SPRING 2005, MAT 542 FINAL EXAM, PART II

Due: May 9, Monday, by 4:00 pm. **STRICT.** Please return your exam to me, 3-114; if I am not there, please slide it under the door.

You may discuss the problems together, but each of you should submit his or her own exam. You may also consult books. However, you are not allowed to discuss the exam with people outside the class.

Justify your answers. While solving, say, problem X, you may quote a result from problem Y, as long as you gave a correct proof of the statement in problem Y.

8. (15 points) Let f be holomorphic in the open unit disk D and continuous on $\overline{D}$. Assume that f is real valued on the boundary of D. Prove that f must be constant.

9. (25 points) Let D be the open unit disk and let $\mathcal{F}$ be the class of functions $f\colon D \to \mathbb{C}$ such that $f(0) = 0$, $f'(0) = 1$ and $|f(z)| \leq 73$ for all $z \in D$. Prove that there exists a constant $c > 0$ such that for any $f \in \mathcal{F}$ the image of the open unit disk $f(D)$ contains the open disk $|z| < c$.

10. (25 points) Let $G \in \mathbb{C}$ be an open, bounded and connected domain. Suppose the boundary of G consists of n smooth simple closed curves, where the boundary is oriented as "the boundary of G". (This orientation is the one that is determined by a chosen orientation of G.) Sometimes such a G is called n-connected. (For example, an open disk is 1-connected and an annulus is 2-connected.)
Suppose f is a non-constant function which is holomophic in G and continuous on the closure $\overline{G}$. Assume that $|f(z)| = 1$ for every $z \in \partial G$. Prove that f has at least n zeros, counted with multiplicities, in G.

11. (10 points) Find a factorization of $\cos(\pi z)$. Justify as many steps as you can.

PART I

1. (20 points) Show that the automorphism group of $\mathbb{C} \setminus \{0\} = \mathbb{C}^*$ is generated by multiplication by nonzero complex numbers and the inversion map $z \mapsto 1/z$. (Here automorphism means a biholomorphic mapping of $\mathbb{C}^*$ to $\mathbb{C}^*$.)

2. (10 points) Is there a conformal mapping of $\mathbb{C}$ that sends $x-$axis to itself and every vertical line $x = a$ to the slanted line $x = y + a$, where $a \in \mathbb{R}$? Justify your answer.

3. (20 points) Let γ_n be the rectangular path defined by $[n + 1/2 + ni,\ -n - 1/2 + ni,\ -n - 1/2 - ni, n + 1/2 - ni,\ n + 1/2 + ni]$, oriented counterclockwise, where n is any positive integer.

(a) Evaluate the integral $\displaystyle\int_{\gamma_n} \pi \frac{\cot(\pi z)}{z^2 - a^2}\, dz$ for $a \neq$ an integer.

(b) Show that $\displaystyle\lim_{n\to\infty} \int_{\gamma_n} \pi \frac{\cot(\pi z)}{z^2 - a^2}\, dz = 0$.

(c) Derive from part (b) that $\pi \cot(\pi a) = \dfrac{1}{a} + \displaystyle\sum_{n=1}^{\infty} \frac{2a}{a^2 - n^2}$.

4. (20 points) Let f be a holomorphic function in $\mathbb{C}$ such that $\int_{\mathbb{C}} |f|^2 < \infty$. Prove that f must be zero.

5. (20 points) Let a be a complex number with $|a| > e$. Show that for every positive integer n the equation $az^n = e^z$ has n different solutions in the open unit disk and has no other solutions in the half-plane $\mathbb{P} = \{z \,|\, \mathrm{Re}(z) < 1\}$.

6. (20 points) Show that every meromorphic function on the Riemann sphere $\mathbb{C}_\infty$ is necessarily rational.

7. (10 points) Let D be the open unit disk and $f\colon D \to D$ be a holomorphic function. Suppose that $f(a) = a$ and $f(b) = b$ for two different complex numbers $a, b \in D$. Show that $f(z) = z$.