# SPRING 2005, MAT 542 PROBLEM SHEET 1

Due: February 10, Thursday.

1. Let $a_n$ and $b_n$ be two sequences. Let $A_n = \sum_{k=0}^{n} a_k$ for $n \geq 0$ and let $A_{-1} = 0$.
   (a) For $0 \leq p \leq q$, verify the "partial summation formula":
   $$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$
   (b) Assume that the sequence $A_n$ is bounded, $b_0 \geq b_1 \geq b_2 \geq \cdots$ and that $\lim_{n\to\infty} b_n = 0$. Prove that $\sum_{n\geq 0} a_n b_n$ is convergent.
   (c) Derive the following from part (b): Suppose that the radius of convergence of $\sum_{n\geq 0} a_n\, z^n$ is one and that $a_0 \geq a_1 \geq a_2 \geq \cdots$ and $\lim_{n\to\infty} a_n = 0$. Then, $\sum_{n\geq 0} a_n\, z^n$ converges at every point on the circle $|z| = 1$, except at $z = 1$.

   Observe that part (c) implies that $\sum_{n\geq 1} \frac{z^n}{n}$ is convergent at all points of $|z| = 1$, except at $z = 1$.
2. (a) Suppose that the power series $\sum_{n\geq 0} a_n\, z^n$ has radius of convergence one and let $f(z) = \sum_{n\geq 0} a_n\, z^n$ on the open unit disk $|z| < 1$. Assume also that the series converges at a point $w$ which lies on the boundary circle $|z| = 1$. Prove that $\lim_{z\to w} f(z) = \sum_{n\geq 0} a_n\, w^n$.
   (b) Prove that the function
   $$\sum_{n\geq 0} 2^{-n}\, z^{2^n}$$
   is holomorphic on the open unit disk $|z| < 1$ and continuous on $|z| \leq 1$. Prove that if $w$ is a $(2^K)^{\text{th}}$ root of unity, then $\lim_{r\to 1^-} |f'(rw)| = \infty$.

   This shows that $f$ cannot be the restriction to the open unit disk of a holomorphic function defined on a connected open set that is strictly larger than the disk.
3. Determine the radius of convergence of each of the following series. Then determine at which points on the boundary of the disk of convergence the series converges.
   (a) $\sum_{n\geq 0} p(n)\, z^n$, where $p$ is a polynomial.
   (b) $\sum_{n\geq 0} n e^{-n}\, z^n$

(c) $\displaystyle\sum_{n\geq 0}(-1)^n(2n+1)^{-1}\,z^{2n+1}$

(d) $\displaystyle\sum_{n\geq 2}\frac{n}{n^2+4}\,z^n$ (*Hint*: Use the "partial summation formula".)

4. The "Fibonacci sequence" $a_n$ is defined recursively by $a_0 = 0$, $a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. Verify that the radius of convergence $R$ of the associated power series $\sum_{n\geq 1} a_n\, z^n$ is positive. Then, through the process of actually finding the sum this series, determine $R$ exactly. (*Hint:* For the first part, check that $0 \leq a_n \leq 2^n$ for every $n$.)

5. Answer the following:

   (a) Let $\sum_{n\geq 0} a_n$ be an absolutely convergent series and let $\sum_{n\geq 0} b_n$ be a convergent series. Let $c_n = \sum_{0\leq k\leq n} a_k b_{n-k}$. Prove that $\sum_{n\geq 0} c_n$ converges and its sum is equal to the product $(\sum_{n\geq 0} a_n)(\sum_{n\geq 0} b_n)$.

   (b) Give an example illustrating the fact that the conclusion in part (a) fails if none of the two convergent series is absolutely convergent. (Hence, in order to ensure that the product converges, it is necessary to assume that at least one of the two series converges absolutely.)

   (c) Let $A$ and $B$ be two convergent power series about the point $a \in \mathbb{C}$ having radii of convergence $R_1$ and $R_2$ respectively. Let $C$ be the series obtained by "taking the product of $A$ and $B$". Then, prove that $C$ is convergent with the radius of convergence $\geq \min\{R_1, R_2\}$. Moreover, show that $C$ sums to the product of the sums of $A$ and $B$ within its radius of convergence. (Here use part (a).)

   (d) Use part (c) to prove that $e^{z+w} = e^z\, e^w$ for any $z, w \in \mathbb{C}$, where $e^z = \sum_{n\geq 0} \frac{z^n}{n!}$. Hence, $e^z \neq 0$ for all $z \in \mathbb{C}$.