Research Description (non-technical)

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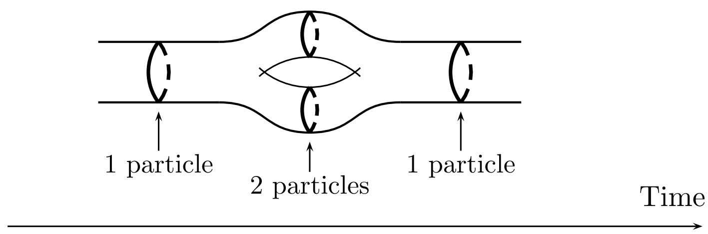
String theory is a physical model that represents elementary particles by vibrating strings with the aim of unifying the four forces of nature (gravitational, electromagnetic, strong nuclear, and weak nuclear). As a closed string (i.e a loop) moves through space, it may split into two (corresponding to fission), which may later recombine into one (corresponding to fusion). The path traced by a closed string in space-time is a (Riemann) surface, which has one hole (i.e. is of genus 1) and two ends in the just mentioned example:



While string theory is one of the main paradigms in physics today, strictly speaking it is not a physical theory at all as it has yet to make any experimentally testable *physical* predictions. However, string theory has made (and continues to make) plenty of *mathematical* predictions. Thus, mathematics (especially geometry) has so far been the only "testing ground" for string theory and the source of much corroboration.

Among the most striking mathematical predictions of string theory are the so-called mirror symmetry formulas for Gromov-Witten invariants (or GW-invariants). Physically, GW-invariants of a space (symplectic manifold) X represent states of a system; mathematically, they are rational numbers obtained by counting, with certain rational coefficients, Riemann surfaces (complex curves) that lie in X. GW-invariants are classified by the maximum genus, i.e. the maximum number of holes, and the number of ends a Riemann surface may have, and by a measure, called degree, of how twisted inside X a Riemann surface may be. For example, the genus 0 closed GW-invariants are weighted counts of spheres, i.e. Riemann surfaces without holes or ends, in X; similarly, the genus 1 closed GW-invariants are weighted counts of spheres and tori, i.e. donuts or Riemann surfaces with one hole, in X.

Most string theory predictions concern spaces of particular importance in physics, called Calabi-Yau manifolds. While some of these spaces are relatively simple from the physics point of view, they are generally highly non-trivial mathematically. The paradigmatic example is a quintic 3-fold $Q3$, i.e. the set of all tuples (x_1, x_2, x_3, x_4) that satisfy a degree 5 polynomial equation in four variables (a degree 5 hypersurface in $\mathbb{C}P^4$). The 1991 prediction of [3] for the genus 0 GW-invariants of $Q3$ was first verified in [11, 15]. This was of great significance not only in physics, but also in mathematics, as the first two proofs were later followed by at least three more [2, 6, 14], as well as by expository accounts of the first two [20, 5, 13]. The next case in terms of complexity, the 1993 prediction of [1] for the genus 1 GW-invariants of $Q3$ remained essentially unapproachable for about 10 years. Different approaches to at least partial verification of [1] were described in [7] and [17]. Separately, a thorough analysis of fundamental properties of the genus 1 GW-invariants was conducted in [29, 30, 16, 24], leading to a third approach to [1]. In [32], I finally gave a mathematical proof of the full prediction of [1], using the last approach. This long-awaited confirmation provides further support for string theory.

The genus 0 GW-invariants of $Q3$, which is a subspace of a Euclidean (projective) space, are the same as certain twisted genus 0 GW-invariants of the ambient space. In turn, the latter can be described combinatorially; the classical Atiyah-Bott localization theorem reduces these invariants to certain sums over graphs. Thus, the problem of verifying the genus 0 mirror prediction for $Q3$ consisted of analyzing certain (complicated) sums. On the other hand, in the genus 1 case, two new issues arose and have now been resolved:

- The algebraically expected relation between the genus 1 GW-invariants of $Q3$ and those of the ambient space does not hold. This drawback of the (standard) genus 1 GW-invariants is addressed in [29, 30, 16] by defining new, reduced, genus 1 GW-invariants and showing that they behave as expected (but from a geometric, rather than algebraic, perspective). As the standard and reduced genus 1 GW-invariants differ by genus 0 GW-invariants, computing either of the genus 1 GW-invariants is in many cases equivalent to computing the other.

- The relevant twisted reduced genus 1 invariants of a Euclidean (projective) space do not readily reduce to sums over graphs due to the presence of singularities in spaces of genus 1 curves. This issue is addressed in [24] by resolving the singularities, i.e. replacing them with smooth patches. The localization theorem is fully applicable in this new setting and turns the twisted reduced genus 1 GW-invariants of a Euclidean space into sums over graphs, similarly to the genus 0 case.

The final step in the proof of the prediction of [1] was to analyze the resulting sums; this was carried out in [32].

All four papers [29, 30, 16, 24], rely on the sharp description in [28] of limiting behavior of families of genus 1 Riemann surfaces. It had been speculated since at least the mid-1990s that there exists a sharp version of the most fundamental result in GW-theory (Gromov's Compactness Theorem for pseudoholomorphic curves [12]) for positive-genera Riemann surfaces (donuts with one or more holes) and relatedly there exist reduced GW-invariants; this is finally confirmed in [28, 29]. The reduced genus 1 GW-invariants defined in [29] are in fact more geometric than the standard ones, as under ideal circumstances they simply count genus 1 Riemann surfaces. Another application of [28] is [24], which provides natural smooth compactifications (capping off of infinite ends) for spaces of genus 1 Riemann surfaces (complex curves in projective spaces). While [28] is a work in symplectic topology, the application just mentioned concerns algebraic geometry. My student is currently working to extend these results to higher genus, especially genus 2.

There has been a long-running interest, inspired by both the classical enumerative geometry of the nineteenth century and the string theory of the past two decades, in constructing the so-called real analogues of GW-invariants that should count complex curves in an (almost) Kahler manifold X preserved by a conjugation. The foundations of the real GW-theory have long lagged well behind developments in the "classical" GW-theory due to fundamental topological obstacles arising in the former. These were overcome in some settings in [26, 27] to define counts of genus 0 real curves in low-dimensional manifolds X, now known as Welschinger's invariants. Building on three years of prior work, P. Georgieva and I finally constructed *arbitrary*-genus analogues of Welschinger's invariants for many symplectic manifolds in [8]. Unlike many other general constructions in this field, ours is accompanied by comparisons and computations in [9, 10, 19] and an appendix obtaining all such invariants for $\mathbb{CP}^3$ for genus $g \le 5$ and degree $d \le 8$; these followup works supply concrete corroborative evidence for our construction in [8]. In addition to providing the long-awaited lower bounds for counts of real algebraic *positive*-genus curves, [8] has made it possible to study the mirror formulas for real GW-invariants mathematically. In particular, it should now be fairly straightforward to complete the verification of the prediction for the real genus 1 GW-invariants of $Q3$ made in [25] by adapting the reasoning in [28, 29, 30, 16] to the real setting of [8].

The analytic and topological methods employed in [28, 29, 30, 16, 24] have found a variety of further applications, with additional work in progress. In particular, they are used in [34] to relate GW-invariants of a manifold to a submanifold in some cases and apply the main theorem to confirm the so-called Fano case of the Gopakumar-Vafa conjecture regarding integer counts of curves. The analogue of this result for the real positive-genus GW-invariants of [8] is obtained in [19]. In [22], a similar approach is used to extend the Gopakumar-Vafa conjecture to the genus 1 invariants of Calabi-Yau 5-folds. Other potentially approachable problems include searching for rigidity properties that carry over from the rigid setting of algebraic geometry to the more flexible setting of symplectic topology. Such properties may be of fundamental importance in GW-theory as well as in birational algebraic geometry.

The methods for handling graphs in localization computations developed in [31, 32] have already been applied in a variety of other settings. In [35], they are used to describe the growth rate of the closed genus 0 GW-invariants in a range of important cases and to obtain vanishing results for them. In [4], these methods are used to obtain the first ever mirror formula for a cousin of GW-invariants, called stable quotients invariants, introduced in [18], and to show that in fact they give a simpler mirror formula. The same principles are used in [23] to confirm the predictions of [25] for counts of annuli and Klein bottles, i.e. genus 1 invariants with two different types of reflection symmetry. This parallels the results of [21] for counts of spheres with a reflection symmetry.

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