

MATH 205, REVIEW EXERCISES FOR MIDTERM I

- (1) Given nonzero vectors  $\vec{v}$  and  $\vec{w}$ . What are the possible angles between the vectors  $\vec{v}$  and  $\vec{w}$  if
- $\vec{v} \cdot \vec{w} = -\|\vec{v}\|\|\vec{w}\|$ ?
  - $\|\vec{v} \times \vec{w}\| = \frac{1}{2}\|\vec{v}\|\|\vec{w}\|$ ?
- (2) Given the vectors  $\vec{a} = \langle -1, 0, 3 \rangle$ ,  $\vec{b} = \langle 2, 1, 0 \rangle$ ,  $\vec{c} = \langle 5, 1, -2 \rangle$  and  $\vec{d} = \langle -2/\sqrt{5}, -1/\sqrt{5}, 0 \rangle$ . For the questions (a)-(c), indicate if the statement is true or false. Justify your answer.
- $\vec{a}$  is parallel to  $\vec{d}$ .
  - $\vec{a} + \vec{b}$  is perpendicular to  $\vec{c}$ .
  - $\vec{d}$  is a unit vector in the direction of  $\vec{b}$ .
- Answer the questions (d)-(f).
- Find a vector perpendicular to both  $\vec{a}$  and  $\vec{b}$ .
  - Find the area of the parallelogram spanned by  $\vec{a}$  and  $\vec{c}$ .
  - Find which two of them do not span a parallelogram. Why?
- (3) Show that the two lines  $l_1 : x = 4 - t, y = 2t, z = 1 + t$  and  $l_2 : x = 18 + 2t, y = -1 - 4t, z = 1 - 2t$  are parallel and find an equation for the plane that contains them.
- (4) The planes  $-y + 2z = 7$  and  $x + z = 1$  meet in a line.
- Find a vector  $\vec{n}$  which is parallel to that line.
  - Find the equation of the plane that contains the point  $P(1, -1, 0)$  and is perpendicular to the line of intersection of the planes.
  - Find the component of  $\vec{v} = \langle 2, 1, 0 \rangle$  which is perpendicular to  $\vec{n}$  in (a).
- (5) Describe the solid region bounded by the unit sphere  $x^2 + y^2 + z^2 = 1$  and the cone  $z = \sqrt{x^2 + y^2}$  in  $\mathbb{R}^3$  using spherical coordinates, i.e., give the ranges of the coordinates  $\rho, \theta$  and  $\phi$ .
- (6) Find the tangent vector  $\gamma'(t)$  and the parametric equations for the tangent line to each of the following curves at the specified value of  $t$ . Determine whether the curve is smooth for each of the curve.
- $\gamma(t) = \langle \sin 3t, \cos 3t, 2t^{3/2} \rangle; t = \pi$
  - $\gamma(t) = \langle \cos^2 t, 3t - t^3, t \rangle; t = 0$
- (7) Let  $\gamma(t) = \langle t, t \sin t, t \cos t \rangle$ . Set up the integral that computes the arc length of  $\gamma(t)$  between the points  $(0, 0, 0)$  and  $(\pi, 0, -\pi)$ .
- (8) Reparametrize the curve

$$\gamma(t) = \left\langle \frac{2}{1+t^2} - 1, \frac{2t}{1+t^2} \right\rangle$$

with respect to the arc length measured from the point  $(1, 0)$  in the direction of increasing  $t$ . Express the arc length parametrization in its **simplest** form. You should be able to recognize this curve, what is it?

- (9) Find the curvature of  $y = e^x$ . At what points does the curve have maximum curvature? Explain what happens as  $x \rightarrow \infty$ .
- (10) Prove that the helix  $\gamma(t) = \langle 2t, 5 \sin t, 5 \cos t \rangle$  has constant curvature.
- (11) Do **problem 32** of section **10.5** in the text book. That is: Find a parametric representation for the torus obtained by rotating about the  $z$ -axis the circle in the  $xz$ -plane with center  $(b, 0, 0)$  and radius  $a < b$ . (*Hint*: Take as parameters the angles  $\theta$  and  $\alpha$  shown in the figure on page 742 of the text book.)
- (12) Describe (sketch) the surface in  $\mathbb{R}^3$  determined by the equation  $y^2 = x - 1$ .
- (13) Explain how to construct the graph of a function of two variables using its level curves.
- (14) Let  $f: \mathbb{R}^3 \rightarrow \mathbb{R}$  be a function defined by  $f(x, y, z) = z^2 + 9x^2$ .
- Find the domain and range of the function  $f$ .
  - Describe the level surfaces of the function  $f$  corresponding to the constant values  $k > 0$ ,  $k = 0$  and  $k < 0$ . What are they geometrically?
  - Restrict the function  $f$  to the  $xy$ -plane (or  $z = 0$  plane). Call this new function as  $g$ . Then describe the level curves of the function  $g$  for the constant values  $k > 0$ ,  $k = 0$  and  $k < 0$ .
- (15) Evaluate the limits in exercises (a)-(f), or explain why the limit fails to exist.
- $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2}$
  - $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$
  - $\lim_{(x,y) \rightarrow (-1,1)} \frac{x^2 e^{y(x+1)} - \cos(-\pi x)}{x^3 + xy^2}$
  - $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin y}{x}$
  - $\lim_{(x,y) \rightarrow (0,0)} yx \cos(1/x)$
  - $\lim_{(x,y,z) \rightarrow (2,1,0)} \sin(yxz) + x^3 e^{zy} - 5$
- (16) Examine the behavior of  $f(x, y) = \frac{x^4 y^4}{(x^2 + y^4)^3}$  as  $(x, y) \rightarrow (0, 0)$  along various straight lines. From your observations, what might you conjecture  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  to be? Next, consider what happens when  $(x, y)$  approaches  $(0, 0)$  along the curve  $x = y^2$ . Does  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  exist? Why or why not?