SPRING 2005, MAT 542 PROBLEM SHEET 2

Due: February 25, Friday.

1. Let $G \subseteq \mathbb{C}$ be an open, connected domain. A function $u(x, y)\colon G \to \mathbb{R}$ is said to be *harmonic* in G if it is C^2-smooth and satifies the *Laplace* equation:
$$\Delta u := u_{xx} + u_{yy} = 0$$
.
   (a) Let $f\colon G \to \mathbb{C}$ be holomorphic and $f(z) = u + iv$. Prove that $u(x, y)$ is harmonic. Note that, similarly, $v(x, y)$ has to be harmonic.
   (b) Use part (a) to show that $u(x, y) = \log(x^2 + y^2)^{1/2}$ is harmonic on $G = \mathbb{C} \setminus \{0\}$.
   (c) Let $u\colon G \to \mathbb{R}$ be harmonic. If there exists a harmonic function $v\colon G \to \mathbb{R}$ such that $f = u + iv$ is holomorphic, then such a function v is called a *harmonic conjugate* of u.
      (i) Note that v is certainly not unique since adding any real constant creates another. Conversely, prove that any two harmonic conjugates of u will differ by a real constant.
      (ii) Does $u(z) = \log|z|$ have a harmonic conjugate in the domain $G = \mathbb{C} \setminus \{0\}$? Justify your answer.
      (iii) Let G be either $\mathbb{C}$ or some open disk in $\mathbb{C}$ and let $u\colon G \to \mathbb{R}$ be harmonic. Show that u has a harmonic conjugate.
2. Prove that the Laplacian $\Delta = 4\,\partial\bar{\partial} = 4\,\bar{\partial}\partial$. Here the operators ∂ and $\bar{\partial}$ are defined as follows:
$$\partial := \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \bar{\partial} := \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$
3. Let $u\colon \mathbb{C} \to \mathbb{R}$ be defined by
$$u(re^{i\theta}) = a_0 + \sum_{k=1}^{n} r^k \left[a_k \cos(k\theta) + b_k \sin(k\theta)\right]$$
for $r \geq 0$ and for real θ, where a_0, a_k and b_k, $1 \leq k \leq n$, are real constants. Show that u is harmonic in $\mathbb{C}$, and determine the harmonic conjugate v of u in the plane such that $v(0) = 0$.
4. Construct a Mobius transformation f such that
   (a) f maps the circle $\Gamma = \Gamma(i; 1)$ to $\Omega = \{w \mid (1 + i)w + (1 - i)\bar{w} = 0\}$.
   (b) f maps $\Gamma = \mathbb{R} \cup \{\infty\}$ to the circle $\Omega = \Omega(0; 1)$, leaving the point -1 fixed.

5. Let $\mathbb{H} = \{z \mid \mathrm{Im(z)} > 0\}$ be the upper half-plane and let $f = (az+b)/(cz+d)$ be a Mobius transformation. Show that $f(\mathbb{H}) = \mathbb{H}$ if and only if a, b, c, d are real numbers.

6. Let $a \in \mathbb{C}$ such that $|a| < 1$. Define
$$\Phi_a(z) = \frac{z-a}{1-\bar{a}z}.$$
Prove that each Φ_a is a conformal self map of the unit disk. Find the inverse of Φ_a.

7. Determine the largest open set G in which the function $f(z) = \log(z^3 + 1)$ is holomorphic. Here log denotes the principal branch of the logarithm. Compute $f'(z)$ in that set.

8. Give a formula for the branch of the fourth root function in the domain $G = \mathbb{C} \setminus [0, \infty)$ that satisfies $f(-1) = (\sqrt{2} + i\sqrt{2})/2$. Calculate $f'(-i)$. What are the other three branches in G?

9. Let $f(z) = \log z$ be the principal branch of the logarithm.
   (a) Find the Taylor series expansion of f about $a = -1 + i$. Find the domain on which this series converges to f.
   (b) Find the radius of convergence of the power series from part (a). Find the function to which this power series converges within its disk of convergence. Is it the same as f?

10. (a) Let f be an entire function and suppose that there is a constant M, an $R > 0$, and an integer $n \geq 1$ such that $|f(z)| \leq M\,|z|^n$ for $|z| > R$. Show that f is a polynomial of degree $\leq n$.
   (b) Suppose that f is bounded and holomorphic on $\mathbb{C} \setminus \{0\}$. Prove that f is constant. (*Hint:* Try to use part (a).)

11. Let $f\colon G \to \mathbb{C}$ be holomorphic, where G is open and connected. Suppose that there exists an $a \in G$ such that $|f(a)| \leq |f(z)|$ for all $z \in G$. Prove that either $f(a) = 0$ or f is constant.

12. Let $u\colon \mathbb{C} \to \mathbb{R}$ be harmonic such that $u(z) \geq 0$ for all $z \in \mathbb{C}$. Prove that u is constant.