A solution to problem 2.(a) in Homework I

Problem 2.(a)

Suppose that the power series $\sum_{n\geq 0} a_n\, z^n$ has radius of convergence one and let $f(z) = \sum_{n\geq 0} a_n\, z^n$ on the open unit disk $|z| < 1$. Assume also that the series converges at a point w which lies on the boundary circle $|z| = 1$. Prove that $\lim_{z\to w} f(z) = \sum_{n\geq 0} a_n\, w^n$.

Solution:

Step 1. We first write the function f in a slightly different form as follows: Define $s_n = \sum_{k=0}^{n} a_k\, w^k$ for $n \geq 1$ and let $s_{-1} = 0$. In particular, we have $s_n - s_{n-1} = a_n\, w^n$ for $n \geq 0$. Then,

$$\begin{aligned}
\sum_{n=0}^{m} a_n\, z^n &= \sum_{n=0}^{m} \underbrace{(a_n w^n)}_{s_n - s_{n-1}} \left(\frac{z}{w}\right)^n \\
&= \sum_{n=0}^{m} (s_n - s_{n-1}) \left(\frac{z}{w}\right)^n \\
&= \sum_{n=0}^{m} s_n \left[\left(\frac{z}{w}\right)^n - \left(\frac{z}{w}\right)^{n+1}\right] + s_m \left(\frac{z}{w}\right)^m \\
&= \left(1 - \frac{z}{w}\right) \sum_{n=0}^{m} s_n \left(\frac{z}{w}\right)^n + s_m \left(\frac{z}{w}\right)^m \qquad (*)
\end{aligned}$$

For $|z| \leq 1$ and $|w| = 1$, i.e. $|z/w| \leq 1$, let $m \to \infty$. We get

$$\lim_{m\to\infty} \sum_{n=0}^{m} a_n\, z^n = \sum_{n=0}^{\infty} a_n\, z^n = f(z) = \lim_{m\to\infty} (*)$$

Hence,

$$f(z) = \left(1 - \frac{z}{w}\right) \sum_{n=0}^{m} s_n \left(\frac{z}{w}\right)^n,$$

since the second summand in $(*)$ approaches zero as $m \to \infty$.

Step 2. Yet another trick. Write

$$1 = \left(1 - \frac{z}{w}\right) \sum_{n=0}^{\infty} \left(\frac{z}{w}\right)^n, \text{ for } |z/w| < 1.$$

Step 3. Recall that the series converges at the point w and that s_n is, by definition, the sequence of partial sums at w. Thus, for some finite number s, we have

$$s = \lim_{n\to\infty} s_n.$$

Let $\epsilon > 0$. Choose $N > 0$ such that $|s - s_n| < \sqrt{\epsilon}/2$ for all $n > N$.

Step 4. Finally we look at the limit $\lim_{z\to w} |f(z) - s|$, which we claim is zero. Here is the proof:

$$\begin{aligned}
|f(z) - s| &= \left|\left(1 - \frac{z}{w}\right)\sum_{n=0}^{\infty} s_n\left(\frac{z}{w}\right)^n - \left(1 - \frac{z}{w}\right)\sum_{n=0}^{\infty} s\left(\frac{z}{w}\right)^n\right| \\
&= \left|\left(1 - \frac{z}{w}\right)\sum_{n=0}^{\infty}(s_n - s)\left(\frac{z}{w}\right)^n\right| \\
&= \left|1 - \frac{z}{w}\right|\left|\sum_{n=0}^{N}(s_n - s)\left(\frac{z}{w}\right)^n + \sum_{n=N+1}^{\infty}(s_n - s)\left(\frac{z}{w}\right)^n\right| \\
&\le \left|1 - \frac{z}{w}\right|\left(\left|\sum_{n=0}^{N}(s_n - s)\left(\frac{z}{w}\right)^n\right| + \left|\sum_{n=N+1}^{\infty}(s_n - s)\left(\frac{z}{w}\right)^n\right|\right) \\
&\le \left|1 - \frac{z}{w}\right|\left(\sum_{n=0}^{N}|s_n - s|\left|\frac{z}{w}\right|^n + \sum_{n=N+1}^{\infty}|s_n - s|\left|\frac{z}{w}\right|^n\right) \qquad (**)
\end{aligned}$$

About the summands in paranthesis: Since $|z/w| < 1$, the first summand will be $\le \sum_{n=0}^{N} |s_n - s|$. But this is a just finite sum and therefore $\le M$ for some constant $M > 0$. The second summand is $\le \sqrt{\epsilon}/2 \sum_{n=N+1}^{\infty} |z/w|^n$, by our choice of N. But $\sum_{n=N+1}^{\infty} |z/w|^n$ must converge to zero since it is the tail of a convergent series.

At this point we need to modify N and M so as to make everthing appropriately small. So let $N' \ge N$ be such that $\sum_{n=N'+1}^{\infty} |z/w|^n < \sqrt{\epsilon}$ and let $M' \ge M$ be such that $\sum_{n=0}^{N'} |s_n - s| < M'$ and that $M' \ge \epsilon/2$.

Following these observations and after resplitting $(**)$ at N' rather than at N, we have

$$|f(z) - s| < \left|1 - \frac{z}{w}\right|\left(M' + \frac{\sqrt{\epsilon}}{2}\sqrt{\epsilon}\right).$$

Our final step is making $|1 - z/w|$ small. Recall that $z \to w$, which means that there exists a $\delta > 0$ such that $|1 - z/w| < \epsilon/(2M')$ whenever $|z - w| < \delta$.

This also means that

$$|f(z) - s| < \frac{\epsilon}{2M'}\left(M' + \frac{\epsilon}{2}\right) = \frac{\epsilon}{2} + \frac{\epsilon^2}{4M'} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

whenever $|z - w| < \delta$.