Enumerative Geometry: from Classical to Modern

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Summary

- Classical enumerative geometry: examples
- Modern tools: Gromov-Witten invariants
  counts of holomorphic maps
- Insights from string theory:
  - quantum cohomology: refinement of usual cohomology
  - mirror symmetry formulas
    duality between symplectic/holomorphic structures
  - integrality predictions for GW-invariants
    geometric explanation yet to be discovered

What is Classical EG about?

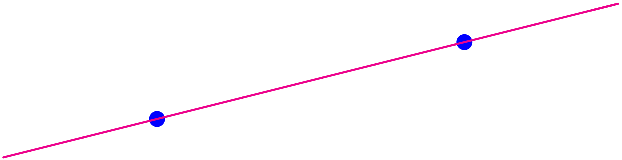
How many geometric objects satisfy given geometric conditions?

objects = curves, surfaces, ...

conditions = passing through given points, curves,...
tangent to given curves, surfaces,...
having given shape: genus, singularities, degree

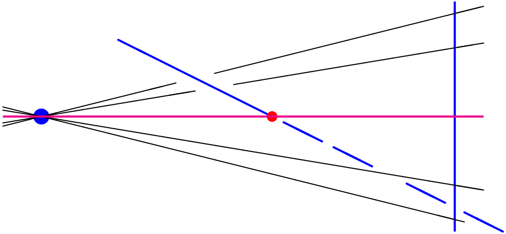
Example 0

Q: How many lines pass through 2 distinct points? 1



Example 1

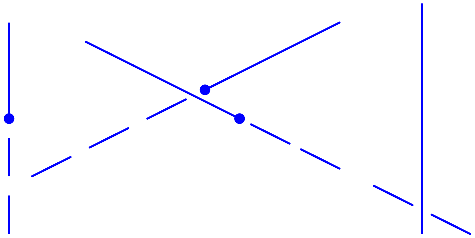
Q: How many lines pass thr 1 point and 2 lines in 3-space? 1



lines thr. the point and 1st line form a plane
2nd line intersects the plane in 1 point

Example 2

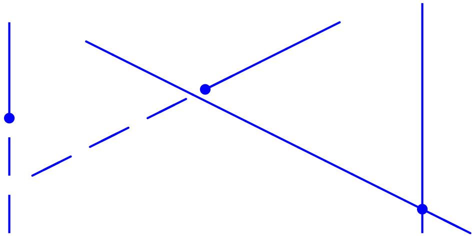
Q: How many lines pass thr 4 general lines in 3-space?



bring two of the lines together so that
they intersect in a point and form a plane

Example 2

Q: How many lines pass thr 4 general lines in 3-space? 2



1 line passes thr the intersection pt and lines #3,4

1 line lies in the plane and passes thr lines #3,4

General Line Counting Problems

All/most line counting problems in vector space V reduce to computing intersections of cycles on

$$\begin{aligned} G(2,V) &\equiv \{\text{2 dim linear subspaces of } V\} \\ &\cong \{\text{(affine) lines in } V\} \end{aligned}$$

This is a special case of Schubert Calculus (very treatable)

Example 0 (a semi-modern view)

Q: How many lines pass through 2 distinct points?

A line in the plane is described by $(A, B, C) \neq 0$:

$$Ax + By + C = 0.$$

(A, B, C) and (A', B', C') describe the same line iff

$$(A', B', C') = \lambda(A, B, C)$$

$$\therefore \quad \{\text{lines in } (x, y)\text{-plane}\} = \{\text{1dim lin subs of } (A, B, C)\text{-space}\}$$
$$\equiv \mathbb{P}^2.$$

Example 0 (a semi-modern view)

Q: How many lines pass through 2 distinct points? 1

$\therefore = \#$ of lines $[A, B, C] \in \mathbb{P}^2$ solving

$$\begin{cases} Ax_1 + By_1 + C = 0 \\ Ax_2 + By_2 + C = 0 \end{cases}$$

$(x_1, y_1), (x_2, y_2)$=fixed points

The system has 1 1dim lin space of solutions in (A, B, C)

Example $0'$ (higher-degree plane curves)

degree d curve in (x, y)-plane

$\equiv$ 0-set of nonzero degree d polynomial in (x, y)

polynomials Q and Q' determine same curve iff $Q' = \lambda Q$

coefficients of Q is $\binom{d+2}{2}$ $\Longrightarrow$

$$\{\text{deg d curves in } (x, y)\text{-plane}\} = \{\text{1dim lin subs of } \binom{d+2}{2}\text{-dim v.s.}\}$$

$$\equiv \mathbb{P}^{N(d)} \qquad N(d) \equiv \binom{d+2}{2} - 1$$

Example 0′ (higher-degree plane curves)

$$\{\text{deg d curves in } (x,y)\text{-plane}\} = \mathbb{P}^{N(d)}$$

"Passing thr a point" = 1 linear eqn on coefficients of Q
$\Longrightarrow$ get hyperplane in $\binom{d+2}{2}$-dim v.s. of coefficients

$$\{\text{deg d curves in } (x,y)\text{-plane thr. } (x_i,y_i)\} \approx \mathbb{P}^{N(d)-1} \subset \mathbb{P}^{N(d)}$$

intersection of $\binom{d+2}{2}-1$ HPs in $\binom{d+2}{2}$-dim v.s. is 1 1dim lin subs
intersection of $N(d)$ HPs in $\mathbb{P}^{N(d)}$ is 1 point

Example 0′ (higher-degree plane curves)

$\exists!$ degree d plane curve thr $N(d) \equiv \binom{d+2}{2} - 1$ general pts

$d = 1$: $\exists!$ line thr 2 distinct pts in the plane
$d = 2$: $\exists!$ conic thr 5 general pts in the plane
$d = 3$: $\exists!$ cubic thr 9 general pts in the plane

Typical Enumerative Problems

Count complex curves = (singular) Riemann surfaces Σ
of fixed genus g, fixed degree d
in $\mathbb{C}^n$, $\mathbb{C}P^n = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \ldots \sqcup \mathbb{C}^0$
in a hypersurface $Y \subset \mathbb{C}^n, \mathbb{C}P^n$ (0-set of a polynomial)

$g(\Sigma) \equiv$ genus of $\Sigma-$ singular points
$d(\Sigma) \equiv$ # intersections of Σ with a generic hyperplane

Adjunction Formula

If $\Sigma \subset \mathbb{C}P^2$ is smooth and of degree d,

$$g(\Sigma) = \binom{d-1}{2}$$

every line, conic is of genus 0
every smooth plane cubic is of genus 1
every smooth plane quartic is of genus 3

Classical Problem

$n_d \equiv$ # genus 0 degree d plane curves thr. (3d-1) general pts

$n_1 = 1$: # lines thr 2 pts
$n_2 = 1$: # conics thr 5 pts
$n_3 = 12$: # nodal cubics thr 8 pts $\Longrightarrow$ $\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{24}$

$n_3 =$ # zeros of transverse bundle section over $\mathbb{C}P^1 \times \mathbb{C}P^2$
= euler class of rank 3 vector bundle over $\mathbb{C}P^1 \times \mathbb{C}P^2$
$\mathbb{C}P^1 =$ cubics thr. 8 general pts; $\mathbb{C}P^2 =$ possibilities for node

Genus 0 Plane Quartics thr 11 pts

$n_4 = \#$ plane quartics thr 11 pts with 3 non-separating nodes
Zeuthen'1870s: $n_4 = 620 = 675 - 55$

$3! \cdot 675$ = euler class of rank 9 vector bundle over $\mathbb{C}P^3 \times (\mathbb{C}P^2)^3$
minus excess contributions of a certain section
$\mathbb{C}P^3 =$ quartics thr 11 pts; $\mathbb{C}P^2$ = possibilities for i-th node

Details in
Counting Rational Plane Curves: Old and New Approaches

Kontsevich's Formula (Ruan-Tian'1993)

$n_d \equiv$ # genus 0 degree d plane curves thr. (3d-1) general pts
$n_1 = 1$

$$n_d = \frac{1}{6(d-1)} \sum_{d_1+d_2=d} \left(d_1 d_2 - 2\frac{(d_1-d_2)^2}{3d-2} \right) \binom{3d-2}{3d_1-1} d_1 d_2 n_{d_1} n_{d_2}$$

$n_2 = 1$, $n_3 = 12$, $n_4 = 620$, $n_5 = 87,304$, $n_6 = 26,312,976$, ...

Gromov's 1985 paper

Consider equivalence classes of maps $f\colon (\Sigma, j) \longrightarrow \mathbb{C}P^n$
(Σ, j)=connected Riemann surface, possibly with nodes

$f : (\Sigma, j) \longrightarrow \mathbb{C}P^n$ and $f' : (\Sigma', j') \longrightarrow \mathbb{C}P^n$ are equivalent if

$$f = f' \circ \tau \text{ for some } \tau\colon (\Sigma, j) \longrightarrow (\Sigma', j')$$

$f\colon (\Sigma, j) \longrightarrow \mathbb{C}P^n$ is stable if

$$\mathrm{Aut}(f) \equiv \big\{\tau\colon (\Sigma, j) \longrightarrow (\Sigma, j) | f \circ \tau = f\big\} \qquad \text{is finite}$$

non-constant holomorphic $f\colon (\Sigma, j) \longrightarrow \mathbb{C}P^n$ is stable iff
the restr. of f to any $S^2 \subset \Sigma$ w. fewer than 3 nodes is not const

Gromov's Compactness Theorem

genus of $f : (\Sigma, j) \longrightarrow \mathbb{C}P^n$ is # of holes in Σ ($\geq g(\Sigma)$)

degree d of $f \equiv |f^{-1}(H)|$ for a generic hyperplane:

$$f_*[\Sigma] = d[\mathbb{C}P^1] \in H_2(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[\mathbb{C}P^1]$$

Theorem: With respect to a natural topology,

$$\overline{\mathfrak{M}}_g(\mathbb{C}P^n, d) \equiv \big\{[f\colon (\Sigma, j) \longrightarrow \mathbb{C}P^n]\colon g(f) = g,\ d(f) = d,\ f \text{ holomor}\big\}$$

is compact

Maps vs. Curves

Image of holomorphic $f\colon (\Sigma, j) \longrightarrow \mathbb{C}P^n$ is a curve
genus of $f(\Sigma) \leq g(f)$; degree of $f(\Sigma) \leq d(f)$

$$\Longrightarrow \quad n_d \equiv \#\text{ genus 0 degree d curves thr. (3d-1) pts in } \mathbb{C}P^2$$
$$= \#\text{ degree d } f\colon (S^2, j) \longrightarrow \mathbb{C}P^2 \text{ s.t. } p_i \in f(\mathbb{C}P^1)$$
$$i = 1, \ldots, 3d-1$$
$$= \#\left\{[f\colon (\Sigma, j) \longrightarrow \mathbb{C}P^2] \in \overline{\mathfrak{M}}_0(\mathbb{C}P^2, d) \colon p_i \in f(\Sigma)\right\}$$

Physics Insight I: Quantum (Co)homology

Use counts of genus 0 maps to $\mathbb{C}P^n$ to deform $\cup$-product on H^*,

$$H^*(\mathbb{C}P^n) = \mathbb{Z}[x]/x^{n+1}, \qquad x^a \cup x^b = x^{a+b},$$

to $*$-product on $H^*(\mathbb{C}P^n)[q_0, \ldots, q_n]$
$x^a * x^b = x^{a+b} +$ q-corrections counting genus 0 maps
thr. $\mathbb{C}P^{n-a}$, $\mathbb{C}P^{n-b}$

Theorem (McDuff-Salamon'93, Ruan-Tian'93, ...)

The product $*$ is associative

$*$ generalizes to all cmpt algebraic/symplectic manifolds

Physics Insight I: Quantum (Co)homology

Associativity of quantum multiplication is equivalent to

- Kontsevich's formula for $\mathbb{C}P^2$, extension to $\mathbb{C}P^n$
- gluing formula for counts of genus 0 maps

Remark: Classical proof of Kontsevich's formula for $\mathbb{C}P^2$ only:
Z. Ran'95, elaborating on '89

Other Enumerative Applications of Stable Maps

- Genus 0 with singularities: Pandharipande, Vakil, Z.-
- Genus 1: R. Pandharipande, Ionel, Z.-
- Genus 2,3: Z.-

Gromov-Witten Invariants

$Y = \mathbb{C}P^n$, = hypersurface in $\mathbb{C}P^n$ (0-set of a polynomial),...
$\mu_1, \ldots, \mu_k \subset Y$ cycles

$$\mathrm{GW}^Y_{g,d}(\mu) \equiv "\#" \{[f\colon (\Sigma, j) \longrightarrow Y] \in \overline{\mathfrak{M}}_g(Y,d)\colon f(\Sigma) \cap \mu_i \neq \emptyset\}$$

$g = 0$, $Y = \mathbb{C}P^n$: $\overline{\mathfrak{M}}_g(Y,d)$ is smooth, of expected dim, "#"=#

Typically, $\overline{\mathfrak{M}}_g(Y,d)$ is highly singular, of wrong dim

Example: Quintic Threefold

$Y_5 \subset \mathbb{C}P^4$ 0-set of a degree 5 polynomial Q
Schubert Calculus: Y_5 contains 2,875 (isolated) lines
S. Katz'86 (via Schubert): Y_5 contains 609,250 conics

For each line $L \subset Y_5$ and conic $C \subset Y_5$,

$$\{[f\colon (\Sigma, j) \longrightarrow Y_5] \in \overline{\mathfrak{M}}_0(Y_5, 2) : f(\Sigma) \subset L\} \approx \overline{\mathfrak{M}}_0(\mathbb{C}P^1, 2)$$
$$\{[f\colon (\Sigma, j) \longrightarrow Y_5] \in \overline{\mathfrak{M}}_0(Y_5, 2) : f(\Sigma) \subset C\} \approx \overline{\mathfrak{M}}_0(\mathbb{C}P^1, 1)$$

are connected components of $\overline{\mathfrak{M}}_0(Y_5, 2)$ of dimensions 2 and 0

Expected dimension of $\overline{\mathfrak{M}}_0(Y_5, d)$

$Y_5 = Q^{-1}(0) \subset \mathbb{C}P^4$ for a degree 5 polynomial Q

$\Longrightarrow \quad \overline{\mathfrak{M}}_0(Y_5, d) = \big\{[f\colon (\Sigma, j) \longrightarrow \mathbb{C}P^4] \in \overline{\mathfrak{M}}_0(\mathbb{C}P^4, d) : Q \circ f = 0\big\}$

holomorphic degree d $f\colon \mathbb{C}P^1 \longrightarrow \mathbb{C}P^4$ has the form

$$f([u, v]) = \big[R_1(u, v), \ldots, R_5(u, v)\big]$$

$R_1, \ldots, R_5 =$ homogeneous polynomials of degree d

$$\Longrightarrow \qquad \dim \overline{\mathfrak{M}}_0(\mathbb{C}P^4, d) = 5 \cdot (d+1) - 1 - 3$$

$Q \circ f$ is homogen of degree $5d$

$\Longrightarrow \quad Q \circ f = 0$ is $5d+1$ conditions on $R_1, \ldots, R_5$

Expected dimension of $\overline{\mathfrak{M}}_0(Y_5, d)$

$$\implies \quad \dim^{vir} \overline{\mathfrak{M}}_0(Y_5, d) = \dim \overline{\mathfrak{M}}_0(\mathbb{C}P^4, d) - (5d+1) = 0$$

A more elaborate computation gives

$$\dim^{vir} \overline{\mathfrak{M}}_g(Y_5, d) = 0 \qquad \forall\, g$$

$\implies$ want to define

$$N_{g,d} \equiv \mathrm{GW}^{Y_5}_{g,d}() \equiv \left|\overline{\mathfrak{M}}_g(Y_5, d)\right|^{vir}$$

GW-Invariants of $Y_5 \subset \mathbb{C}P^4$

$$\overline{\mathfrak{M}}_g(Y_5, d) = \{[f\colon (\Sigma, j) \longrightarrow Y_5] \mid g(f)=g, d(f)=d, \bar{\partial}_j f = 0\}$$

$$\bar{\partial}_j f \equiv df + J_{Y_5} \circ df \circ j$$

$$N_{g,d} \equiv \left|\overline{\mathfrak{M}}_g(Y_5, d)\right|^{vir}$$

$$\equiv \#\{[f\colon (\Sigma, j) \longrightarrow Y_5] \mid g(f)=g, d(f)=d, \bar{\partial}_j f = \nu(f)\}$$

ν = small generic deformation of $\bar{\partial}$-equation

ν multi-valued $\Longrightarrow N_{g,d} \in \mathbb{Q}$

What is special about Y_5?

Y_5 is Calabi-Yau 3-fold:

- $c_1(TY_5) = 0$
- Y_5 is "flat on average": $\mathrm{Ric}_{Y_5}=0$

CY 3-folds are central to string theory

Physics Insight II: Mirror Symmetry

A-Model partition function for Calabi-Yau 3-fold Y

MIRROR principle

B-Model partition function for mirror (family) of Y

generating function for GWs of Y:
$$F_g^A(q) = \sum_{d=1}^{\infty} N_{g,d} q^d$$

?

F_g^B "measures" geometry of moduli spaces of mirror CYs

B-Side Computations for $Y = Y_5$

- Candelas-de la Ossa-Green-Parkes'91
  construct mirror family, compute F_0^B
- Bershadsky-Cecotti-Ooguri-Vafa'93 (BCOV)
  compute F_1^B using physics arguments
- Fang-Z. Lu-Yoshikawa'03 compute F_1^B mathematically
- Huang-Klemm-Quackenbush'06
  compute F_g^B, $g \leq 51$ using physics

Mirror Symmetry Predictions and Verifications

Predictions

$$F_g^A(q) \equiv \sum_{d=1}^{\infty} N_{g,d} q^d \stackrel{?}{=} F_g^B(q).$$

Theorem (Givental'96, Lian-Liu-Yau'97,.........∼'00)

$g = 0$ predict. of Candelas-de la Ossa-Green-Parkes'91 holds

Theorem (Z.'07)

$g = 1$ predict. of Bershadsky-Cecotti-Ooguri-Vafa'93 holds

General Approach to Verifying $F_g^A = F_g^B$ (works for $g = 0, 1$)

Need to compute each $N_{g,d}$ and all of them (for fixed g):

Step 1: relate $N_{g,d}$ to GWs of $\mathbb{C}P^4 \supset Y_5$

Step 2: use $(\mathbb{C}^*)^5$-action on $\mathbb{C}P^4$ to compute each $N_{g,d}$ by localization

Step 3: find some recursive feature(s) to compute $N_{g,d}$ $\forall d$ $\iff F_g^A$

From $Y_5 \subset \mathbb{C}P^4$ to $\mathbb{C}P^4$

$$\begin{array}{ccc} & & \mathcal{L} \equiv \mathcal{O}(5) \\ & & Q \uparrow \downarrow \pi \\ Y_5 \equiv Q^{-1}(0) & \subset & \mathbb{C}P^4 \end{array} \qquad \begin{array}{ccc} & & \mathcal{V}_{g,d} \equiv \overline{\mathfrak{M}}_g(\mathcal{L}, d) \\ & & \tilde{Q} \uparrow \downarrow \tilde{\pi} \\ \overline{\mathfrak{M}}_g(Y_5, d) \equiv \tilde{Q}^{-1}(0) & \subset & \overline{\mathfrak{M}}_g(\mathbb{C}P^4, d) \end{array}$$

$$\tilde{\pi}\big([\xi \colon \Sigma \longrightarrow \mathcal{L}]\big) = [\pi \circ \xi \colon \Sigma \longrightarrow \mathbb{C}P^4]$$

$$\tilde{Q}\big([f \colon \Sigma \longrightarrow \mathbb{C}P^4]\big) = [Q \circ f \colon \Sigma \longrightarrow \mathcal{L}]$$

From $Y_5 \subset \mathbb{C}P^4$ to $\mathbb{C}P^4$

$$\begin{array}{c} \mathcal{L} \equiv \mathcal{O}(5) \\ Q \uparrow \downarrow \pi \\ Y_5 \equiv Q^{-1}(0) \subset \mathbb{C}P^4 \end{array} \qquad \begin{array}{c} \mathcal{V}_{g,d} \equiv \overline{\mathfrak{M}}_g(\mathcal{L}, d) \\ \tilde{Q} \uparrow \downarrow \tilde{\pi} \\ \overline{\mathfrak{M}}_g(Y_5, d) \equiv \tilde{Q}^{-1}(0) \subset \overline{\mathfrak{M}}_g(\mathbb{C}P^4, d) \end{array}$$

This suggests: *Hyperplane Property*

$$N_{g,d} \equiv \left|\overline{\mathfrak{M}}_g(Y_5, d)\right|^{vir} \equiv \left|\tilde{Q}^{-1}(0)\right|^{vir} \stackrel{?}{=} \left\langle e(\mathcal{V}_{g,d}), [\overline{\mathfrak{M}}_g(\mathbb{C}P^4, d)]^{vir} \right\rangle$$

Genus 0 vs. Positive Genus

$g = 0$ everything is as expected:

- $\overline{\mathfrak{M}}_g(\mathbb{C}P^4, d)$ is smooth
- $[\overline{\mathfrak{M}}_g(\mathbb{C}P^4, d)]^{vir} = [\overline{\mathfrak{M}}_g(\mathbb{C}P^4, d)]$
- $\mathcal{V}_{g,d} \longrightarrow \overline{\mathfrak{M}}_g(\mathbb{C}P^4, d)$ is vector bundle
- hyperplane prop. makes sense and holds

$g \geq 1$ none of these holds

Genus 1 Analogue

Thm. A (J. Li–Z.'04): HP holds for reduced genus 1 GWs

$$\left|\overline{\mathfrak{M}}_1^0(Y_5,d)\right|^{vir} = e(\mathcal{V}_{1,d}) \cap \overline{\mathfrak{M}}_1^0(\mathbb{C}P^4,d).$$

This generalizes to complete intersections $Y \subset \mathbb{C}P^n$.

- $\overline{\mathfrak{M}}_1^0(\mathbb{C}P^4,d) \subset \overline{\mathfrak{M}}_1(\mathbb{C}P^4,d)$ main irred. component closure of $\{[f\colon \Sigma \longrightarrow \mathbb{C}P^4] \in \overline{\mathfrak{M}}_1(\mathbb{C}P^4,d)\colon \Sigma \text{ is smooth}\}$
- $\mathcal{V}_{1,d} \longrightarrow \overline{\mathfrak{M}}_1^0(\mathbb{C}P^4,d)$ not vector bundle, but $e(\mathcal{V}_{1,d})$ well-defined (0-set of generic section)

Standard vs. Reduced GWs

$$\text{Thm. A} \implies N_{1,d}^0 \equiv \left|\overline{\mathfrak{M}}_1^0(Y_5,d)\right|^{vir} = \int_{\overline{\mathfrak{M}}_1^0(\mathbb{C}P^4,d)} e(\mathcal{V}_{1,d})$$

$$\overline{\mathfrak{M}}_1^0(Y_5,d) \equiv \overline{\mathfrak{M}}_1^0(\mathbb{C}P^4,d) \cap \overline{\mathfrak{M}}_1(Y_5,d)$$

Thm. B (Z.'04,'07): $N_{1,d} - N_{1,d}^0 = \frac{1}{12} N_{0,d}$

This generalizes to all symplectic manifolds:

[standard] − [reduced genus 1 GW] = F(genus 0 GW)

$\therefore$ to check BCOV, enough to compute $\int_{\overline{\mathfrak{M}}_1^0(\mathbb{C}P^4,d)} e(\mathcal{V}_{1,d})$

Torus Actions

- $(\mathbb{C}^*)^5$ acts on $\mathbb{C}P^4$ (with 5 fixed pts)
- $\Longrightarrow$ on $\overline{\mathfrak{M}}_g(\mathbb{C}P^4, d)$ (with simple fixed loci)
  and on $\mathcal{V}_{g,d} \longrightarrow \overline{\mathfrak{M}}_g(\mathbb{C}P^4, d)$
- $\int_{\overline{\mathfrak{M}}_g^0(\mathbb{C}P^4,d)} e(\mathcal{V}_{g,d})$ localizes to fixed loci

$g = 0$: Atiyah-Bott Localization Thm reduces $\int$ to $\sum_{graphs}$

$g = 1$: $\overline{\mathfrak{M}}_g^0(\mathbb{C}P^4, d), \mathcal{V}_{g,d}$ singular $\Longrightarrow$ AB does not apply

Genus 1 Bypass

Thm. C (Vakil–Z.'05): $\mathcal{V}_{1,d} \longrightarrow \overline{\mathfrak{M}}_1^0(\mathbb{C}P^4, d)$ admit natural desingularizations:

$$\begin{array}{ccc} \widetilde{\mathcal{V}}_{1,d} & \longrightarrow & \mathcal{V}_{1,d} \\ \big\downarrow & & \big\downarrow \\ \widetilde{\mathfrak{M}}_1^0(\mathbb{C}P^4, d) & \longrightarrow & \overline{\mathfrak{M}}_1^0(\mathbb{C}P^4, d) \end{array}$$

$$\Longrightarrow \qquad \int_{\overline{\mathfrak{M}}_1^0(\mathbb{C}P^4,d)} e(\mathcal{V}_{1,d}) = \int_{\widetilde{\mathfrak{M}}_1^0(\mathbb{C}P^4,d)} e(\widetilde{\mathcal{V}}_{1,d})$$

Computation of Genus 1 GWs of CIs

Thm. C generalizes to all $\mathcal{V}_{1,d} \longrightarrow \overline{\mathfrak{M}}^0_{1,k}(\mathbb{C}P^n, d)$:

$$\begin{array}{ccc} \mathcal{L} \equiv \mathcal{O}(a) & \qquad & \mathcal{V}_{1,d} \equiv \overline{\mathfrak{M}}_{1,k}(\mathcal{L}, d) \\ \downarrow \pi & & \downarrow \tilde{\pi} \\ \mathbb{C}P^n & & \overline{\mathfrak{M}}_{1,k}(\mathbb{C}P^n, d) \end{array}$$

$\therefore$ Thms A,B,C provide an algorithm for computing genus 1 GWs of complete intersections $X \subset \mathbb{C}P^n$

Computation of $N_{1,d}$ for all d

- split genus 1 graphs into many genus 0 graphs at special vertex
- make use of good properties of genus 0 numbers to eliminate infinite sums
- extract non-equivariant part of elements in $H^*_{\mathbb{T}}(\mathbb{P}^4)$

Key Geometric Foundation

A Sharp Gromov's Compactness Thm in Genus 1 (Z.'04)

- describes limits of sequences of pseudo-holomorphic maps
- describes limiting behavior for sequences of solutions of a $\bar{\partial}$-equation with limited perturbation
- allows use of topological techniques to study genus 1 GWs

Main Tool

Analysis of Local Obstructions

- study obstructions to smoothing pseudo-holomorphic maps from singular domains
- not just potential existence of obstructions