Mean Value Theorem, for some c between 0 and 3

$$f'(c) = \frac{f(3) - f(0)}{3 - 0} = \frac{1}{3} < 1.$$

(f) *If* f *is differentiable and has a local maximum at* c, *then* $f''(c) < 0$.
F: The function $f(x) = -x^4$ has a local maximum $c=0$, but $f''(0)=0$.

(g) *The function* $f(x) = \sqrt{x^4 + e^{x^3}}\,\sin(\pi x)$ *has at least one critical number in the interval* $(-1, 3)$.
T: Since $f(0)=0=f(1)$, by the Mean Value Theorem for some c in the interval $[0,1]$

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = 0.$$

(h) *If* f *is continuous on* $[a,b]$ *and* $f(x) \geq 0$, *then* $\int_a^b \sqrt{f(x)}\,dx = \sqrt{\int_a^b f(x)\,dx}$.
F: If $f(x)=1$, $a=0$, and $b=2$, then

$$\int_a^b \sqrt{f(x)}\,dx = 2 \neq \sqrt{2} = \sqrt{\int_a^b f(x)\,dx}.$$

(i) $\int_{-2}^{2} x\sqrt{14 + x^6 + x^8}\,dx > 0$.
F: The integrand is an odd function; so the integral is 0.

(j) *The Fundamental Theorem of Calculus asserts that*

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \quad \textit{for all } x \in (a,b),$$

provided that f *is continuous on* $[a,b]$.
T: This is one of the two statements of FTC.

2. (**10+5 pts**) *Use Newton's method starting with the initial approximation $x_1 = 1$ to find the third approximation x_3 to the number $\sqrt[4]{3}$. Is your estimate bigger or smaller than the actual value of $\sqrt[4]{3}$? Justify your answer by picture.*

Since $x = \sqrt[4]{3}$ is a solution of $f(x) = x^4 - 3 = 0$, apply Newton's Method to approximate a solution of $f(x) = 0$ starting with $x_1 = 1$. Since $f'(x) = 4x^3$, we obtain

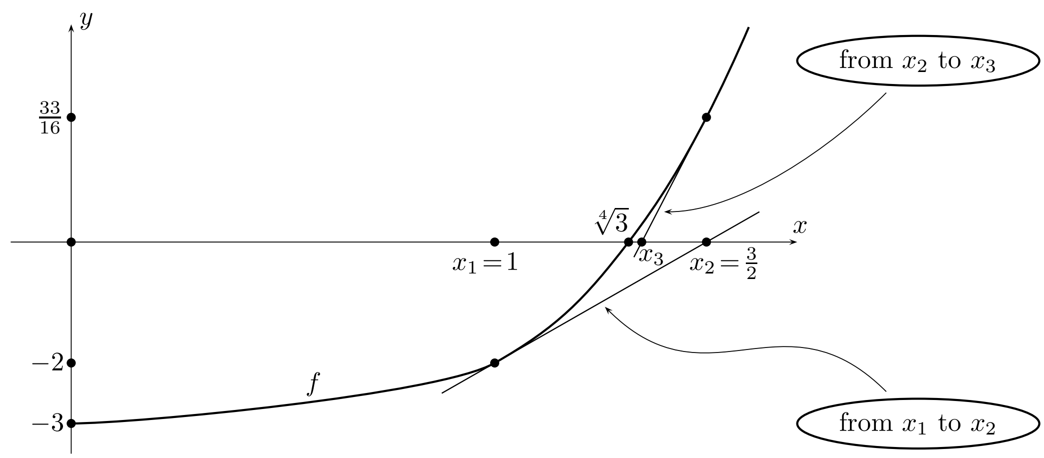
$$x_1 = 1 \qquad f(x_1) = 1^4 - 3 = -2 \qquad f'(x_1) = 4 \cdot 1^3 = 4$$

$$x_2 = x_1 - \frac{f'(x_1)}{f(x_1)} = 1 - \frac{-2}{4} = 1 + \frac{1}{2} = \frac{3}{2}$$

$$f(x_2) = \frac{3^4}{2^4} - 3 = \frac{81}{16} - 3 = \frac{33}{16} \qquad f'(x_2) = 4\frac{3^3}{2^3} = \frac{27}{2}$$

$$x_3 = x_2 - \frac{f'(x_2)}{f(x_2)} = \frac{3}{2} - \frac{33/16}{27/2} = \frac{3}{2} - \frac{11}{9 \cdot 8} = \frac{3 \cdot 36 - 11}{72} = \boxed{\frac{97}{72}}$$

The estimate x_3 is $\boxed{\text{bigger}}$ than the actual value of $\sqrt[4]{3}$, as illustrated in the diagram below. The estimate x_2 is obtained from x_1 by drawing the line tangent to the graph of f at $(x_1, f(x_1))$ and taking its x-intercept; this intercept is x_2. Since the graph of f is concaved up and increasing, x_2 must be larger than $\sqrt[4]{3}$. Similar, x_3 is the x-intercept of the line tangent to the graph of f at $(x_2, f(x_2))$; it is larger than $\sqrt[4]{3}$ for the same reasons.



3. (**40 pts**) *Find the following limits.*

(a) (**10 pts**) $\lim_{x\to 0} \left(1 + \dfrac{x^2 + 2^x - 5}{|x^3 - 27| + \sec x}\right)$

Plug in $x = 0$:

$$\lim_{x\to 0} \left(1 + \frac{x^2 + 2^x - 5}{|x^3 - 27| + \sec x}\right) = 1 + \frac{0^2 + 2^0 - 5}{|0^3 - 27| + \sec 0} = 1 + \frac{-4}{28} = 1 - \frac{1}{7} = \boxed{\frac{6}{7}}$$

(b) (**10 pts**) $\lim_{x\to 0} \dfrac{e^x - 1}{x^2 - \sin(3x)}$

If we plug in $x = 0$, we obtain

$$\frac{e^0 - 1}{0^2 - \sin(3 \cdot 0)} = \frac{0}{0}.$$

This is not defined, but we can use L'Hospital's Rule:

$$\lim_{x\to 0} \frac{e^x - 1}{x^2 - \sin(3x)} = \lim_{x\to 0} \frac{(e^x - 1)'}{(x^2 - \sin(3x))'} = \lim_{x\to 0} \frac{e^x}{2x - 3\cos(3x)}$$
$$= \frac{e^0}{2 \cdot 0 - 3\cos(3 \cdot 0)} = \frac{1}{-3} = \boxed{-\frac{1}{3}}$$

(c) (**10 pts**) $\lim_{x\to 1} x^{1/(1-x)}$

If we plug in $x=1$, we obtain

$$1^{1/(1-1)} = 1^{\pm\infty}.$$

This is not defined. We cannot use L'Hospital's Rule directly, so we take the natural log:

$$\lim_{x\to 1}\left(\ln x^{1/(1-x)}\right) = \lim_{x\to 1}\left(\frac{1}{1-x}\ln x\right) = \lim_{x\to 1}\left(\frac{\ln x}{1-x}\right).$$

If we plug in $x=1$, we obtain

$$\frac{\ln 1}{1-1} = \frac{0}{0}.$$

This is not defined, but we can now use L'Hospital's Rule:

$$\lim_{x\to 1}\left(\frac{\ln x}{1-x}\right) = \lim_{x\to 1}\left(\frac{(\ln x)'}{(1-x)'}\right) = \lim_{x\to 1}\frac{1/x}{-1} = \frac{1/1}{-1} = -1.$$

Finally,

$$\lim_{x\to 1} x^{1/(1-x)} = \lim_{x\to 1} e^{\ln x^{1/(1-x)}} = e^{\lim_{x\to 1}\ln x^{1/(1-x)}} = \boxed{e^{-1}} = \boxed{\frac{1}{e}}$$

(d) (**10 pts**) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sin\left(\pi i\frac{1}{n}\right)$

This is a Riemann sum. Here $\Delta x = 1/n$ and $x_i = i/n = a + i\Delta x$. Thus, $a=0$, $b=1$, and $f(x)=\sin(\pi x)$:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sin\left(\pi i\frac{1}{n}\right) = \int_0^1 \sin(\pi x)\,dx.$$

To compute the integral, substitute $u=\pi x \implies du = \pi\,dx$:

$$\int_0^1 \sin(\pi x)\,dx = \int_0^{\pi}\sin u\,\frac{du}{\pi} = -\frac{\cos u}{\pi}\bigg|_{u=0}^{u=\pi} = -\frac{1}{\pi}\big(\cos\pi - \cos 0\big) = -\frac{1}{\pi}\big(-1-1\big) = \boxed{\frac{2}{\pi}}$$

4. (**25 pts**) *Let*

$$f(x) = \int_{-1}^{x^3} \sqrt{1+t^4}\, dt.$$

(i) *Find the critical numbers, if any, and intervals of decrease and increase of f.*

(ii) *Find the inflection points, if any, and intervals of concavity of f.*

(iii) *Sketch accurately the graph of f. Find the x-intercept of the graph and mark it on the x-axis. Is $f(0)$ bigger or smaller than 1? Mark the point $(0,1)$ on the y-axis, positioning it with respect to the point $(0, f(0))$.*

(iv) *Does f have any local maximum or minimum values? What about absolute extrema?*

(i) We need to compute $f'(x)$. In order to apply FTC, write $f(x) = F(x^3)$, where

$$F(u) = \int_{-1}^{u} \sqrt{1+t^4}\, dt.$$

We can now apply FTC to compute $F'(u)$ and then chain rule to compute $f'(x)$:

$$F'(u) = \sqrt{1+u^4} \qquad \Longrightarrow \qquad f'(x) = F'(x^3)\cdot(x^3)' = \sqrt{1+(x^3)^4}\cdot(3x^2) = 3x^2\sqrt{1+x^{12}}.$$

The derivative f' of f is defined for all x and $f'(x) = 0$ if and only if $x = 0$. Thus, the only $\boxed{\text{critical number for } f \text{ is } x=0}$ Since $f'(x) > 0$ for all $x \neq 0$ and $f'(0) = 0$, f is always increasing, i.e. the $\boxed{\text{interval of increase is } (-\infty,\infty)}$

(ii) We now need to compute f'':

$$f''(x) = \big(f'(x)\big)' = \big(3x^2(1+x^{12})^{1/2}\big)' = 3\big(2x(1+x^{12})^{1/2} + x^2\cdot\frac{1}{2}(1+x^{12})^{-1/2}\cdot 12x^{11}\big)$$
$$= \frac{6x}{\sqrt{1+x^{12}}}\big(1+x^{12}+3x^{12}\big) = \frac{6(4x^{12}+1)}{\sqrt{1+x^{12}}}x.$$

Thus, $f''(x)>0$ and thus $\boxed{\text{concaved up if } x>0}$; $f''(x)<0$ and thus $\boxed{\text{concaved down if } x<0}$; $\boxed{x=0 \text{ is the inflection point}}$

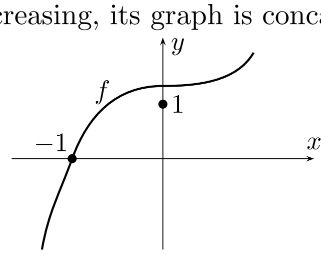
(iii) Since

$$f(-1) = \int_{-1}^{(-1)^3} \sqrt{1+t^4}\, dt = \int_{-1}^{-1} \sqrt{1+t^4}\, dt = 0,$$

an x-intercept is $x=-1$. Since $\sqrt{1+t^4}>1$ for all $t\neq 0$,

$$f(0) = \int_{-1}^{0^3} \sqrt{1+t^4}\, dt > \int_{-1}^{0} 1\, dt = 1\cdot\big(0-(-1)\big) = 1.$$

Thus, the point $(0,1)$ lies below the y-intercept of the graph of f. In addition, by (i) and (ii), f is always increasing, its graph is concaved down if $x<0$ and up if $x>0$:



(iv) Since f is always increasing, f has $\boxed{\text{no}}$ local max/min values or absolute extrema.

5. (40 **pts**) *Compute the following integrals.*

(a) (**10 pts**) $\int \left(1 + 3x^5 + \frac{1}{1+x^2}\right) dx$

$$\int \left(1 + 3x^5 + \frac{1}{1+x^2}\right) dx = \int 1\, dx + 3\int x^5\, dx + \int \frac{1}{1+x^2}\, dx$$
$$= x + 3\frac{x^6}{6} + \tan^{-1}(x) + C = \boxed{x + \tfrac{1}{2}x^6 + \tan^{-1}(x) + C}$$

(b) (**10 pts**) $\int_1^e \left(\frac{2}{x} + \frac{1}{x^2}\right) dx$

Since an anti-derivative for $2x^{-1}+x^{-2}$ is $2\ln|x| - x^{-1}$, by FTC

$$\int_1^e \left(\frac{2}{x} + \frac{1}{x^2}\right) dx = \left(2\ln|x| - x^{-1}\right)\Big|_{x=1}^{x=e} = \left(2\ln e - e^{-1}\right) - \left(2\ln 1 - 1^{-1}\right)$$
$$= \left(2\cdot 1 - e^{-1}\right) - \left(2\cdot 0 - 1\right) = \boxed{3 - e^{-1}} = \boxed{3 - \tfrac{1}{e}}$$

(c) (**10 pts**) $\displaystyle\int \frac{e^{(1+\sqrt{x}\,)}}{\sqrt{x}}\,dx$

Substitute $u = 1+\sqrt{x} = 1+x^{1/2} \implies du = \frac{1}{2}x^{-1/2}dx = dx/2\sqrt{x}$:

$$\int e^{(1+\sqrt{x}\,)}\,\frac{dx}{\sqrt{x}} = \int e^{u}\,2du = 2e^{u} + C = \boxed{2e^{(1+\sqrt{x}\,)} + C}$$

(d) (**10 pts**) $\displaystyle\int_0^{\pi} \sin^3\left(\frac{x}{4}\right)\cos\left(\frac{x}{4}\right)\,dx$

Substitute

$$u{=}\sin(x/4) \qquad \implies \qquad du = \cos(x/4)\cdot\frac{1}{4}\,dx = \cos(x/4)\frac{dx}{4}.$$

Furthermore,

$$\sin(0/4) = \sin 0 = 0, \qquad \sin(\pi/4) = 1/\sqrt{2} \qquad \implies$$

$$\int_0^{\pi} \sin^3\left(\frac{x}{4}\right)\cos\left(\frac{x}{4}\right)\,dx = \int_0^{1/\sqrt{2}} u^3\,4du$$

$$= 4\frac{u^4}{4}\Big|_{u=0}^{u=1/\sqrt{2}} = \frac{1}{\sqrt{2}^4} - 0^4 = \frac{1}{2^2} = \boxed{\frac{1}{4}}$$

6. (**40 pts**) *Compute the following.*

(a) (**10 pts**) $\frac{d}{dx}\big(\tan(x^7)+\ln(1+\sin x)\big)$

By Chain Rule,

$$\begin{aligned}\frac{d}{dx}\big(\tan(x^7)+\ln(1+\sin x)\big) &= \frac{1}{\cos^2(x^7)}\cdot 7x^6+\frac{1}{1+\sin x}\cdot\cos x\\ &= \boxed{\frac{7x^6}{\cos^2(x^7)}+\frac{\cos x}{1+\sin x}}\end{aligned}$$

(b) (**10 pts**) y' if $y^3 = xy+\cos y+e^{-x}$

Use Implicit Differentiation:

$$\begin{aligned} y^3 = xy+\cos y+e^{-x} \quad &\Longrightarrow \quad 3y^2\cdot y' = y+xy'-\sin y\cdot y'-e^{-x}\\ &\Longrightarrow \quad \big(3y^2-x+\sin y\big)y' = y-e^{-x}\\ &\Longrightarrow \quad \boxed{y' = \frac{y-e^{-x}}{3y^2-x+\sin y}}\end{aligned}$$

(c) (**10 pts**) y' if $y = (x+2)^{1/x}$
Use Logarithmic Differentiation:

$$y = (x+2)^{1/x} \implies \ln y = \ln\left((x+2)^{1/x}\right) = \frac{1}{x}\ln(x+2) = x^{-1}\ln(x+2)$$

$$\implies \frac{y'}{y} = -x^{-2}\ln(x+2) + x^{-1}\frac{1}{x+2}$$

$$\implies y' = \big(-x^{-2}\ln(x+2) + x^{-1}\frac{1}{x+2}\big)y$$

$$\implies \boxed{y' = \left(-\frac{\ln(x+2)}{x^2} + \frac{1}{x(x+2)}\right)(x+2)^{1/x} = x^{-1}\left(\frac{1}{x+2} - \frac{\ln(x+2)}{x}\right)(x+2)^{1/x}}$$

(d) (**10 pts**) $f'(-1)$ if $f(x) = 3 + \int_{x^2}^{1} \sec(1-t)\,dt$

Write $f(x) = 3 - F(x^2)$, where

$$F(u) = -\int_u^1 \sec(1-t)\,dt = \int_1^u \sec(1-t)\,dt.$$

By FTC and the Chain Rule:

$$F'(u) = \sec(1-u) \implies$$
$$f'(x) = -F'(x^2)\cdot(x^2)' = -\sec(1-x^2)\cdot 2x \implies$$
$$f'(-1) = -\sec(1-(-1)^2)\cdot 2\cdot(-1) = 2\frac{1}{\cos 0} = \boxed{2}$$

7. (**20 pts**) *A box with a square base and open top is to have a volume of* 32 in^3. *Find the dimensions of the box that minimize the amount of material used. Make sure to show that the dimensions you have found actually give rise to an absolute minimum.*

Denote by a the length and width of the square base and by h the height of the box. Then,

$$V = a^2 h = 32.$$

We need to minimize the surface area of the box, which consists of 4 sides of area ah each and the bottom of area a^2:

$$S = 4ah + a^2.$$

Using the first equation to solve for h in terms of a and plugging into the second, we obtain

$$h = \frac{32}{a^2} \qquad \Longrightarrow \qquad S(a) = 4a\frac{32}{a^2} + a^2 = 128a^{-1} + a^2.$$

We need to minimize this function for a in $(0,\infty)$. Its derivative is given by

$$S'(a) = -128a^{-2} + 2a.$$

It is defined on $(0,\infty)$, i.e. wherever S is defined. Thus, the only critical numbers for S are the zeros of S':

$$S'(a) = -128a^{-2} + 2a = 0 \quad \Longrightarrow \quad 2a = 128a^{-2} \quad \Longrightarrow \quad a^3 = 64 \quad \Longrightarrow \quad a = 64^{1/3} = 4.$$

There are two ways to see that $S(a)$ achieves the absolute minimum at $a=4$. Since S is continuous on $(0,\infty)$,

$$\lim_{a\to 0^+} S(a) = \frac{128}{0^+} + 0 = \infty, \qquad \text{and} \qquad \lim_{a\to\infty} S(a) = 0 + \infty = \infty,$$

$S(a)$ must achieve its absolute minimum at some point a on $(0,\infty)$. Since such a point must be a critical number for S, $a=4$ is the only possibility. Alternatively, note that

$$S'(a) = 2a^{-2}(a^3 - 64) = 2a^{-2}(a-4)(a^2+4a+16) = 2a^{-2}(a^2+4a+16)\,(a-4).$$

Thus, $S'(a)<0$ if $0<a<4$ and $S'(a)>0$ if $a>4$. The first derivative test for absolute extrema then implies that $a=4$ is the absolute minimum of the function $S(a)$ on $(0,\infty)$.

The corresponding height for $a=4$ is

$$h = \frac{32}{4^2} = 2.$$

Thus, the desired dimensions are

$$\boxed{\text{length=width= 4 in, height= 2 in}}$$

8. (**10 pts**) *Suppose that a car starts moving from stationary position and that it has acceleration given by* $a(t) = 6e^t + 20t^3$. *Find the function which gives the distance that the car has traveled after* t *hours.*

Denote by $x(t)$ the position of the car, relative to its starting point, at time t and by $v(t)$ its velocity at time t. Since $x(t)$ is the position relative to the starting point and the car is stationary at $t=0$,

$$x(0) = 0 \qquad \text{and} \qquad v(0) = 0.$$

Since $v'(t)=a(t)$, $v(t)$ is an anti-derivative for $a(t)$:

$$a(t) = 6e^t + 20t^3 \qquad \Longrightarrow \qquad v(t) = 6e^t + 20\frac{t^4}{4} + C = 6e^t + 5t^4 + C.$$

Plug in $v(0)=0$ to find C:

$$v(0) = 6e^0 + 5 \cdot 0^4 + C = 0 \quad \Longrightarrow \quad 6 + C = 0 \quad \Longrightarrow \quad C = -6 \quad \Longrightarrow \quad v(t) = 6e^t + 5t^4 - 6.$$

Since $x'(t)=v(t)$, $x(t)$ is an anti-derivative for $v(t)$:

$$v(t) = 6e^t + 5t^4 - 6 \qquad \Longrightarrow \qquad x(t) = 6e^t + 5\frac{t^5}{5} - 6t + C = 6e^t + t^5 - 6t + C.$$

Plug in $x(0)=0$ to find C:

$$x(t) = 6e^0 + 0^5 - 6 \cdot 0 + C = 0 \qquad \Longrightarrow \qquad 6 + C = 0 \qquad \Longrightarrow \qquad C = -6$$

$$\Longrightarrow \qquad \boxed{x(t) = 6e^t + t^5 - 6t - 6}$$