MAT 131, Calculus I   Fall 2005
Sample Final Exam

(1) *For each of the questions below, indicate if the statement is true* (**T**) *or false* (**F**).

a	f
b	g
c	h
d	i
e	j

(a) $f(x) = x\ln(x) - x$ *is an antiderivative of* $\ln x$ *on* $(0,\infty)$.
T: $f'(x) = \ln(x) + \frac{x}{x} - 1$.

(b) *The horizontal axis is a horizontal asymptote for* $f(x) = \frac{x^3}{3e^x}$.
T: Compute $\lim_{x\to+\infty} f(x)$. Use L'Hospital Rule three times to end up with the limit of $2/e^x$ which is 0.

(c) $f(x) = (\sin x).e^x$ *is concave upward on* $(0,\pi/2)$.
T: $f'(x) = \cos x.e^x + \sin x.e^x$ and $f''(x) = -\sin x.e^x + \cos x.e^x + \cos x.e^x + \sin x.e^x = 2\cos x.e^x$ which is non negative on $(0,\pi/2)$.

(d) *If f,g are continuous on* $(0,\infty)$ *then* $\int_1^2 f(t).g(t)dt = (\int_1^2 f(t)dt).(\int_1^2 g(t)dt)$.
F: Take as a counterexample $f(x) = g(x) = x$.

(e) $g(x) = \frac{3(x+1)(x-2)}{(x-7)(x-1)}$ *is differentiable on* $(-\infty, \frac{1}{2})$.
T: $g(x)$ is differentiable on any interval that doesn't contain 1 and 7.

(f) $\frac{d}{dx}\ln(1+x^2) = \frac{2x}{(1+x^2)^2}$.
F: The true answer is $\frac{2x}{1+x^2}$.

(g) $f(x) = (x-1)^5$ *has one local minimum and one local maximum on* $(-2,2)$.
F: The derivative is $5(x-1)^4 \geq 0$ so the function is increasing everywhere so there is no local max or min.

(h) $\int_1^2 \frac{3\sqrt{x}+x^3}{x} = 6(\sqrt{2}-1) + \frac{7}{3}$.
T: Write $\frac{3\sqrt{x}+x^3}{x} = 3x^{-1/2} + x^2$ and then use the fact that an atiderivative for x^n is $\frac{x^{n+1}}{n+1}$.

(i) *The tangent line to the graph of* $f(x) = \sqrt{1+x^3}$ *at* $(a = 2)$ *has an equation given by* $y = 2x - 1$.
T: The tangent line is given by $y = (\sqrt{1+2^3}) + \frac{1}{2\sqrt{1+2^3}}.3.2^2.(x-2) = 2x - 1$.

(j) $\int_{-2}^2 \frac{1}{t^4}dt = -\frac{1}{3}\left(\frac{1}{8} - \frac{1}{-8}\right) = -\frac{1}{12}$.
F: The function is not continuous on this interval, so you cannot apply the Evaluation Theorem.

(2) (a) **Find** $\lim_{x\to+\infty} x^2.\ln(1+\frac{1}{x})$.
This is an indeterminate form of type $\infty.0$. Write the function as a quotient $\frac{\ln(1+1/x)}{1/x^2}$, and apply L'Hospital rule to get $\frac{-1/x^2}{(1+1/x)(-2/x^3)}$ which has limit $+\infty$.

(b) **Find the derivative of** $f(x) = (cos3x)^x + x\ln x - x$**.**
As usual, write $(cos3x)^x = e^{x.\ln(\cos 3x)}$. Then $f'(x) = (cos3x)^x.(\ln(\cos 3x)+x.\frac{-3\sin 3x}{\cos 3x})+\ln x$.

(c) **Compute** $\lim_{x\to+\infty}(x+\frac{\cos x}{x^2}) - x.(\sqrt{1+\frac{1}{x}})$**.**
First get rid of $\frac{\cos x}{x^2}$ whose limit is zero. Then you get an indeterminate form $\infty.0$. Factor by x and write $x(1-\sqrt{1+1/x}) = x.\frac{1-(1+1/x)}{1+\sqrt{1+1/x}} = -\frac{1}{1+\sqrt{1+1/x}}$ whose limit is $-\frac{1}{2}$.

(d) **Find an antiderivative** $F(x)$ **of** $f(x)$ **on** $(0,+\infty)$ **when** $f(x) = \sin(7x+12)+\sqrt{2x}$.
An antiderivative of $\sin(7x+12)$ is $-\frac{1}{7}\cos(7x+12)$ and write $\sqrt{2x} = \sqrt{2}.x^{1/2}$ to get one possible antiderivative $F(x) = -\frac{1}{7}\cos(7x+12) + \sqrt{2}.\frac{2}{3}x^{3/2}$.

(e) **Find** $\frac{dy}{dx}$ **when you know that** $y^3 + 3xy + x^2 = 0$**.**
We differentiate implicitly the equation with respect to x, by considering y as a function of x:
$3y^2.\frac{dy}{dx} + 3y + 3x.\frac{dy}{dx} + 2x = 0$,
so $\frac{dy}{dx} = -\frac{2x+3y}{3y^2+3x}$.

(3) **Let** $f(x) = x.(x-2)^3$

(a) **Find the absolute maximum, the absolute minimum and the local ones on** $[-3,3]$.

We have $f'(x) = (x-2)^3 + x.3(x-2)^2 = (x-2)^2(4x-2)$, so the sign of the derivative is the sign of (x-1/2). Therefore f is decreasing on $[-3,1/2]$, increasing on $[1/2,3]$. Since $f(-3) = 3.125 = 375$, $f(1/2) = -\frac{27}{16}$ and $f(3) = 3$, we conclude that f has an absolute max at -3, an absolute min at $1/2$, which is also the only local min. There is no local max.

(b) **Find the largest intervals included in** $[-3,3]$ **where f is increasing, decreasing, and then concave upward.**
We answered one part of it already. Then we compute $f''(x) = 2(x-2).(4x-2) + 4(x-2)^2 = 16(x-2)(x-3/4)$. So the function is concave upward on $(-3,3/4)\cup(2,3)$.

(4) **Newton's method.**

(a) **Write down the two first steps** x_1, x_2 **for Newton's method applied to the equation** $f(x) = x^3 - 2$ **with starting point** $x_0 = 1$.

We know that $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. So here we have $x_{n+1} = x_n - \frac{x_n^3-2}{3x_n^2}$. Therefore $x_1 = 1 - \frac{-1}{3} = \frac{4}{3}$, and then $x_2 = \frac{4}{3} - \frac{(4/3)^3-2}{3(4/3)^2} = \frac{91}{72}$ after simplification.

(b) **Redo the exercise from a previous homework 4 page 325.**

(5) (a) **Compute** $\int_1^5 (\cos(3x+12) + \frac{\sqrt{x}+x^3}{x^4})dx$.
By the fundamental theorem of calculus, you just need to find an antiderivative of $\cos(3x+12)$ (for example $\frac{1}{3}\sin(3x+12)$) and of $\frac{\sqrt{x}+x^3}{x^4} = x^{-7/2} + \frac{1}{x}$: using that $\frac{x^{n+1}}{n+1}$ is an antiderivative of x^n for $n \neq -1$, you can see that $\frac{x^{-5/2}}{-5/2} + \ln x$ works.
Then the integral is equal to : $\frac{1}{3}(\sin(27) - \sin(15)) - \frac{2}{5}(5^{-5/2} - 1)$.

(b) **Show that** $\int_2^7 \sqrt{3 + x^2 + 7x^4}dx \geq 5\sqrt{3}$.
Just remark that $\sqrt{3 + x^2 + 7x^4} \geq \sqrt{3}$ and then integrate: you will get $\int_2^7 \sqrt{3 + x^2 + 7x^4}dx \geq \int_2^7 \sqrt{3}dx = 5\sqrt{3}$.

(c) **Find** $\frac{d}{dx}\int_{-1}^{x^2} \sin(3+t^2)dt$.
Call $F(x) = \int_{-1}^{x} \sin(3+t^2)dt$, then we need to find $\frac{d}{dx}F(x^2)$ which is $F'(x^2).2x$ by the Chain rule formula, but now $F'(x) = \sin(3+x^2)$ by the fundamental theorem of calculus. Therefore the answer is $\sin(3+x^4).2x$.

(d) **Compute** $\int_0^1 \frac{7e^{3x}}{1+e^{3x}}dx$.
Let $u = e^{3x}$ then $du = 3e^{3x}dx$ and then by the substitution rule the integral is now $\int_1^{e^3} \frac{7}{3}\frac{1}{1+u}du = \frac{7}{3}(\ln(1+e^3) - \ln(1)) = \frac{7}{3}\ln(1+e^3)$.

(e) **Compute** $\int_0^{\pi/2} (\cos t. \sin^3 t + 2\sin t. \cos t)dt$.
Simply set $u = \sin t$ and then $du = \cos t dt$ and the integral becomes $\int_0^1 (u^3 + u)du = 1/4 + 1 = 5/4$.

(f) **What is the following limit:** $\lim_{n\to+\infty} \sum_{i=1}^{n} \frac{i^7}{n^8}$**?**
Here you can recognize a Riemann sum for $f(x) = x^7$ on the interval $[0,1]$. Therefore this limit is equal to $\int_0^1 x^7 dx = \frac{1}{8}$.

(6) **Among all the right angle triangles of fixed area (equal to 1/2), find the one with the shortest hypotenuse.**
Let's call z the length of the hypotenuse and x, y the others sides of the triangle. Then we want to minimize $z = \sqrt{x^2 + y^2}$ knowing that $x.y = 2(area) = 1$. Therefore we want to minimize $z(x) = \sqrt{x^2 + \frac{1}{x^2}}$. First we notice that $z(x)$ is minimal when $z^2(x)$ is minimal. But $\frac{d}{dx}z^2(x) = 2x - \frac{2}{x^3} = \frac{2}{x^3}(x^4 - 1)$. So z^2 is decreasing on $[0,1]$ and then increasing: this means that the absolute minimum is reached at $x = y = 1$.