SPRING 2005, MAT 542 PROBLEM SHEET 3

Due: April 14, Thursday.

1. How many Laurent series representations of the function

$$f(z) = \frac{1}{1 - z^2} + \frac{1}{5 - z}$$

   are there? Find each of them in the region where the series converges.

2. Find and classify the singularities of the following functions:

   (a) $f(z) = \dfrac{1 + z^2}{(z^2 + z + 1)(z - 1)^2}$

   (b) $f(z) = \dfrac{z}{\sin z}$

3. Calculate the following integrals using residues.

   (a) $\displaystyle\int_0^\pi \frac{dx}{(a + \cos x)^2}$, where $a > 1$.

   (b) $\displaystyle\int_0^\infty \frac{dx}{(a + b\,x^2)^n}$, where $a, b > 0$.

   (c) $\displaystyle\int_0^\infty \frac{\log x}{(1 + x)^3}$. (*Note:* We have not done this kind of integration in class, so you may need to first read about the ways to treat them.)

4. (a) Let n be an integer ≥ 2. Show that

$$\int_0^\infty \frac{dx}{1 + x^n} = \frac{\pi/n}{\sin(\pi/n)},$$

   by integrating the function $1/(1 + z^n)$ along the contour formed by the segment $[0, R]$ of the positive real axis, the arc represented by Re^{it} for $0 \leq t \leq 2\pi/n$, and the segment represented by $re^{2\pi i/n}$ for $0 \leq r \leq R$.

   (b) Let n be an integer ≥ 2 and let α be a real number such that $n > 1 + \alpha > 0$. Evaluate, by the same method as in part (a), the integral

$$\int_0^\infty \frac{x^\alpha}{1 + x^n}\, dx.$$

5. Let $f(z) = e^{(z + 1/z)}$. Prove that

$$\mathrm{Res}(f;\, 0) = \sum_{k=0}^{\infty} \frac{1}{k!(k + 1)!}.$$

6. Let $f(z)$ be a holomorphic function for $|z| > R$. Define the *residue (at the point) at infinity* by
$$\operatorname{Res}(f;\,\infty) = -\frac{1}{2\pi i}\int_{|z|=r} f(z)\,dz,$$
where r is any number $> R$. Verify the following statements:
   (a) $\operatorname{Res}(f;\,\infty) = -a_{-1}$ if $\sum_{n=-\infty}^{\infty} a_n\,z^n$ is the Laurent series expansion of $f(z)$ for $|z| > R$. (The minus sign makes sense since a positively oriented closed circle about ∞ is negatively oriented with respect to the origin and vice-versa.)
   (b) If $g(z) = -z^{-2}\,f(1/z)$, then $\operatorname{Res}(f;\,\infty) = \operatorname{Res}(g;\,0)$. (Here the idea is to send ∞ to 0 using the map $z \mapsto 1/z$, hence the factor $-1/z^2$.)
   (c) If $f(z)$ is a rational function with poles $p_1, p_2, \ldots, p_n$ in $\mathbb{C}$, then
$$\operatorname{Res}(f;\,\infty) + \sum_{k=1}^{n}\operatorname{Res}(f;\,p_k) = 0.$$
   (d) Calculate $\operatorname{Res}(f;\,\infty)$ for
      (i) $f(z) = z^3 - 7z^2 + 8$
      (ii) $f(z) = e^z/p(z)$, where $p(z)$ is a polynomial.
7. Use Rouché's Theorem to show that $f(z) = 1 + 15z + z^5$ has one of its roots in the disk $B(0; 3/2)$ and all of its roots in the disk $B(0; 2)$.
8. Give a proof of the Fundamental Theorem of Algebra using Rouché's Theorem.
9. Let e_1 and e_2 be two non-zero complex numbers such that $e_1/e_2 \notin \mathbb{R}$, i.e., they are linearly independent over $\mathbb{R}$. We say that a meromorphic function $f(z)$ is *doubly periodic* with periods e_1 and e_2 if $f(z) = f(z+e_1) = f(z+e_2)$ for all z. (Or, equivalently, $f(z) = f(z + n_1e_1 + n_2e_2)$ for all z and for all integers n_1, n_2.)
   (a) Choose a complex number a such that $f(z)$ has no poles on the boundary of the (closed) parallelogram with vertices a, $a + e_1$, $a + e_2$, $a + e_1 + e_2$. Prove that $\int_\gamma f(z)\,dz = 0$, where γ denotes the boundary of the paralellogram.
   (b) Use part (a) and the Argument Principle to prove the following: Let f be a non-constant, doubly periodic meromorphic function with periods e_1 and e_2. Let $a \in \mathbb{C}$ and consider the paralellogram P as described in part (a). Then, the number of zeros of f contained in P is equal to the number of poles contained in P, provided that no zeros or poles of f occur on the boundary of the parallelogram. (Poles and zeros are counted with multiplicity.)
   (c) Prove that every doubly periodic entire function is constant.