MATH 205; Spring 2004; B. Gürel
Review Exercises for the Final

1. For each of the ten questions below, state whether the assertion is *true* or *false*. (You do not need to justify your answer.)

   (a) The area of the portion of the graph of $z = f(x, y)$ over a region D in the (xy)-plane is equal to

   $$\iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA.$$

   (b) The integral

   $$\int_0^{2\pi}\int_0^2\int_r^2 dz\, dr\, d\theta$$

   represents the volume enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 2$.

   (c) Let $f(x, y)$ be a continuous function defined on a smooth curve C. Then $\int_{-C} f(x, y)\, ds = -\int_C f(x, y)\, ds$.

   (d) Let C be a smooth curve given by the vector fuction $\mathbf{r}(t)$, $a \le t \le b$. Let $f(x, y, z)$ be a smooth function defined on $\mathbb{R}^3$. The Fundamental Theorem for Line Integrals asserts that

   $$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(a)) - f(\mathbf{r}(b)).$$

   (e) The Jacobian of the map $T(u, v) = (au + bv, cu + dv)$ is $ad + bc$.

   (f) The integral $\iint_S f\, dS$, where f is a function, changes sign when the orientation of S is changed.

   (g) Let D be a region to which Green's theorem applies. Then the area of D is equal to $\frac{1}{2}\int_{\partial D}(xdy - ydx)$.

   (h) Let a surface S be oriented by the unit normal $\mathbf{n}$. Then $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{F} \cdot \mathbf{n})\, dS$, for any vector field $\mathbf{F}$.

   (i) The vector field

   $$\mathbf{F}(x, y) = \frac{-y}{x^2 + y^2}\,\mathbf{i} + \frac{x}{x^2 + y^2}\,\mathbf{j}$$

   is conservative.

   (j) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ for any C^2 vector field $\mathbf{F}$.

2. Evaluate the integral $\iint_D (\cos x - y)\, dx\, dy$, where the region D is bounded by $y = \sin x$, $y = 0$, $x = 0$, and $x = \pi$.

3. Evaluate the integral $\int_0^1 \int_x^1 e^{x/y} dy\, dx$.

4. Evaluate the integral $\iiint_E z\, dx\, dy\, dz$, where E is the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

5. Find the area of the portion of the paraboloid $z = 4 - x^2 - y^2$ that lies over the plane $z = 0$.

6. Find the volume of the solid that is bounded by the paraboloid $z = 9 - x^2 - y^2$, the xy-plane and the cylinder $x^2 + y^2 = 4$ and lies inside the cylinder $x^2 + y^2 = 4$.

7. Evaluate the line integral $\int_C f\, ds$, where $f(x, y) = y \cos(2\pi x)$ and C is the triangle with sides $x = 0$, $y = 0$, and $x + y = 1$.

8. Evaluate the following line integrals:

   (a) $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = -2(y + x)\mathbf{i} + (3x + 2y)\mathbf{j}$ and C is the ellipse $x^2/9 + y^2/3 = 1$ oriented counter-clockwise.

   (b) $\int_C (x\, dx + y\, dy + z^2\, dz)$, where C is the curve parametrized as $C(t) = (\cos t, \sin t, t)$ with $0 \le t \le 1$.

9. Is $\int_C 2x \sin y\, dx + (x^2 \cos y - 3y^2)\, dy$ independent of path in $\mathbb{R}^2$, where C is any path from $(-1, 0)$ to $(5, 1)$? Evaluate the integral.

10. Evaluate the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$ and S is the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \ge 0$.

11. Let $\mathbf{F}(z, y, z) = yz^2\mathbf{i} + xz^2\mathbf{j} + 2xyz\mathbf{k}$.

    (a) Calculate $\nabla \times \mathbf{F}$ and $\nabla \cdot \mathbf{F}$.

    (b) Is $\mathbf{F}$ conservative? Find a function f such that $\mathbf{F} = \nabla f$.

12. Use Stokes' theorem to evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = -xy\mathbf{i} - xz\mathbf{j} - yz\mathbf{k}$ and C is the triangle with vertices $(0, 1, 0)$, $(0, 1, 5)$, and $(3, 1, 0)$ oriented by the ordering of the points.

13. Use the Divergence theorem to evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ and S is the cylinder bounded by $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 1$ (the top and the bottom are included), oriented outward.