Research Description

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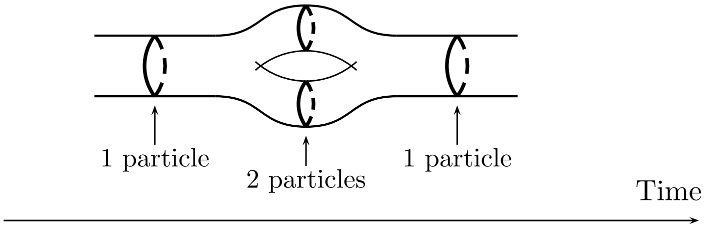
`String theory` is a physical model that represents elementary particles by vibrating `strings` with the aim of unifying the four forces of nature (gravitational, electromagnetic, strong nuclear, and weak nuclear). As a `closed` string (i.e a loop) moves through space, it may split into two (corresponding to fission), which may later recombine into one (corresponding to fusion). The path traced by a closed string in space-time is a (`Riemann`) `surface`, which has one hole (i.e. `is of genus one`) and two ends in the just mentioned example:



While string theory is one of the main paradigms in physics today, strictly speaking it is not a physical theory at all as it has yet to make any experimentally testable *physical* predictions. However, string theory has made (and continues to make) plenty of *mathematical* predictions. Thus, mathematics (especially geometry) has so far been the only "testing ground" for string theory and the source of much corroboration.

Among the most striking mathematical predictions of string theory are so called `mirror symmetry` formulas for Gromov-Witten invariants (or `GW-invariants`). Physically, GW-invariants of a space (symplectic manifold) X represent states of a system; mathematically, they are rational numbers obtained by counting, with certain rational coefficients, Riemann surfaces (complex curves) that lie in X and have no ends. GW-invariants are classified by the maximum `genus` i.e. the maximum number of holes, a Riemann surface may have, and a measure, called `degree`, of how twisted inside X a Riemann surface may be. For example, `genus-zero GW-invariants` are weighted counts of spheres, i.e. Riemann surfaces without holes, in X; similarly, `genus-one GW-invariants` are weighted counts of spheres and `tori`, i.e. donuts or Riemann surfaces with one hole, in X.

Most string theory predictions concern spaces of particular importance in physics, called Calabi-Yau manifolds. While some of these spaces are relatively simple from the physics point of view, they are generally highly non-trivial mathematically. Perhaps the most treatable example is a `quintic 3-fold` $Q3$, i.e. the set of all tuples (x_1, x_2, x_3, x_4) that satisfy a degree-five polynomial equation in four variables (a degree-five hypersurface in $\mathbb{C}P^4$). Even in this case, the 1991 prediction of [CDGP] for genus-*zero* GW-invariants was first verified only in the mid-1990s, in [Gi] and [LLY]. This was of great significance not only in physics, but also in mathematics, as the first two proofs were later followed by at least three more ([Be], [Ga1], [Le]) as well as by expository accounts of the first two (*Séminaire Bourbaki* notes [Pa]; books [CoK] and [MirSym]). The next case in terms of complexity, the 1993 prediction of [BCOV] for genus-*one* GW-invariants of $Q3$ remained essentially unapproachable for about 10 years. Over the past several years, different approaches to at least partial verification of [BCOV] were described in [Ga2] and [MPa]. Separately, a thorough analysis of fundamental properties of genus-one GW-invariants was conducted in [Z2], [Z3], [LZ], and [VaZ], leading to a third approach to [BCOV]. In May 2007, I finally gave a mathematical proof of the full prediction of [BCOV] in [Z4], using the last approach. This long-awaited confirmation provides further support for string theory.

The genus-*zero* GW-invariants of $Q3$, which is a subspace of a Euclidean (projective) space, are the same as certain `twisted` genus-zero GW-invariants of the ambient space. In turn, the latter can be described combinatorially; the classical Atiyah-Bott `localization theorem` reduces these invariants to certain sums over graphs. Thus, the problem of verifying the genus-*zero* mirror prediction for a quintic 3-fold consisted of analyzing certain (complicated) sums. On the other hand, in the genus-*one* case, two new issues arose and have now been resolved:

- The algebraically expected relation between the genus-one GW-invariants of $Q3$ and those of the ambient space does not hold. This drawback of the (standard) genus-one GW-invariants is addressed in [Z2], [Z3], and [LZ] by defining new, `reduced`, genus-one GW-invariants and showing that they behave as expected (but from a geometric, rather than algebraic, perspective). As the standard and reduced genus-one GW-invariants differ by genus-zero GW-invariants, computing either of the genus-one GW-invariants is in many cases equivalent to computing the other.
- The relevant twisted reduced genus-one invariants of a Euclidean (projective) space do not readily reduce to sums over graphs due to the presence of singularities in spaces of genus-one curves. This issue is addressed in [VaZ] by resolving the singularities, i.e. replacing them with smooth patches. The localization theorem is fully applicable in this new setting and turns the twisted reduced genus-one GW-invariants of a Euclidean space into sums over graphs, similarly to the genus-zero case.

The final step in the proof of the [BCOV] prediction was to analyze the resulting sums; this was carried out in [Z4].

All four papers, [Z2], [Z3], [LZ], and [VaZ], rely on my sharp description in [Z1] of limiting behavior for families of genus-one Riemann surfaces. It had been speculated since at least the mid-1990s that there exists a sharp version of the most fundamental result in GW-theory (Gromov's Compactness Theorem for pseudoholomorphic curves, [Gr]) for positive-genera Riemann surfaces (donuts with one or more holes) and relatedly there exist reduced GW-invariants. My papers, [Z1] and [Z2], finally confirm this speculation in the genus-one case. The reduced genus-one GW-invariants defined in [Z2] are in fact more geometric than the standard ones, as under ideal circumstances they simply count genus-one Riemann surfaces. Another application of [Z1] is [VaZ], which provides natural smooth compactifications (capping off of infinite ends) for spaces of genus-one Riemann surfaces (complex curves in projective spaces). It is interesting to note that while [Z1] is a work in symplectic topology, the application just mentioned concerns algebraic geometry.

The analytic and topological methods employed in [Z1], [Z2], [Z3], [LZ], and [VaZ] should have a variety of further applications. In particular, they should apply to higher-genera cases (especially the genus-two case) of the problems discussed above, with the aim of testing higher-genera mirror-symmetry predictions of string theory. Other potentially approachable problems include searching for rigidity properties that carry over from the rigid setting of algebraic geometry to the more flexible setting of symplectic topology. Such properties may be of fundamental importance in GW-theory and may provide confirmation of other predictions of string theory, including integrality of certain counts of Riemann surfaces that involve rational coefficients.

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