Enumeration of Genus-Two Curves with a Fixed Complex Structure in $\mathbb{P}^2$ and $\mathbb{P}^3$

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Abstract

The main concrete result of this paper is enumeration of genus-two curves with complex structure fixed in $\mathbb{P}^2$ and $\mathbb{P}^3$. Along the way, rational curves with certain simple singularities are counted as well. While the methods described can be used to count positive-genus curves in some other cases, the most powerful direct applications of the machinery developed are to enumeration of rational curves with a very large class of singularities in projective spaces.

Contents

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1 Introduction

1.1 Background and Results

Let (Σ,j_Σ) be a nonsingular Riemann surface of genus $g\geq 2$, and let d,n be positive integers with $d\geq 1$ and $n\geq 2$. Denote by $\mathcal{H}_{\Sigma,d}(\mathbb{P}^n)$ the set of simple holomorphic maps from Σ to $\mathbb{P}^n$ of degree d. Let $\mu=(\mu_1,\ldots,\mu_N)$ be an N-tuple of proper complex submanifolds of $\mathbb{P}^n$ such that

$$\sum_{l=1}^{l=N}\mathrm{codim}_{\mathbb{C}}\mu_l = d(n+1)-n(g-1)+N. \tag{1.1}$$

If these submanifolds are in general position, the cardinality of the set

$$\mathcal{H}_{\Sigma,d}(\mu)=\{(y_1,\ldots,y_N;u)\colon u\in\mathcal{H}_{\Sigma,d}(\mathbb{P}^n);\ y_l\in\Sigma,\ u(y_l)\in\mu_l\ \forall l=1,\ldots,N\} \tag{1.2}$$

is finite, and its cardinality depends only on the homology classes of $\mu_1,\ldots,\mu_N$. The group $\mathrm{Aut}(\Sigma)$ of holomorphic automorphisms of Σ acts freely on $\mathcal{H}_{\Sigma,d}(\mu)$. For this reason, algebraic geometers prefer to consider the ratio of the cardinality of the set $\mathcal{H}_{\Sigma,d}(\mu)$ and the order of the group $\mathrm{Aut}(\Sigma)$. For a dense open subset of complex structures on Σ, the cardinality of the set $\mathcal{H}_{\Sigma,d}(\mu)$ has the same order. The same is true of the set $\mathrm{Aut}(\Sigma)$. If j_Σ lies in this open subset, we denote the above ratio by $n_{g,d}(\mu)$. This number is precisely the number of irreducible, nodal degree-d genus-g curves in $\mathbb{P}^n$ with a fixed generic complex structure on the normalization and passing through the constraints $\mu_1,\ldots,\mu_N$.

For $g=0,1$, one can define the numbers $n_{g,d}(\mu)$ for constraints of appropriate total codimension by counting the number of equivalence classes under the action of the now infinite group $\mathrm{Aut}(\Sigma)$ on the set $\mathcal{H}_{\Sigma,d}(\mu)$ defined as in (1.2) above. It is shown in [RT] that

$$n_{0,d}(\mu) = \mathrm{RT}_{0,d}(\mu_1, \mu_2, \mu_3; \mu_4, \ldots, \mu_N),$$

where $\mathrm{RT}_{0,d}(\cdot;\cdot)$ denotes the symplectic invariant of $\mathbb{P}^n$ as defined in [RT]. For $g=1$, in [I] the difference

$$\mathrm{RT}_{g,d}(\mu_1; \mu_2, \ldots, \mu_N) - 2n_{g,d}(\mu)$$

is expressed as an intersection number on a blowup of the space of degree-d $(N+1)$-marked rational curves passing through the constraints $\mu_1, \ldots, \mu_N$. This number is shown to be computable, and explicit formulas are given in the $n=2,3$ cases. On the other hand, the symplectic invariant is easily computable from the two composition laws of [RT]. A completely different approach for the $n=2$, $g=1$ case is given in [P1]. Using this algebraic approach, [KQR] express $n_{2,d}$ in the $n=2$ case in terms of the numbers $n_{0,d'}$ with $d' \leq d$.

In this paper, we extend the approach of [I] to compute the difference

$$\mathrm{RT}_{2,d}(\cdot; \mu_1, \ldots, \mu_N) - 2n_{2,d}(\mu)$$

in the $n=2,3$ cases. The reason for the factor of two above is that the automorphism group of a generic genus-two Riemann surface has order two. The following two theorems are the main results of this paper. The two tables list some low-degree genus-two numbers. Evidence in support of the two formulas is described in Subsection 5.8, where more low-degree numbers for $\mathbb{P}^3$ are also given.

Theorem 1.1 *Let $n_{2,d}$ denote the number of genus-two degree-d curves that pass through $3d-2$ points in general position in $\mathbb{P}^2$ and have a fixed generic complex structure. With $n_d = n_{0,d}$,*

$$n_{2,d} = 3(d^2-1)n_d + \frac{1}{2}\sum_{d_1+d_2=d}\Big(d_1^2 d_2^2 + 28 - 16\frac{9d_1d_2-1}{3d-2}\Big)\binom{3d-2}{3d_1-1} d_1 d_2 n_{d_1} n_{d_2}.$$

d	1	2	3	4	5	6	7
$n_{2,d}$	0	0	0	14,400	6,350,400	3,931,128,000	3,718,909,209,600

Theorem 1.2 *If d is a positive integer and μ is a tuple of p points and q lines in general position in $\mathbb{P}^3$ with $2p+q=4d-3$,*

$$2n_{2,d}(\mu) = RT_{2,d}(\cdot; \mu) - CR(\mu),$$

where $CR(\mu)$ is the sum of the intersection numbers of explicit tautological classes in the space of stable rational maps into $\mathbb{P}^3$.

degree	4			5		6
(p,q)	(3,7)	(2,9)	(1,11)	(8,1)	(0,17)	(10,1)
$n_{2,d}(\mu)$	14,400	307,200	4,748,160	9,600	7,494,574,433,280	1,301,760

A formula for $CR(\mu)$ is given in Theorem 5.28. Intersection numbers of tautological classes are shown to be computable in [P2]. In fact, we give a method of computing these numbers along the lines of that in [I], which is slightly different from the method of [P2]; see Subsection 5.7.

The numbers we obtain in the $n\!=\!2$ case are different from the numbers given in [KQR]. However, our numbers can be recovered via the approach of [KQR]. In particular,

$$n_{2,d} = 6\big(n_{2,d}^{KQR} + \tau_d\big),$$

where τ_d is the number of degree-d tacnodal rational curves passing through $(3d-2)$ points in general position in $\mathbb{P}^2$. The factor of six is a minor omission on the authors' part. The contribution of $6\tau_d$ arises from a three-component stratum [KQR] rule out by Remark 3.12, which is stated without a proof. Details can be found in [Z2].

This paper combines the topological tools of Section 3 with the explicit analytic structure theorems of [Z1]. Together these give a general framework that will hopefully provide a way of computing positive-genus enumerative invariants from the symplectic ones in any homogeneous Kahler manifold. In fact, the methods of this paper should apply, with very little change, at least up to genus seven in $\mathbb{P}^2$, to the $g\!=\!3$ case in $\mathbb{P}^3$, and to the $g\!=\!2$ case in $\mathbb{P}^4$. Genus-three plane fixed-complex-structure curves have been enumerated; see [Z3].

Along the way, we enumerate cuspidal rational curves in $\mathbb{P}^2$ and two-component rational curves connected at a tacnode in $\mathbb{P}^3$; see Lemmas 5.4 and 5.5. The formula of Lemma 5.4 is not new. However, the methods of this paper can be used to count rational curves with singularities of "local nature." By "local nature," we mean that a description of the singularities can be given that involves at most one point of each component of the normalization of the curve. For example, a tacnode on a one-component curve is not of "local nature," but a tacnode at the node common to two irreducible components of a curve is. So is a cusp of any arbitrary pre-specified form. Unlike many approaches in algebraic geometry, our methods are not limited to $\mathbb{P}^2$ and apply just as well to arbitrary-dimensional projective spaces. In fact, the machinery itself can be used on other homogeneous manifolds to express counts of singular rational curves in terms of intersections of tautological classes on moduli spaces of rational maps. However, there is no general method of computing these intersections for homogeneous manifolds other than the projective spaces.

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1.2 Summary

If $\nu\in\Gamma(\Sigma\times\mathbb{P}^n;\Lambda^{0,1}\pi_\Sigma^*T^*\Sigma\otimes\pi_{\mathbb{P}^n}^*T\mathbb{P}^n)$, let $\mathcal{M}_{\Sigma,\nu,d}$ denote the set of all smooth maps u from Σ to $\mathbb{P}^n$ of degree d such that $\bar\partial u|_z=\nu|_{(z,u(z))}$ for all $z\in\Sigma$. If μ is as above, put

$$\mathcal{M}_{\Sigma,\nu,d}(\mu)=\big\{(y_1,\ldots,y_N;u)\colon u\in\mathcal{M}_{\Sigma,\nu,d};\ y_l\in\Sigma,\ u(y_l)\in\mu_l\ \forall l=1,\ldots,N\big\}.$$

For a generic ν, $\mathcal{M}_{\Sigma,\nu,d}$ is a smooth finite-dimensional oriented manifold, and $\mathcal{M}_{\Sigma,\nu,d}(\mu)$ is a zero-dimensional finite submanifold of $\mathcal{M}_{\Sigma,\nu,d}\times\Sigma^N$, whose cardinality (with sign) depends only the homology classes of $\mu_1, \ldots, \mu_N$; see [RT]. The symplectic invariant $\mathrm{RT}_{g,d}(;\mu)$ is the signed cardinality of the set $\mathcal{M}_{\Sigma,\nu,d}(\mu)$.

If $\|\nu_i\|_{C^0} \longrightarrow 0$ and $(\underline{y}_i; u_i) \in \mathcal{M}_{\Sigma,\nu_i,d}(\mu)$, then a subsequence of $\{(\underline{y}_i; u_i)\}_{i=1}^{\infty}$ must converge in the stable-map topology to one of the following:
(1) an element of $\mathcal{H}_{\Sigma,d}(\mu)$;
(2) $(\Sigma_T, \underline{y}, u)$, where Σ_T is a bubble tree of S^2's attached to Σ with marked points $y_1, \ldots, y_N$, and $u\colon \Sigma_T \longrightarrow \mathbb{P}^n$ is a holomorphic map such that $u(y_l) \in \mu_l$ for $l = 1, \ldots, N$, and
  (2a) $u|_\Sigma$ is simple and the tree contains at least one S^2;
  (2b) $u|_\Sigma$ is multiply-covered;
  (2c) $u|_\Sigma$ is constant and the tree contains at least one S^2.
By Proposition 6.6, the case (2a) does not occur if the constraints are in general position. Furthermore, if $g=2$, (2b) cannot occur either if $n=2,3$ or if $n=4$ and $d\neq 2$. It is well-known that $n_{2,2}(\mu)=0$, and thus the case $n=4$ and $g=d=2$ presents no interest. Our approach will be to take t very small and to count the number of elements of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$ that lie near the maps of type (2c). The rest of the elements of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$ must lie near the space $\mathcal{H}_{\Sigma,d}(\mu)$. By Proposition 3.30 in [Z1] and Corollary 6.5, there is a one-to-one correspondence between the elements of $\mathcal{H}_{\Sigma,d}(\mu)$ and the nearby elements of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$, at least if $d \geq 3$. If $d=1,2$, $\mathcal{H}_{\Sigma,d}(\mu)=\emptyset$; see the proof of Proposition 6.6. Thus, we are able to compute the cardinality of $\mathcal{H}_{\Sigma,d}(\mu)$ by computing the total number of elements of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$ that lie near the maps of type (2c).

In Subsection 1.3, we summarize our notation for spaces of bubble maps and vector bundle over them. For details, the reader is referred to [Z1]. In Section 2, we describe an obstruction-bundle setup and state Theorem 2.7, which relates the elements of $\mathcal{M}_{\Sigma,d,t\nu}(\mu)$ lying near the maps of type (2c) to the zero set of a map between two bundles. We also describe the local structure of certain spaces of stable rational maps. These spaces are very familiar in algebraic geometry, but for our computations in Section 5 we need the analytic estimate of Theorem 2.8.

In Section 3, we introduce a category of mostly smooth (ms) objects and maps and present the topological tools used in Section 4. We view moduli spaces of rational maps as ms-manifolds, rather than as stacks. This approach allows to study the behavior of certain bundle sections over these topological spaces using the analytic estimate of Theorem 2.8.

In Section 4, we use the topological tools of Subsection 3.1 to show that the number of zeros of the maps of Theorem 2.7 is the same as the number of zeros of explicit affine maps between vector bundles over cartesian products of spaces of rational maps with Σ^k. The results of this simplification are summarized in Subsection 4.9. In Section 5, we relate the zeros of these affine maps to the intersection numbers of spaces of stable rational maps into $\mathbb{P}^n$. We use Theorem 2.8 and Section 3 for *local* excess-intersection type of computations. We conclude with the very explicit formula of Theorem 1.1 in the $n=2$ case and a somewhat less explicit one of Theorem 5.28 in the $n=3$ case.

1.3 Notation

In this subsection, we give a brief description of the most important notation used in this paper. See Section 2 in [Z1] for more details.

Let $q_N, q_S : \mathbb{C} \longrightarrow S^2 \subset \mathbb{R}^3$ be the stereographic projections mapping the origin in $\mathbb{C}$ to the north and south poles, respectively. Explicitly,

$$q_N(z) = \Big(\frac{2z}{1+|z|^2}, \frac{1-|z|^2}{1+|z|^2}\Big) \in \mathbb{C}\times\mathbb{R}, \quad q_S(z) = \Big(\frac{2z}{1+|z|^2}, \frac{-1+|z|^2}{1+|z|^2}\Big). \tag{1.3}$$

We denote the south pole of S^2, i.e. the point $(0,0,-1)\in\mathbb{R}^3$, by ∞. Let

$$e_\infty = (0,0,1) = dq_S\Big|_0\Big(\frac{\partial}{\partial s}\Big) \in T_\infty S^2, \tag{1.4}$$

where we write $z = s+it \in \mathbb{C}$. We identify $\mathbb{C}$ with $S^2-\{\infty\}$ via the map q_N. If N is any nonnegative integer, let $[N]=\{1,\ldots,N\}$.

Definition 1.3 *(1) A finite partially ordered set I is a* <u>linearly ordered set</u> *if for all $i_1, i_2, h\in I$ such that $i_1, i_2 < h$, either $i_1 \le i_2$ or $i_2 \le i_1$.*
(2) A linearly ordered set I is a <u>rooted tree</u> *if I has a unique minimal element, i.e. there exists $\hat{0}\in I$ such that $\hat{0}\le i$ for all $i\in I$.*

If I is a linearly ordered set, let $\hat{I}$ be the subset of the non-minimal elements of I. For every $h \in \hat{I}$, denote by $\iota_h \in I$ the largest element of I which is smaller than h. We call $\iota : \hat{I} \longrightarrow I$ the attaching map of I. Suppose $I = \bigsqcup_{k\in K} I_k$ is the splitting of I into rooted trees such that k is the minimal element of I_k. If $\hat{1}\notin I$, we define the linearly ordered set $I\sqcup_k \hat{1}$ to be the set $I\sqcup\{\hat{1}\}$ with all partial-order relations of I along with the relations

$$k < \hat{1}, \qquad \hat{1} < h \;\text{ if }\; h\in\hat{I}_k.$$

If I is a rooted tree, we write $I\sqcup\hat{1}$ for $I\sqcup_k\hat{1}$.

If $S=\Sigma$ or $S=S^2$ and M is a finite set, a $\mathbb{P}^n$*-valued bubble map with M-marked points* is a tuple

$$b = \big(S, M, I; x, (j,y), u\big),$$

where I is a linearly ordered set, and

$$x\colon \hat{I} \longrightarrow S\cup S^2, \quad j\colon M\longrightarrow I, \quad y\colon M\longrightarrow S\cup S^2, \quad\text{and}\quad u\colon I\longrightarrow C^\infty(S;\mathbb{P}^n)\cup C^\infty(S^2;\mathbb{P}^n)$$

are maps such that

$$x_h \in \begin{cases} S^2-\{\infty\}, & \text{if } \iota_h\in\hat{I};\\ S, & \text{if } \iota_h\notin\hat{I},\end{cases} \qquad y_l \in \begin{cases} S^2-\{\infty\}, & \text{if } j_l\in\hat{I};\\ S, & \text{if } j_l\notin\hat{I},\end{cases} \qquad u_i \in \begin{cases} C^\infty(S^2;\mathbb{P}^n), & \text{if } i\in\hat{I};\\ C^\infty(S;\mathbb{P}^n), & \text{if } i\notin\hat{I},\end{cases}$$

and $u_h(\infty)=u_{\iota_h}(x_h)$ for all $h\in\hat{I}$. We associate such a tuple with Riemann surface

$$\Sigma_b = \Big(\bigsqcup_{i\in I}\Sigma_{b,i}\Big)\Big/\sim, \quad\text{where}\qquad \Sigma_{b,i} = \begin{cases}\{i\}\times S^2, & \text{if } i\in\hat{I};\\ \{i\}\times S, & \text{if } i\notin\hat{I},\end{cases} \qquad\text{and}\quad (h,\infty)\sim(\iota_h,x_h)\;\;\forall h\in\hat{I},$$

with marked points $(j_l, y_l) \in \Sigma_{b,j_l}$, and continuous map $u_b: \Sigma_b \longrightarrow \mathbb{P}^n$, given by $u_b|_{\Sigma_{b,i}} = u_i$ for all $i \in I$. We require that all the singular points of Σ_b, i.e. $(\iota_h, x_h) \in \Sigma_{b,\iota_h}$ for $h \in \hat{I}$, and all the marked points be distinct. In addition, if $\Sigma_{b,i} = S^2$ and $u_{i*}[S^2] = 0 \in H_2(\mathbb{P}^n;\mathbb{Z})$, then $\Sigma_{b,i}$ must contain at least two singular and/or marked points of Σ_b other than (i,∞). Two bubble maps b and b' are *equivalent* if there exists a homeomorphism $\phi\colon \Sigma_b \longrightarrow \Sigma_{b'}$ such that $u_b = u_{b'} \circ \phi$, $\phi(j_l, y_l) = (j'_l, y'_l)$ for all $l \in M$, $\phi|_{\Sigma_{b,i}}$ is holomorphic for all $i \in I$, and $\phi|_{\Sigma_{b,i}} = Id$ if $S = \Sigma$ and $i \in I - \hat{I}$.

The general structure of bubble maps is described by tuples $\mathcal{T} = (S, M, I; j, \underline{d})$, with $d_i \in \mathbb{Z}$ describing the degree of the map u_b on $\Sigma_{b,i}$. We call such tuples *bubble types*. Bubble type $\mathcal{T}$ is *simple* if I is a rooted tree; $\mathcal{T}$ is *is basic* if $\hat{I} = \emptyset$; $\mathcal{T}$ is *semiprimitive* if $\iota_h \notin \hat{I}$ for all $h \in \hat{I}$. We call semiprimitive bubble type $\mathcal{T}$ *primitive* if $j_l \in \hat{I}$ for all $j_l \in M$. The above equivalence relation on the set of bubble maps induces an equivalence relation on the set of bubble types. For each $h, i \in I$, let

$$D_i\mathcal{T} = \{h \in \hat{I} : i < h\}, \quad \bar{D}_i\mathcal{T} = D_i\mathcal{T} \cup \{i\}, \quad H_i\mathcal{T} = \{h \in \hat{I} : \iota_h = i\}, \quad M_i\mathcal{T} = \{l \in M : j_l = i\},$$

$$\chi_{\mathcal{T}} h = \begin{cases} 0, & \text{if } d_i = 0 \ \forall i \le h; \\ 1, & \text{if } d_h \neq 0, \text{ but } d_i = 0 \ \forall i < h; \\ 2, & \text{otherwise.} \end{cases}$$

Let $\mathcal{H}_{\mathcal{T}}$ denote the space of all holomorphic bubble maps with structure $\mathcal{T}$.

The automorphism group of every bubble type $\mathcal{T}$ we encounter in Sections 4 and 5 is trivial. Thus, every bubble type discussed below is presumed to be automorphism-free.

If $S = \Sigma$, we denote by $\mathcal{M}_{\mathcal{T}}$ the set of equivalence classes of bubble maps in $\mathcal{H}_{\mathcal{T}}$. Then there exists $\mathcal{M}^{(0)}_{\mathcal{T}} \subset \mathcal{H}_{\mathcal{T}}$ such that $\mathcal{M}_{\mathcal{T}}$ is the quotient of $\mathcal{M}^{(0)}_{\mathcal{T}}$ by an $(S^1)^{\hat{I}}$-action. Corresponding to this action, we obtain $|\hat{I}|$ line orbi-bundles $\{L_h\mathcal{T} \longrightarrow \mathcal{M}_{\mathcal{T}} : h \in \hat{I}\}$. The bundle of gluing parameters in this case is

$$F\mathcal{T} = \bigoplus_{h \in \hat{I}} F_h\mathcal{T}, \quad \text{where} \qquad F_{h,[b]}\mathcal{T} = \begin{cases} L_{h,[b]}\mathcal{T} \otimes L^*_{\iota_h,[b]}\mathcal{T}, & \text{if } \iota_h \in \hat{I}; \\ L_{h,[b]}\mathcal{T} \otimes T_{x_h}\Sigma, & \text{if } \iota_h \notin \hat{I}. \end{cases}$$

Let $F^{\emptyset}\mathcal{T} = \{\upsilon = (\upsilon_h)_{h \in \hat{I}} \in F\mathcal{T} : \upsilon_h \neq 0 \ \forall h \in \hat{I}\}$. Each line orbi-bundle $F_h\mathcal{T} \longrightarrow \mathcal{M}_{\mathcal{T}}$ is the quotient of a line bundle $F^{(0)}_h\mathcal{T} \longrightarrow \mathcal{M}^{(0)}_{\mathcal{T}}$ by a $G_{\mathcal{T}} \equiv (S^1)^{\hat{I}}$-action. We denote by $F^{(\emptyset)}\mathcal{T}$ the preimage of $F^{\emptyset}\mathcal{T}$ in $F^{(0)}\mathcal{T} \equiv \bigoplus_{h \in \hat{I}} F^{(0)}_h\mathcal{T}$. The bundles $F^{\emptyset}\mathcal{T}$, $F^{(\emptyset)}\mathcal{T}$, and $F^{(0)}_h\mathcal{T}$ are defined even if the automorphism group of $\mathcal{T}$ is nontrivial.

For each bubble type $\mathcal{T} = (S^2, M, I; j, \underline{d})$, let

$$\mathcal{U}_{\mathcal{T}} = \big\{[b] : b = \big(S^2, M, I; x, (j,y), u\big) \in \mathcal{H}_{\mathcal{T}},\ u_{i_1}(\infty) = u_{i_2}(\infty) \ \forall i_1, i_2 \in I - \hat{I}\big\}.$$

Similarly to the $S = \Sigma$ case above, $\mathcal{U}_{\mathcal{T}}$ is the quotient of a subset $\mathcal{B}_{\mathcal{T}}$ of $\mathcal{H}_{\mathcal{T}}$ by a $\tilde{G}_{\mathcal{T}} \equiv (S^1)^I$-action. Denote by $\mathcal{U}^{(0)}_{\mathcal{T}}$ the quotient of $\mathcal{B}_{\mathcal{T}}$ by $G_{\mathcal{T}} \equiv (S^1)^{\hat{I}} \subset \tilde{G}_{\mathcal{T}}$. Then $\mathcal{U}_{\mathcal{T}}$ is the quotient of $\mathcal{U}^{(0)}_{\mathcal{T}}$ by the residual $G^*_{\mathcal{T}} \equiv (S^1)^{I - \hat{I}} \subset \tilde{G}_{\mathcal{T}}$ action. Corresponding to these quotients, we obtain line orbi-bundles

$\{L_h\mathcal{T}\longrightarrow\mathcal{U}_{\mathcal{T}}^{(0)}\colon h\in\hat{I}\}$ and $\{L_i\mathcal{T}\longrightarrow\mathcal{U}_{\mathcal{T}}\colon i\in I\}$. Let

$$F\mathcal{T}=\bigoplus_{h\in\hat{I}}F_h\mathcal{T}\longrightarrow\mathcal{U}_{\mathcal{T}}^{(0)},\quad\text{where}\quad F_{h,[b]}\mathcal{T}=\begin{cases}L_{h,[b]}\mathcal{T}\otimes L^*_{\iota_h,[b]}\mathcal{T}, & \text{if } \iota_h\in\hat{I};\\ L_{h,[b]}\mathcal{T}, & \text{if } \iota_h\notin\hat{I};\end{cases}$$
$$\mathcal{F}\mathcal{T}=\bigoplus_{h\in\hat{I}}\mathcal{F}_h\mathcal{T}\longrightarrow\mathcal{U}_{\mathcal{T}},\quad\text{where}\quad \mathcal{F}_{h,[b]}\mathcal{T}=L_{h,[b]}\mathcal{T}\otimes L^*_{\iota_h,[b]}\mathcal{T}.$$

The orbi-bundles $F_h\mathcal{T}$ and $\mathcal{F}_i\mathcal{T}$ are quotients of line bundles over $\mathcal{B}_{\mathcal{T}}$ similarly to the $S=\Sigma$ case.

The stable-map topology on the space of equivalence classes of bubble maps induces a partial ordering on the set of bubble types and their equivalence classes such that the spaces

$$\bar{\mathcal{M}}_{\mathcal{T}}=\bigcup_{\mathcal{T}'\le\mathcal{T}}\mathcal{M}_{\mathcal{T}'},\quad \bar{\mathcal{U}}_{\mathcal{T}}^{(0)}=\bigcup_{\mathcal{T}'\le\mathcal{T}}\mathcal{U}_{\mathcal{T}'}^{(0)},\quad\text{and}\quad \bar{\mathcal{U}}_{\mathcal{T}}=\bigcup_{\mathcal{T}'\le\mathcal{T}}\mathcal{U}_{\mathcal{T}'}$$

are compact and Hausdorff. The $G^*_{\mathcal{T}}$-action on $\mathcal{U}_{\mathcal{T}}^{(0)}$ extends to an action on $\bar{\mathcal{U}}_{\mathcal{T}}^{(0)}$, and thus line orbi-bundles $L_i\mathcal{T}\longrightarrow\mathcal{U}_{\mathcal{T}}$ with $i\in I-\hat{I}$ extend over $\bar{\mathcal{U}}_{\mathcal{T}}$. The evaluation maps

$$\mathrm{ev}_l\colon\mathcal{H}_{\mathcal{T}}\longrightarrow\mathbb{P}^n,\quad \mathrm{ev}_l\big((S,M,I;x,(j,y),u)\big)=u_{j_l}(y_l),$$

descend to all the quotients and induce continuous maps on $\bar{\mathcal{M}}_{\mathcal{T}}$, $\bar{\mathcal{U}}_{\mathcal{T}}$, and $\bar{\mathcal{U}}_{\mathcal{T}}^{(0)}$. If $\mu=\mu_M$ is an M-tuple of submanifolds of $\mathbb{P}^n$, let

$$\mathcal{M}_{\mathcal{T}}(\mu)=\{b\in\mathcal{M}_{\mathcal{T}}\colon\mathrm{ev}_l(b)\in\mu_l\ \forall l\in M\}$$

and define spaces $\mathcal{U}_{\mathcal{T}}(\mu)$, $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$, etc. in a similar way. If $S=S^2$, we define another evaluation map,

$$\mathrm{ev}\colon\mathcal{B}_{\mathcal{T}}\longrightarrow\mathbb{P}^n\quad\text{by}\quad \mathrm{ev}\big((S^2,M,I;x,(j,y),u)\big)=u_{\hat{0}}(\infty),$$

where $\hat{0}$ is any minimal element of I. This map descends to $\mathcal{U}_{\mathcal{T}}^{(0)}$ and $\mathcal{U}_{\mathcal{T}}$. If $\mu=\mu_{\{\hat{0}\}\sqcup M}$ is a tuple of constraints, let

$$\mathcal{U}_{\mathcal{T}}(\mu_{\hat{0}};\mu_M)=\{b\in\mathcal{U}_{\mathcal{T}}(\mu_M)\colon\mathrm{ev}(b)\in\mu_{\hat{0}}\}$$

and define $\mathcal{U}_{\mathcal{T}}^{(0)}(\mu_{\hat{0}};\mu_M)$, etc. similarly. If $S=\Sigma$, $\mathcal{T}$ is a simple bubble type, and $d_{\hat{0}}=0$, define

$$\mathrm{ev}\colon\mathcal{H}_{\mathcal{T}}\longrightarrow\mathbb{P}^n\quad\text{by}\quad \mathrm{ev}\big((\Sigma,M,I;x,(j,y),u)\big)=u_{\hat{0}}(\Sigma).$$

This map is well-defined, since $u_{\hat{0}}$ is a degree-zero holomorphic map and thus is constant.

If $\mathcal{T}$ is any bubble type, let $\langle\mathcal{T}\rangle$ be the basic bubble such that $\mathcal{T}\le\langle\mathcal{T}\rangle$. If $\mathcal{T}$ is a simple bubble type, let $\bar{\mathcal{T}}$ be the bubble type obtained from $\mathcal{T}$ by dropping the minimal element $\hat{0}$ from the indexing set I and the subset $M_{\hat{0}}\mathcal{T}$ from M. Note that if $\mathcal{T}$ is primitive, $\bar{\mathcal{T}}$ is basic.

Finally, if X is any space, $F\longrightarrow X$ a normed vector bundle, and $\delta\colon X\longrightarrow\mathbb{R}$ is any function, let

$$F_\delta=\big\{(b,v)\in F\colon|v|_b<\delta(b)\big\}.$$

Similarly, if Ω is a subset of F, let $\Omega_\delta=F_\delta\cap\Omega$. If $\upsilon=(b,v)\in F$, denote by b_υ the image of υ under the bundle projection map, i.e. b in this case.

2 Analysis

2.1 The Basic Setup

In this section, we focus on bubble types $\mathcal{T}=\big(S,M,I;j,\underline{d}\big)$ such that either $S=S^2$ or $d_{\hat{0}}=0$. In the first case, we describe a small neighborhood of $\mathcal{U}_{\mathcal{T}}(\mu)$ in $\bar{\mathcal{U}}_{\langle\mathcal{T}\rangle}(\mu)$ and the behavior of sections of certain bundles over $\bar{\mathcal{U}}_{\langle\mathcal{T}\rangle}(\mu)$ near $\mathcal{U}_{\mathcal{T}}(\mu)$; see Theorem 2.8. This theorem is deduced from Theorem 3.33 in [Z1]. If $\mathcal{T}$ is a simple bubble type, $S=\Sigma$, and $d_{\hat{0}}=0$, we describe the elements of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$ lying near $\mathcal{M}_{\mathcal{T}}(\mu)$ as the zero set of a map defined on an open subset of the bundle $F\mathcal{T}$; see Theorem 2.7. The map takes values in a bundle over $\mathcal{M}_{\mathcal{T}}(\mu)$, which is the analogue of Taubes's obstruction bundle of [T] in this setting. Theorem 2.7 is a consequence of Theorem 3.29 in [Z1], which requires us to make two major choices. This is done in the next two subsections.

If $\mathcal{T}=\big(S,M,I;j,\underline{d}\big)$ and $S=S^2$, by Corollaries 6.3 and 6.5, $\mathcal{T}$ is a $(\mathbb{P}^n,J)$-regular bubble type in the sense of Definition 3.1 in [Z1]. This regularity property implies that
(R1) $\mathcal{H}_{\mathcal{T}}$ is a smooth manifold;
(R2) for any $b=\big(S,M,I;x,(j,y),u\big)\in\mathcal{H}_{\mathcal{T}}$, a neighborhood of b in $\mathcal{H}_{\mathcal{T}}$, is modeled on

$$\ker\big(D_b\colon\Gamma(b)\longrightarrow\Gamma^{0,1}(b)\big)\oplus\bigoplus_{h\in I}T_{x_h}\Sigma_{b,\iota_h}\oplus\bigoplus_{l\in M}T_{y_l}\Sigma_{b,j_l}.$$

(R3) $D_b\colon\Gamma(b)\longrightarrow\Gamma^{0,1}(b)$ is surjective for all $b\in\mathcal{H}_{\mathcal{T}}$.
Here $\Gamma^{0,1}(b)$ denotes the space of $u_b^*T\mathbb{P}^n$-valued $(0,1)$-forms on the components of Σ_b, while $\Gamma(b)$ is the set of vector fields ξ on the components of Σ_b that agree at the nodes and such that $\xi(i_1,\infty)=\xi(i_2,\infty)$ for all $i_1,i_2\in I-\hat{I}$. The operator D_b is the linearization of the $\bar{\partial}$-operator with respect to a connection in $T\mathbb{P}^n$. Along $\mathcal{H}_\Sigma$, it is independent of the choice of the connection. On the other hand, if $\mathcal{T}$ is a simple bubble type, $S=\Sigma$, and $d_{\hat{0}}=0$, by the same two corollaries, $\mathcal{T}$ is a $(\mathbb{P}^n,J)$-semiregular bubble type in the sense of Definition 3.2 in [Z1]. This means that (R1) and (R2) are satisfied, with $\Gamma(b)$ defined as above but omitting the last condition. Property (R3) is not satisfied, and in fact by the two corollaries,

$$\text{coker } D_b\approx\mathcal{H}_\Sigma^{0,1}\otimes T_{\text{ev}(b)}\mathbb{P}^n\qquad\forall b\in\mathcal{H}_{\mathcal{T}},$$

where $\mathcal{H}_\Sigma^{0,1}$ is the space of harmonic $(0,1)$-forms on Σ. This cokernel bundle descends to a bundle $\Gamma_-^{0,1}\longrightarrow\mathcal{M}_{\mathcal{T}}$, which will be our obstruction bundle.

If $S=\Sigma$, for the gluing construction in [Z1], we choose a smooth family $\{g_{b,\hat{0}}\colon b\in\mathcal{H}_{\mathcal{T}}\}$ of metrics on Σ such that for all

$$b=\big(\Sigma,M,I;x,(j,y),u\big)\in\mathcal{H}_{\mathcal{T}},$$

the metric $g_{b,\hat{0}}$ is flat on a neighborhood of x_h in Σ for all $h\in\hat{I}$ such that $\iota_h=\hat{0}$. This family of metrics, in fact, depends only on the sets $\{x_h:\iota_h=\hat{0}\}$. Along with the standard metric on S^2, the metric $g_{b,\hat{0}}$ induces a Riemannian metric $g_b=(g_{b,i})_{i\in I}$ on $\Sigma_b=\bigcup_{i\in I}\Sigma_{b,i}$. If $S=S^2$, we take $g_{b,i}$ to be the standard metric on $\Sigma_{b,i}=S^2$ for all $i\in I$. With notation as above, if $x_h,z\in\Sigma_{b,\hat{0}}=\Sigma$, let $r_{b,h}(z)=d_{g_{b,\hat{0}}}(x_h,z)$. If $x_h,z\in\Sigma_{b,i}=S^2$ and $z\neq\infty$, let $r_{b,h}(z)=|z-x_h|$.

For each $\upsilon=(b,v_h)_{h\in\hat{I}}\in F^{(0)}\mathcal{T}$ sufficiently small, in [Z1] we then define a complex curve Σ_υ, smooth maps $q_\upsilon\colon\Sigma_\upsilon\longrightarrow\Sigma_b$ and $q_{\upsilon,i}\colon\Sigma_{\upsilon,\hat{0}}\longrightarrow\Sigma_b$ for $i\in I$, and Riemannian metric g_υ on Σ on Σ_υ such that

(G1) the linearly ordered set corresponding to Σ_υ is $I(\upsilon)\equiv I-\{h\in\hat{I}: \upsilon_h\neq 0\}$;
(G2) the map $q_\upsilon|_{\Sigma_{\upsilon,\hat{0}}}$ factors through each of the maps $q_{\upsilon,i}$;
(G3) $q_\upsilon\colon(\Sigma_\upsilon,g_\upsilon)\longrightarrow(\Sigma_b,g_b)$ is an isometry (and thus holomorphic) outside of the annuli

$$\begin{aligned} A^+_{\upsilon,h} &= q^{-1}_{\upsilon,\iota_h}\Big(\big\{z\in\Sigma_{b,\iota_h}: |\upsilon_h|^{\frac{1}{2}}\le r_{b,h}(z)\le 2|\upsilon_h|^{\frac{1}{2}}\big\}\Big);\\ A^-_{\upsilon,h} &= q^{-1}_{\upsilon,\iota_h}\Big(\big\{z\in\Sigma_{b,\iota_h}: \frac{1}{2}|\upsilon_h|^{\frac{1}{2}}\le r_{b,h}(z)\le |\upsilon_h|^{\frac{1}{2}}\big\}\Big). \end{aligned} \tag{2.1}$$

(G4) $q_{\upsilon,\iota_h}\colon(A^\pm_{\upsilon,h},g_\upsilon)\longrightarrow\big(q_{\upsilon,\iota_h}(A^\pm_{\upsilon,h}),g_b\big)$ is an isometry.
The map q_υ collapses disjoint circles on Σ_υ and identifies the resulting surfaces with S^2 in a manner encoded by υ. Alternatively, $(\Sigma_\upsilon,g_\upsilon)$ can be viewed as the surface obtained by smoothing (some of) the nodes of Σ_b. The maps q_υ and $q_{\upsilon,i}$ are constructed explicitly by fixing a smooth function $\beta\colon\mathbb{R}\longrightarrow[0,1]$ such that

$$\beta(t)=\begin{cases}0, & \text{if } t\le 1;\\ 1, & \text{if } t\ge 2,\end{cases} \qquad\text{and}\qquad \beta'(t)>0 \ \text{ if } t\in(1,2). \tag{2.2}$$

If $r>0$, let $\beta_r\in C^\infty(\mathbb{R};\mathbb{R})$ be given by $\beta_r(t)=\beta(r^{-\frac{1}{2}}t)$. Note that

$$\text{supp}(\beta_r)=[r^{\frac{1}{2}},2r^{\frac{1}{2}}],\quad \|\beta_r'\|_{C^0}\le C_\beta r^{-\frac{1}{2}},\quad\text{and}\quad \|\beta_r''\|_{C^0}\le C_\beta r^{-1}. \tag{2.3}$$

These cutoff functions will not appear in the main statements of this paper, but they do show up in the proofs of Lemma 2.1, Theorem 2.8, and Proposition 4.4. Having constructed the maps q_υ, we let $b(\upsilon)=(\Sigma_\upsilon,u_\upsilon)=(\Sigma_\upsilon,u_b\circ q_\upsilon)$. The marked points on Σ_υ are the preimages of the marked points of Σ_b under the map q_υ.

We also need to choose a smooth family $\{g_{\mathbb{P}^n,b}\colon b\in\mathcal{M}_{\mathcal{T}}^{(0)}\}$ of metrics on $\mathbb{P}^n$ invariant under the equivalence relation on $\mathcal{M}_{\mathcal{T}}^{(0)}$ if $S=\Sigma$ and on $\mathcal{B}_{\mathcal{T}}$ if $S=S^2$. While taking $g_{\mathbb{P}^n,b}$ to be the standard metric on $\mathbb{P}^n$ may be the canonical choice, for computational reasons it is more convenient to take $g_{\mathbb{P}^n,b}=g_{\mathbb{P}^n,\text{ev}(b)}$, where $\{g_{\mathbb{P}^n,q}\colon q\in\mathbb{P}^n\}$ is the family of metrics of Lemma 2.1.

Lemma 2.1 *There exist $r_{\mathbb{P}^n}>0$ and a smooth family of Kahler metrics $\{g_{\mathbb{P}^n,q}: q\in\mathbb{P}^n\}$ on $\mathbb{P}^n$ with the following property. If $B_q(q',r)\subset\mathbb{P}^n$ denotes the $g_{\mathbb{P}^n,q}$-geodesic ball about q' of radius r, the triple $(B_q(q,r_{\mathbb{P}^n}),J,g_{\mathbb{P}^n,q})$ is isomorphic to a ball in $\mathbb{C}^n$ for all $q\in\mathbb{P}^n$.*

Proof: On the open set $U_0=\{[X_0:\ldots:X_n]\in\mathbb{P}^n: X_0\neq 0\}$, the Fubini-Study symplectic form is given by

$$\omega_{\mathbb{P}^n}=\frac{i}{2\pi}\partial\bar{\partial}\,\ln(1+f_0),\quad\text{where}\quad f_0([X_0:\ldots:X_n])=\sum_{k\in[n]}|X_k/X_0|^2; \tag{2.4}$$

see [GH, p31]. Let $q=[1:0:\ldots:0]$. Set

$$\omega_{\mathbb{P}^n,q,\epsilon}=\frac{i}{2\pi}\partial\bar{\partial}\big\{f_0+(\beta_{\epsilon^2}\circ f_0)\big(\ln(1+f_0)-f_0\big)\big\}. \tag{2.5}$$

Note that $\omega_{\mathbb{P}^n,q,\epsilon}$ agrees with $\omega_{\mathbb{P}^n}$ outside of the set $\{f_0\le 2\epsilon\}$ and with the standard symplectic form $\omega_{\mathbb{C}^n}$ on $\{f_0\le\epsilon\}$. Here we view $\omega_{\mathbb{C}^n}$ as a form on U_0 via the coordinates $z_{0,k}=X_k/X_0$, $k\in[n]$.

In particular, $\omega_{\mathbb{P}^n,q,\epsilon}$ is globally defined, and the corresponding Riemannian metric on $\{f_0 \le \epsilon\}$ is flat. Furthermore,

$$\begin{aligned}\omega_{\mathbb{P}^n,q,\epsilon} =& \big\{(1-\beta_{\epsilon^2}\circ f_0)\omega_{\mathbb{C}^n} + (\beta_{\epsilon^2}\circ f_0)\omega_{\mathbb{P}^n}\big\} \\ &+ \frac{i}{2\pi}\Big\{\big(\partial(\beta_{\epsilon^2}\circ f_0)\big)\big(\bar\partial \tilde f_0\big) - \big(\bar\partial(\beta_{\epsilon^2}\circ f_0)\big)\big(\partial\tilde f_0\big) + \big(\partial\bar\partial(\beta_{\epsilon^2}\circ f_0)\big)\tilde f_0\Big\},\end{aligned} \tag{2.6}$$

where $\tilde f_0 = \ln(1+f_0) - f_0$. On the set $\{f_0 \le 2\epsilon\}$ with $\epsilon \le \frac{1}{2}$,

$$\big\|\tilde f_0\big\|_{C^0} \le C\epsilon^2 \quad \text{and} \quad \big\|d\tilde f_0\big\|_{C^0} \le C\epsilon^{\frac{3}{2}}, \tag{2.7}$$

where $\big\|d\tilde f_0\big\|_{C^0}$ denotes the C^0-norm with respect to the standard metric on $\mathbb{C}^n$. Furthermore, by (2.3),

$$\big\|d(\beta_{\epsilon^2}\circ f_0)\big\|_{C^0} \le C\epsilon^{-1}\epsilon^{\frac{1}{2}}, \quad \big\|\nabla^2(\beta_{\epsilon^2}\circ f_0)\big\|_{C^0} \le C\big(\epsilon^{-2}\epsilon^{\frac{1}{2}}\epsilon^{\frac{1}{2}} + \epsilon^{-1}\big), \tag{2.8}$$

where again all the norms are computed with respect to the standard metric on $\mathbb{C}^n$. Equations (2.7) and (2.8) imply that the term on the second line of (2.6) tends to 0 as ϵ goes to 0. Thus by (2.6), we can choose $\epsilon > 0$ such that $\omega_{\mathbb{P}^n,q} \equiv \omega_{\mathbb{P}^n,q,\epsilon}$ is a symplectic form on all of $\mathbb{P}^n$. Note that $\omega_{\mathbb{P}^n,q}$ is invariant under the action of the stabilizer of q in SU_{n+1}, which is the subgroup

$$\mathrm{Stab}_p(SU_{n+1}) = \left\{\begin{pmatrix} \overline{\det(h)} & 0 \\ 0 & h\end{pmatrix} : h \in U_n\right\} \subset SU_{n+1}.$$

We can define a smooth family of symplectic Kahler forms on $\mathbb{P}^n$ by

$$\omega_{\mathbb{P}^n,g\cdot q} = g^*\omega_{\mathbb{P}^n,q}, \qquad g \in SU_{n+1}.$$

The above invariance property of $\omega_{\mathbb{P}^n,q}$ insures that $\omega_{\mathbb{P}^n,g\cdot q}$ depends only on $g\cdot q$. We can now take $g_{\mathbb{P}^n,g\cdot q}$ to be the metric corresponding to the symplectic form $\omega_{\mathbb{P}^n,g\cdot q}$ and the standard complex structure J on $\mathbb{P}^n$.

We denote by $\exp_b$ and $\Pi_{b,X}$ for $X \in T\mathbb{P}^n$ the $g_{\mathbb{P}^n,b}$-exponential map and $g_{\mathbb{P}^n,b}$-parallel transport along the $g_{\mathbb{P}^n,b}$-geodesic for X, respectively. If $\upsilon \in F^{(0)}\mathcal{T}$, let

$$g_{\mathbb{P}^n,\upsilon} = g_{\mathbb{P}^n,b_\upsilon}, \quad \exp_\upsilon = \exp_{b_\upsilon}, \quad \Pi_{\upsilon,X} = \Pi_{b_\upsilon,X}.$$

If $\upsilon \in F^{(0)}$ is sufficiently small, we define L^2-norms inner-products on

$$\Gamma(\upsilon) \equiv \Gamma\big(b(\upsilon)\big) \qquad \text{and} \qquad \Gamma^{0,1}(\upsilon) \equiv \Gamma^{0,1}\big(b(\upsilon)\big)$$

via the metrics $g_{\mathbb{P}^n,\upsilon}$ and g_υ in the usual way. Denote by D_υ the linearization of the $\bar\partial$-operator with respect to the metric $g_{\mathbb{P}^n,\upsilon}$ on $\mathbb{P}^n$ and by D_υ^* its formal adjoint with respect to the above (L^2,υ) inner-product. We fix $p>2$ and denote by $\|\cdot\|_{\upsilon,p,1}$ and $\|\cdot\|_{\upsilon,p}$ the modified Sobolev $(L^p_1, g_{\mathbb{P}^n,\upsilon}, g_\upsilon)$ and $(L^p, g_{\mathbb{P}^n,\upsilon}, g_\upsilon)$ norms of [LT] on $\Gamma(\upsilon)$ and $\Gamma^{0,1}(\upsilon)$, respectively. Let $L^p_1(\upsilon)$ and $L^p(\upsilon)$ be the corresponding completions. A description of the modified Sobolev norms in the notation of this paper can be found in [Z1]. They are needed only for certain technical aspects of this paper.

2.2 Obstruction Bundle

In this subsection, in the case $S=\Sigma$, we choose an obstruction bundle over $F^{(\emptyset)}\mathcal{T}_\delta$ in the sense of Definition 3.13 in [Z1] with $\delta\in C^\infty(\mathcal{M}_\mathcal{T};\mathbb{R}^+)$ sufficiently small.

Let $\delta_\mathcal{T}\in C^\infty(\mathcal{M}_\mathcal{T};\mathbb{R}^+)$ be such that

$$4\delta_\mathcal{T}(b)\|du_i\|_{b,C^0} < r_{\mathbb{P}^n} \qquad \forall b=\big(\Sigma,M,I;x,(j,y),u\big)\in\mathcal{M}_\mathcal{T},\ i\in I.$$

We assume that the above function δ is such that $8\delta^{\frac{1}{2}}<\delta_\mathcal{T}$. If $\upsilon\in F^{(\emptyset)}\mathcal{T}_\delta$ and $X\psi\in T_{\mathrm{ev}(b_\upsilon)}\mathbb{P}^n\otimes\mathcal{H}_\Sigma^{0,1}$, define $R_\upsilon X\psi\in\Gamma^{0,1}(u_\upsilon)$ as follows. If $z\in\Sigma_\upsilon=\Sigma$ is such that $q_\upsilon(z)\in\Sigma_{b_\upsilon,h}$ for some $h\in\hat{I}$ with $\chi_\mathcal{T}h=1$ and $\big|q_S^{-1}(q_\upsilon(z))\big|\le 2\delta_\mathcal{T}(b_\upsilon)$, by our assumption on $\delta_\mathcal{T}$, we can define $\bar{u}_\upsilon(z)\in T_{\mathrm{ev}(b_\upsilon)}\mathbb{P}^n$ by

$$\exp_{\upsilon,\mathrm{ev}(b_\upsilon)}\bar{u}_\upsilon(z)=u_\upsilon(z), \quad |\bar{u}_\upsilon(z)|<r_{\mathbb{P}^n}.$$

Given $z\in\Sigma$, let $h_z\in I$ be such that $q_\upsilon(z)\in\Sigma_{b_\upsilon,h_z}$. If $w\in T_z\Sigma$, put

$$R_\upsilon X\psi|_z w=\begin{cases}0, & \text{if } \chi_\mathcal{T}h_z=2;\\ \beta\big(\delta_\mathcal{T}(b_\upsilon)|q_\upsilon z|\big)(\psi|_z w)\Pi_{\upsilon,\bar{u}_\upsilon(z)}X, & \text{if } \chi_\mathcal{T}h_z=1;\\ (\psi|_z w)X, & \text{if } \chi_\mathcal{T}h_z=0.\end{cases}$$

Let $\Gamma_-^{0,1}(\upsilon)$ be the image of $T_{\mathrm{ev}(b_\upsilon)}\mathbb{P}^n\otimes\mathcal{H}_\Sigma^{0,1}$ under the map R_υ. Denote by $\pi_{\upsilon,-}^{0,1}$ the (L^2,υ)-orthogonal projection of $L^p(\upsilon)$ onto $\Gamma_-^{0,1}(\upsilon)$.

The spaces $\Gamma_-^{0,1}(\upsilon)$ form our obstruction bundle over $F^{(\emptyset)}\mathcal{T}$. We need to show that these spaces satisfy the requirements of Definition 3.13 in [Z1]. First, the rate of change of $\pi_{\upsilon,-}^{0,1}$ with respect to changes in υ should be controlled by a function of b_υ only. The proof of this fact is similar to the proof of the second statement of (5) of Lemma 3.6 in [Z1]. The next lemma implies that the remaining conditions are also satisfied. For any $h\in\hat{I}$, put

$$|\upsilon|_h=\prod_{i\in\hat{I},i\le h}|\upsilon_i|.$$

Lemma 2.2 *For any $\upsilon\in F^{(\emptyset)}\mathcal{T}_\delta$ and $X\psi\in T_{ev(b_\upsilon)}\mathbb{P}^n\otimes\mathcal{H}_\Sigma^{0,1}$, $D_\upsilon^*R_\upsilon X\psi$ vanishes outside of the annuli*

$$\tilde{A}_{\upsilon,h}\equiv q_\upsilon^{-1}\big(\{(h,z)\in\Sigma_{b_\upsilon,h}\colon \delta_\mathcal{T}(b_\upsilon)\le|q_S^{-1}(z)|\le 2\delta_\mathcal{T}(b_\upsilon)\}\big)$$

with $h\in\hat{I}$ such that $\chi_\mathcal{T}h=1$. Furthermore, there exists $C\in C^\infty(\mathcal{M}_\mathcal{T};\mathbb{R}^+)$ such that
(1) $\|D_\upsilon^*R_\upsilon X\psi\|_{\upsilon,C^0}\le C(b_\upsilon)\Big(\sum_{\chi_\mathcal{T}h=1}|\upsilon|_h\Big)|X|_\upsilon\|\psi\|_2$;
(2) $\big(1-C(b_\upsilon)^{-1}|\upsilon|^{\frac{2}{\tilde{p}}}\big)\|X\psi\|_{\upsilon,\tilde{p}}\le\|R_\upsilon X\psi\|_{\upsilon,\tilde{p}}\le\big(1+C(b_\upsilon)^{-1}|\upsilon|^{\frac{2}{\tilde{p}}}\big)\|X\psi\|_{\upsilon,\tilde{p}}$, where $\tilde{p}=2,p$.

Proof: The first statement and estimate (2) are immediate from the definition of $R_\upsilon X\psi$ and of the norms; see [Z1]. Let (s,t) be the conformal coordinates on $\tilde{A}_{\upsilon,h}$ given by $q_\upsilon(s,t)=s+it\in\mathbb{C}$. Write $g_\upsilon=\theta^{-2}(s,t)(ds^2+dt^2)$. Then

$$\theta=\frac{1}{2}\left(1+s^2+t^2\right). \tag{2.9}$$

Put

$$\xi(s,t) = \big\{R_v X\psi\big\}_{(s,t)}\partial_s = \beta\big(\delta_{\mathcal{T}}(b_v)\sqrt{s^2+t^2}\big)\big(\psi|_{(s,t)}\partial_s\big)\Pi_{v,\bar{u}_v(s,t)}X. \tag{2.10}$$

Then by [MS, p29],

$$D_v^* R_v X\psi|_z = \theta^2\Big(-\frac{D}{ds}\xi + J\frac{D}{dt}\xi\Big), \tag{2.11}$$

where $\frac{D}{ds}$ and $\frac{D}{dt}$ denote covariant differentiation with respect to the metric $g_{\mathbb{P}^n,v}$ on $\mathbb{P}^n$. Since this metric is flat on the support of ξ and $\psi\in\mathcal{H}_\Sigma^{0,1}$, equations (2.9)-(2.11) give

$$D_v^* R_v X\psi|_z = \frac{\big(1+s^2+t^2\big)^2}{4}\Big\{\beta'|_{\delta_{\mathcal{T}}(b_v)\sqrt{s^2+t^2}}\delta_{\mathcal{T}}(b_v)\frac{-s+it}{\sqrt{s^2+t^2}}\Big\}\big(\psi|_{(s,t)}\partial_s\big)\Pi_{v,\bar{u}_v(s,t)}X. \tag{2.12}$$

Since the right hand-side of (2.12) vanishes unless $\delta_{\mathcal{T}}(b_v)^{-1}\le\sqrt{s^2+t^2}\le 2\delta_{\mathcal{T}}(b_v)^{-1}$, it follows that

$$\big|D_v^* R_v X\psi\big|_{v,z} \le C(b_v)\big|\psi|_{(s,t)}\partial_s\big||X|_v \le C'(b_v)|v|_h\|\psi\|_2|X| \tag{2.13}$$

Claim (1) follows from (2.13).

Let $\tilde{R}_v\colon \mathcal{H}_\Sigma^{0,1}\otimes T_{\mathrm{ev}(b_v)}\mathbb{P}^n \longrightarrow \Gamma_-(v)$ be the adjoint of R_v^{-1}, i.e.

$$\big\langle\!\big\langle \tilde{R}_v X\psi, R_v X'\psi'\big\rangle\!\big\rangle_{v,2} = \big\langle\!\big\langle X\psi, X'\psi'\big\rangle\!\big\rangle_{b_v,2} = \langle X,X'\rangle_{b_v}\langle\psi,\psi'\rangle_2 \tag{2.14}$$

for all $X,X'\in T_{\mathrm{ev}(b_v)}\mathbb{P}^n$ and $\psi,\psi'\in\mathcal{H}_\Sigma^{0,1}$. By Lemma 2.2, $\|\tilde{R}_v - R_v\|_2 \le C(b_v)|v|$.

2.3 Tangent-Bundle Model

We now describe our choice for a tangent-bundle model, which is the subject of Definition 3.11 in [Z1].

For any $v\in F^{(0)}\mathcal{T}$ sufficiently small and $\xi\in\Gamma(b_v)$, define $R_v\xi\in L_1^p(v)$ by $\{R_v\xi\}(z)=\xi\big(q_v(z)\big)$. Let $\Gamma_-(v)$ be the image of $\ker(D_{b_v})$ under the map R_v. Denote by $\Gamma_+(v)$ its (L^2,v)-orthogonal complement in $L_1^p(v)$. Let $\pi_{v,\pm}$ be the (L^2,v)-orthogonal projection onto $\Gamma_\pm(v)$.

If $x\in\Sigma$, let $\mathcal{H}_\Sigma^-(x)=\{\psi\in\mathcal{H}_\Sigma^{0,1}: \psi|_x=0\}$. This is a codimension-one subspace of $\mathcal{H}_\Sigma^{0,1}$ for all $x\in\Sigma$; see [GH]. Denote by $\mathcal{H}_\Sigma^+(x)$ its L^2-orthogonal complement. The space $\mathcal{H}_\Sigma^+(x)$ is independent of the choice of a Kahler metric on (Σ,j_Σ). For any $h\in\hat{I}$, we put $\tilde{x}_h(v)=q_{v,\iota_h}^{-1}(\iota_h,x_h)$. Fix $h^*\in\hat{I}$ such that $\chi_{\mathcal{T}}h^*=1$. Let

$$\bar{\Gamma}_-(v) = D_v^* R_v\big(\mathcal{H}_\Sigma^+(\tilde{x}_{h^*}(v))\otimes T_{\mathrm{ev}(b_v)}\mathbb{P}^n\big).$$

Denote by $\bar{\Gamma}_+(v)$ the (L^2,v)-orthogonal complement of $\bar{\Gamma}_-(v)$ in $L_1^p(v)$ and by $\bar{\pi}_{v,\pm}$ the (L^2,g_v)-orthogonal projections onto $\bar{\Gamma}_\pm(v)$. Let $\tilde{\Gamma}_+(v)$ be the image of $\Gamma_+(v)$ under $\bar{\pi}_{v,+}$ and let $\tilde{\Gamma}_-(v)$ be the (L^2,v)-orthogonal complement of $\tilde{\Gamma}_+(v)$ in $L_1^p(u_v)$.

The spaces $\tilde{\Gamma}_-(v)$ will be our tangent-space model. We need to check that the requirements of Definition 3.11 in [Z1] are satisfied. Let

$$\{h\in\hat{I}: \chi_{\mathcal{T}}(h)=1\} = \{h_1=h^*,h_2,\ldots,h_m\}.$$

If $z\in\Sigma_{b,h_r}$ is such that $|q_S^{-1}(z)|\le 2\delta(b)$, define $\bar{u}_{h_r}(z)\in T_{\mathrm{ev}(b)}\mathbb{P}^n$ by

$$\exp_{b,\mathrm{ev}(b)}\bar{u}_{h_r}(z)=u_{h_r}(z),\quad |\bar{u}_{h_r}(z)|_b<r_{\mathbb{P}^n}.$$

If $X\in T_{\mathrm{ev}(b)}\mathbb{P}^n$, define $R_{b,h_r}X\in\Gamma(u_{h_r})$ by

$$R_{b,h_r}X(z)=\begin{cases}0, & \text{if } |z|\ge 2\delta_{\mathcal{T}}(b)^{-1};\\ \beta'\big|_{\delta_{\mathcal{T}}(b)|z|}\frac{(1+|z|^2)^2z}{|z|}\Pi_{b,\bar{u}_{h_r}(z)}X, & \text{otherwise.}\end{cases}$$

Since $R_{b,h_r}X$ vanishes at all the nodes of Σ_b by assumption on $\delta_{\mathcal{T}}$, we can extend $R_{b,h_r}X$ by zero to an element of $\Gamma(b)$. If $c=c_{[m]}\in\mathbb{C}^{[m]}$ is different from zero, let

$$\bar{\Gamma}_-(b;c)=\Big\{\sum_{r\in[m]}c_rR_{b,h_r}X:X\in T_{\mathrm{ev}(b)}\mathbb{P}^n\Big\}.$$

Denote by $\bar{\Gamma}_+(b;c)$ the (L^2,b)-orthogonal complement of $\bar{\Gamma}_-(b;c)$ in $\Gamma(b)$. Let $\bar{\pi}_{(b;c),\pm}$ be the corresponding (L^2,b)-orthogonal projection maps. Let $\tilde{\Gamma}_+(b;c)=\bar{\pi}_{(b,c),+}\big(\Gamma_+(b)\big)$ and let $\tilde{\Gamma}_-(b;c)$ be its (b,L^2)-orthogonal complement.

Lemma 2.3 *There exist $\delta,C\in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)};\mathbb{R}^+)$ such that for all $\upsilon\in F^{(\emptyset)}\mathcal{T}_\delta$ and $\xi\in\bar{\Gamma}_-(\upsilon)$,*

$$\|\xi\|_{\upsilon,p,1}\le C(b_\upsilon)\|\xi\|_{\upsilon,2}.$$

In addition, $\dim_{\mathbb{C}}\bar{\Gamma}_-(\upsilon)=\dim_{\mathbb{C}}\bar{\Gamma}_-(b_\upsilon;c)=n$ for any nonzero $c\in\mathbb{C}^m$. Furthermore, if $\upsilon_k\longrightarrow b\in\mathcal{M}_{\mathcal{T}}^{(0)}$ and $\xi_k\in\bar{\Gamma}_-(\upsilon)$ is such that $\|\xi_k\|_{\upsilon_k,2}=1$, then there exists a nonzero $c\in\mathbb{C}^m$ and $\xi\in\bar{\Gamma}_-(b;c)$ with $\|\xi\|_{b,2}=1$ such that a subsequence of $\{\xi_k\}$ C^0-converges to ξ.

Remark: The last statement means that a subsequence of $\{\xi_k\}$ C^0-converges to ξ on compact subsets of Σ_b^* and the norms $\|\xi_k\|_{\upsilon_k,p,1}$ are uniformly bounded; see Definition 3.9 in [Z1].

Proof: (1) Let ψ be a generator of $\mathcal{H}^{0,1}_{\Sigma,+}\big(\tilde{x}_{h_1}(\upsilon)\big)$. If $X\in T_{\mathrm{ev}(b_\upsilon)}\mathbb{P}^n$ and $r\in[m]$, define $R_{\upsilon,h_r}X\in\Gamma(u_\upsilon)$ as follows. If $q_\upsilon(z)\in\Sigma_{b_\upsilon,h_r}$, let

$$R_{\upsilon,h_r}X(z)=\Big(\sum_{r\in[m]}\big|\psi_{\tilde{x}_r(\upsilon)}d(q_{\upsilon,h_r}^{-1}\circ q_N)\partial_s\big|\Big)^{-1}\frac{(1+|q_\upsilon z|^2)^2q_\upsilon z}{|q_\upsilon z|}\\ \times\beta'\big|_{\delta_{\mathcal{T}}(b_\upsilon)|q_\upsilon z|}\big(\psi_z d(q_{\upsilon,h_r}^{-1}\circ q_N)\partial_s\big)\Pi_{b_\upsilon,\bar{u}_\upsilon(z)}X.$$

Note that the sum is not zero, since $\psi|_{\tilde{x}_{h_1}(\upsilon)}\ne 0$. If $q_\upsilon(z)\notin\Sigma_{b_\upsilon,h_r}$, we let $R_{\upsilon,h_r}X(z)=0$. Since the modified Sobolev norms are equivalent to the standard ones away from the thin necks of $(\Sigma_\upsilon,g_\upsilon)$,

$$\begin{aligned}\big\|R_{\upsilon,h_r}X\big\|_{\upsilon,p,1}&\le C(b_\upsilon)\Big(\sum_{r\in[m]}\big|\psi_{\tilde{x}_r(\upsilon)}d(q_{\upsilon,h_r}^{-1}\circ q_N)\partial_s\big|\Big)^{-1}\big|\psi_z d(q_{\upsilon,h_r}^{-1}\circ q_N)\partial_s\big||X|_\upsilon\\ &\le C'(b_\upsilon)\|R_{\upsilon,h_r}X\|_{\upsilon,2}.\end{aligned}\tag{2.15}$$

By the proof of Lemma 2.2, if $\xi\in\bar{\Gamma}_-(\upsilon)$,

$$\xi=R_\upsilon X\equiv\sum_{r\in[m]}R_{\upsilon,h_r}X,$$

for some $X\in T_{\mathrm{ev}(b_\upsilon)}\mathbb{P}^n$. Thus, the first two statements of the lemma follow from (2.15).
(2) If $\upsilon_k\longrightarrow b$ and $\xi_k=R_\upsilon X_k\in\bar{\Gamma}_-(\upsilon_k)$ is such that $\|\xi_k\|_{\upsilon_k,2}=1$, then it is immediate from (1) that a subsequence of ξ_k C^0-converges to $\sum\limits_{r\in[m]} c_rR_{b,h_r}X$, where

$$X=\lim_{k\longrightarrow\infty}X_k,\quad c_r=\lim_{k\longrightarrow\infty}\Big(\sum_{r\in[m]}\big|\psi_{\tilde{x}_r(\upsilon)}d(q_{\upsilon,h_r}^{-1}\circ q_N)\partial_s\big|\Big)^{-1}\big(\psi_{\tilde{x}_r(\upsilon)}d(q_{\upsilon,h_r}^{-1}\circ q_N)\partial_s\big). \tag{2.16}$$

The two limits in (2.16) exist after passing to a subsequence of the original sequence. This proves the last statement of the lemma.

Lemma 2.4 *There exist $\delta,C\in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)};\mathbb{R}^+)$ such that for all $\upsilon\in F^{(\emptyset)}\mathcal{T}_\delta$ and $\xi\in\tilde{\Gamma}_-(\upsilon)$,*

$$\|\xi\|_{\upsilon,p,1}\le C(b_\upsilon)\|\xi\|_{\upsilon,2}.$$

Proof: Let $\Gamma_{-+}(\upsilon)$ be the (υ,L^2)-orthogonal complement of $\pi_{\upsilon,-}\big(\bar{\Gamma}_-(\upsilon)\big)$ in $\Gamma_-(\upsilon)$. Then

$$\tilde{\Gamma}_-(\upsilon)=\Gamma_{-+}(\upsilon)\oplus\bar{\Gamma}_-(\upsilon).$$

Since this decomposition is (L^2,υ)-orthogonal, we can assume that either $\xi\in\Gamma_{-+}(\upsilon)$ or $\xi\in\bar{\Gamma}_-(\upsilon)$. In the first case, the statement is obvious, since $\Gamma_{-+}(\upsilon)\subset\Gamma_-(\upsilon)$. The second case is proved in Lemma 2.3.

Corollary 2.5 *Suppose $\upsilon_k\in F^{(0)}\mathcal{T}_\delta$ and $\upsilon_k\longrightarrow b\in\mathcal{M}_{\mathcal{T}}^{(0)}$. If $\{\xi_{\upsilon_k,l}\}$ is an (L^2,υ)-orthonormal basis for $\tilde{\Gamma}_-(\upsilon_k)$, then there exists a nonzero $c\in\mathbb{C}^m$ and an (L^2,b)-orthonormal basis $\{\xi_{b,l}\}$ for $\tilde{\Gamma}_-(b;c)$ such that after passing to a subsequence $\xi_{\upsilon_k,l}$ C^0-converges to $\xi_{b,l}$ for all l.*

Proof: If $\xi_{k,l}\in\bar{\Gamma}_-(\upsilon_k)$, by Lemma 2.3 a subsequence of $\{\xi_{k,l}\}$ C^0-converges to an element of $\xi_l\in\bar{\Gamma}_-(b;c)$ for some nonzero $c\in\mathbb{C}^n$ dependent on the sequence $\{\upsilon_k\}$. Furthermore, orthonormal pairs of such elements C^0-converge to an orthonormal pair in $\bar{\Gamma}_-(b)$. If $\xi_{k,l}\in\bar{\Gamma}_{-+}(\upsilon_k)\subset\Gamma_-(\upsilon_k)$, then by definition of $\Gamma_-(\upsilon_k)$, a subsequence of $\{\xi_{k,l}\}$ C^0-converge to an element $\xi_l\in\Gamma_-(b)$, which must be orthogonal to $\bar{\Gamma}_-(b;c)$; see Lemma 3.10 in [Z1]. Thus, a subsequence of $\big\{\{\xi_{k,l}\}\big\}$ C^0-converges to an orthonormal set of vectors in $\tilde{\Gamma}_-(b)$, which implies that $\dim_{\mathbb{C}}\tilde{\Gamma}_-(b;c)\ge\dim_{\mathbb{C}}\tilde{\Gamma}_-(\upsilon_k)$. However,

$$\begin{aligned}\dim_{\mathbb{C}}\tilde{\Gamma}_-(b;c)&=\dim_{\mathbb{C}}\Gamma_{-+}(b;c)+\dim_{\mathbb{C}}\bar{\Gamma}_-(b;c)\\&=\dim_{\mathbb{C}}\Gamma_-(b)+\big(\dim_{\mathbb{C}}\bar{\Gamma}_-(b;c)-\dim_{\mathbb{C}}\pi_{b,-}\bar{\Gamma}_-(b;c)\big)\,;\\ \dim_{\mathbb{C}}\tilde{\Gamma}_-(\upsilon_k)&=\dim_{\mathbb{C}}\Gamma_{-+}(\upsilon_k)+\dim_{\mathbb{C}}\bar{\Gamma}_-(\upsilon_k)\\&=\dim_{\mathbb{C}}\Gamma_-(\upsilon_k)+\big(\dim_{\mathbb{C}}\bar{\Gamma}_-(\upsilon_k)-\dim_{\mathbb{C}}\pi_{\upsilon_k,-}\bar{\Gamma}_-(\upsilon_k)\big)\,,\end{aligned}$$

where $\Gamma_{-+}(b;c)$ denotes the (L^2,b)-complement of $\pi_{b,-}\bar{\Gamma}_-(b;c)$ in $\Gamma_-(b)$. Since $\Gamma_-(\upsilon_k)$ and $\Gamma_-(b)$ have the same dimension, in order to conclude the proof, it is sufficient to show that

$$\pi_{b,-}:\bar{\Gamma}_-(b;c)\longrightarrow\Gamma_-(b;c)$$

is an isomorphism; see Lemma 2.6.

Lemma 2.6 *There exists $C\in C^\infty(\mathcal{M}_{\mathcal{T}}^{(0)};\mathbb{R}^+)$ such that for all $b\in\mathcal{M}_{\mathcal{T}}^{(0)}$, nonzero $c\in\mathbb{C}^m$, and $\xi\in\bar{\Gamma}_-(b;c)$*

$$\|\xi\|_{b,2}\le C(b_\omega)\|\pi_{b,-}\xi\|_{b,2}.$$

Proof: Suppose $X \in T_{\mathrm{ev}(b)}\mathbb{P}^n$. We define $\tilde{R}_{b,h_r}X \in \Gamma^{0,1}(u_{h_r})$, outside of $\infty \in \Sigma_{b,h_r}$, by

$$\tilde{R}_{b,h_r}X\big|_x = 4\beta\big(\delta_{\mathcal{T}}(b)|q_N^{-1}(x)|\big)\big(q_N^{-1*}d\bar{z}\big)\Pi_{b,\bar{u}_{h_r}(x)}X,$$

where $d\bar{z}$ is the usual $(0,1)$-form on $\mathbb{C}$. By the same computation as in the proof of Lemma 2.2, $R_{b,h_r}X = D^*_{b,u_{h_r}}\tilde{R}_{b,h_r}X$. Thus, if $\xi = \xi_{\hat{I}} \in \ker D_b$ and $2\delta < \delta_{\mathcal{T}}(b)$, by integration by parts,

$$\begin{aligned}
\big\langle\!\big\langle \xi, R_{b,h_r}X \big\rangle\!\big\rangle_b &= \big\langle\!\big\langle \xi_{h_r}, D^*_{b,u_{h_r}}\tilde{R}_{b,h_r}X \big\rangle\!\big\rangle_b \\
&= 2i\delta_{\mathcal{T}}(b)^{-1} \int\limits_{|q_N^{-1}(x)|=\delta^{-1}} \langle \xi_{h_r}(x), \Pi_{b,\bar{u}_{h_r}(x)}X\rangle_b q_N^{-1*}dz,
\end{aligned} \tag{2.17}$$

since $D_{b,u_{h_r}}\xi_{h_r} = 0$. Using the change of variables with $x = q_N(w^{-1})$, we obtain

$$\begin{aligned}
\int\limits_{|q_N^{-1}(x)|=\delta^{-1}} \langle \xi_{h_r}(x), \Pi_{b,\bar{u}_{h_r}(x)}X\rangle_b q_N^{-1*}dz &= -\int\limits_{|w|=\delta} \langle \xi_{h_r}|_{q_N(w^{-1})}, \Pi_{b,\bar{u}_{h_r}(q_N(w^{-1}))}X\rangle_b \frac{dw}{w^2} \\
&= -2\pi i \frac{d}{dw}\langle \xi_{h^*}|_{q_N(w^{-1})}, \Pi_{b,\bar{u}_{h_r}(q_N(w^{-1}))}X\rangle_b\Big|_{w=0} \\
&= -2\pi i \frac{d}{d\bar{z}}\langle \xi_{h_r}|_{q_S(z)}, \Pi_{b,\bar{u}_{h_r}(q_S(z))}X\rangle_b\Big|_{z=0} = -2\pi i\Big\langle \frac{D}{ds}(\xi_{h_r}\circ q_S)\Big|_{z=0}, X\Big\rangle,
\end{aligned} \tag{2.18}$$

since $D_{b,u_{h_r}}\xi_{h_r} = 0$. It follows from (2.17) and (2.18) that for any $\xi = \xi_{[M]} \in \ker(D_b)$,

$$\Big\langle\!\Big\langle \xi, \sum_{r\in[m]} c_r R_{b,h_r}X \Big\rangle\!\Big\rangle_b = 4\pi\delta_{\mathcal{T}}(b)^{-1}\sum_{r\in[m]} c_r\Big\langle \frac{D}{ds}(\xi_{h_r}\circ q_S)\Big|_{z=0}, X\Big\rangle. \tag{2.19}$$

Along with Corollary 6.3, equations (2.19) gives

$$\begin{aligned}
\Big\|\pi_{b,-}\sum_{r\in[m]} c_r R_{b,h_r}X\Big\|_{b,2} &\ge C(b)|c_{r^*}| \sup_{\xi_{[M]}\in\ker(D_b), \|\xi_{[M]}\|=1} \Big\langle \frac{D}{ds}(\xi_{h_{r^*}}\circ q_S)\Big|_{z=0}, X\Big\rangle_b \\
&\ge C'(b)|c_{r^*}||X| \ge C''(b)\Big\|\sum_{r\in[m]} c_r R_{b,h_r}X\Big\|_{b,2},
\end{aligned} \tag{2.20}$$

where $r^* \in [m]$ is such that $|c_{r^*}| = \sup_r |c_r|$. Since the right-hand side of (2.20) must be a continuous function of b, the claim follows.

The statement of Corollary 2.5 is precisely Condition (1) of Definition 3.11 in [Z1]. The other two conditions require that the rate of change of the (L^2, υ)-orthogonal projection onto $\tilde{\Gamma}_-(\upsilon)$ be controlled by a function of b_υ only. This is a consequence of the convergence described in the Corollary 2.5, i.e. we can use the same argument as described in the remark following Lemma 3.6 in [Z1], but with $\Gamma_-(b)$ replaced by the appropriate space $\Gamma_-(b;c)$ (depending on υ).

2.4 Structure Theorem, $S = \Sigma$

If $\mathcal{T} = (\Sigma, [N], I; j, \underline{d})$ is a simple bubble type and μ is an N-tuple of complex submanifolds of $\mathbb{P}^n$ such that the evaluation map,

$$\mathrm{ev}_{[N]} \equiv \mathrm{ev}_1 \times \ldots \times \mathrm{ev}_N : \mathcal{M}_{\mathcal{T}} \longrightarrow (\mathbb{P}^n)^N,$$

is transversal to $\mu_1\times\ldots\times\mu_N$, $\mathcal{M}_{\mathcal{T}}(\mu)$ is a complex submanifold of $\mathcal{M}_{\mathcal{T}}$. Let $\mathcal{N}^\mu\mathcal{T}$ be its normal bundle. If $\mathcal{S}$ is a complex submanifold of $\mathcal{M}$, denote its normal bundle by $\mathcal{N}\mathcal{S}$ and an identification of small neighborhoods of $\mathcal{S}$ in $\mathcal{N}\mathcal{S}$ and in $\mathcal{M}_{\mathcal{T}}$ by $\phi_{\mathcal{S}}$. For any complex vector bundle $V\longrightarrow\mathcal{M}_{\mathcal{T}}$, we denote by $\Phi_{\mathcal{S}}$ an identification of $\phi_{\mathcal{S}}^*V$ and $\pi_{\mathcal{N}\mathcal{S}}^*V$ such that its restriction to the fibers over $\mathcal{S}$ is the identity. We assume that $\Phi_{\mathcal{S}}$ preserves $F^{\emptyset}\mathcal{T}\subset F\mathcal{T}$. Let

$$F^{\emptyset}\mathcal{S}=\big\{(b,\vec{n},\upsilon)\in\mathcal{N}\mathcal{S}\oplus F\mathcal{S}\colon (b,\upsilon)\in F^{\emptyset}\mathcal{T}\big\}.$$

If $\mathrm{ev}_{[N]}|_{\mathcal{S}}$ is transversal to $\mu_1\times\ldots\times\mu_N$, $\mathcal{S}(\mu)\equiv\mathcal{S}\cap\mathcal{M}_{\mathcal{T}}(\mu)$ is a complex submanifold of $\mathcal{S}$ with normal bundle $\mathcal{N}^\mu\mathcal{T}$. Let $\phi_{\mathcal{S}}^\mu$ and $\Phi_{\mathcal{S}}^\mu$ be the analogues of $\phi_{\mathcal{S}}$ and $\Phi_{\mathcal{S}}$ for the bundle $\mathcal{N}^\mu\mathcal{T}\longrightarrow\mathcal{S}(\mu)$. We assume the bundle $\mathcal{N}^\mu\mathcal{T}$ is normed. We call the pair $(\Phi_{\mathcal{S}},\Phi_{\mathcal{S}}^\mu)$ a *regularization* of $\mathcal{S}(\mu)$ if it satisfies a certain minor compatibility condition. For the purposes of this paper, it suffices to say that once $\Phi_{\mathcal{S}}$ is chosen, it is a condition on $\Phi_{\mathcal{S}}^\mu|_{F\mathcal{T}}$; see Subsection 3.8 in [Z1] for details. However, the exact nature of $\Phi_{\mathcal{S}}^\mu|_{F\mathcal{T}}$ is irrelevant for our computational purposes. Finally, we denote by $\bar{C}^\infty_{(d;N)}(\Sigma;\mu)$ the space of all bubble maps $\big(\Sigma,[N],I;x,(j,y),u\big)$ such that $\sum_{i\in I}u_{i*}[\Sigma_{b,i}]=d\lambda$, where $\lambda\in H_2(\mathbb{P}^n;Z)$ is the class of a line, and $u_{j_l}(y_l)\in\mu_l$ for all $l\in[N]$.

Theorem 2.7 *Suppose d is a positive integer, $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ is a simple bubble type with $d_{\hat{0}}=0$ and $\sum\limits_{i\in I}d_i=d$, $\mathcal{S}\subset\mathcal{M}_{\mathcal{T}}$ is a complex submanifold, and*

$$\nu\in\Gamma^{0,1}\big(\Sigma\times\mathbb{P}^n;\Lambda^{0,1}_{J,j}\pi_\Sigma^*T^*\Sigma\otimes\pi_{\mathbb{P}^n}^*T\mathbb{P}^n\big)$$

is a generic section. Let μ be an N-tuple of complex submanifolds of $\mathbb{P}^n$ in general position of total codimension

$$\mathit{codim}_{\mathbb{C}}\mu=d(n+1)-n(g-1)+N,$$

and $(\Phi_{\mathcal{S}},\Phi_{\mathcal{S}}^\mu)$ a regularization of $\mathcal{S}(\mu)$. Then for every precompact open subset K of $\mathcal{S}(\mu)$, there exist a neighborhood U_K of K in $\bar{C}^\infty_{(d;N)}(\Sigma;\mu)$ and $\delta,\epsilon,C>0$ with the following property. For every $t\in(0,\epsilon)$, there exist a section

$$\varphi^\mu_{\mathcal{S},t\nu}\in\Gamma\big(F^{\emptyset}\mathcal{S}_\delta|_K;\pi_{F\mathcal{S}}^*\mathcal{N}^\mu\mathcal{S}\big),\quad \textit{with}\quad \big|\varphi^\mu_{\mathcal{S},t\nu}(\upsilon)\big|_{b_\upsilon}\le C\big(t+|\upsilon|^{\frac{1}{p}}\big),$$

and a sign-preserving bijection between $\mathcal{M}_{\Sigma,t\nu,d}(\mu)\cap U_K$ and the zero set of the section $\psi^\mu_{\mathcal{S},t\nu}$ defined by

$$\begin{aligned}\psi^\mu_{\mathcal{S},t\nu}&\in\Gamma\big(F^{\emptyset}\mathcal{S}_\delta|_K;\pi_{F\mathcal{S}}^*(\mathcal{H}^{0,1}_\Sigma\otimes ev^*T\mathbb{P}^n)\big),\quad \Pi_{b_\upsilon,\phi^\mu_{\mathcal{S}}\varphi^\mu_{\mathcal{S},t\nu}(\upsilon)}\psi^\mu_{\mathcal{S},t\nu}(\upsilon)=\psi_{\mathcal{S},t\nu}\big(\Phi^\mu_{\mathcal{S}}(\varphi^\mu_{\mathcal{S},t\nu}(\upsilon))\big);\\ \psi_{\mathcal{S},t\nu}&\in\Gamma\big(F^{\emptyset}\mathcal{S}_\delta\big|_{\mathcal{S}\cap U_K};\pi_{F\mathcal{S}}^*(\mathcal{H}^{0,1}_\Sigma\otimes ev^*T\mathbb{P}^n)\big),\quad \Pi_{b_\upsilon,\phi_{\mathcal{S}}(\upsilon)}\psi_{\mathcal{S},t\nu}(\upsilon)=\psi_{\mathcal{T},t\nu}\big(\Phi_{\mathcal{S}}(\upsilon)\big);\\ \psi_{\mathcal{T},t\nu}&\in\Gamma\big(F^{\emptyset}\mathcal{T}_\delta\big|_{\mathcal{M}_{\mathcal{T}}\cap U_K};\pi_{F\mathcal{T}}^*(\mathcal{H}^{0,1}_\Sigma\otimes ev^*T\mathbb{P}^n)\big),\quad \tilde{R}_\upsilon\psi_{\mathcal{T},t\nu}(\upsilon)=\pi^{0,1}_{\upsilon,-}\big(t\nu_{\upsilon,t}-\bar{\partial}u_\upsilon-D_\upsilon\xi_{\upsilon,t\nu}\big),\end{aligned}$$

where $\Pi_{b,b'}$ denotes the $g_{\mathbb{P}^n,b}$-parallel transport along the $g_{\mathbb{P}^n,b}$-geodesics from $ev(b)$ to $ev(b')$ whenever $d_{\mathbb{P}^n}\big(ev(b),ev(b')\big)<r_{\mathbb{P}^n}$, $\xi_{\upsilon,t\nu}\in\tilde{\Gamma}_+(\upsilon)$,

$$\big\|\nu_{\upsilon,t}-\nu\big\|_{\upsilon,2}\le C\big(t+|\upsilon|^{\frac{1}{p}}\big),\quad \textit{and}\quad \big\|\xi_{\upsilon,t\nu}\big\|_{\upsilon,p,1}\le C\big(t+|\upsilon|^{\frac{1}{p}}\big).$$

Proof: This theorem follows immediately from Theorem 3.29 in [Z1] applied to the obstruction bundle setup of Subsections 2.2 and 2.3. The only refinement is that we drop the term $\tilde{\eta}_{\upsilon,t\nu}$ from the definition of $\psi_{\mathcal{T},t\nu}$. This is because it vanishes on the support of the $(0,1)$-forms in $\Gamma^{0,1}_-(\upsilon)$, provided δ is sufficiently small. Thus, $\pi^{0,1}_{\upsilon,-}\tilde{\eta}_{\upsilon,t\nu}=0$.

2.5 Structure Theorem, $S=S^2$

In this subsection, we define sections $\mathcal{D}^{(m)}_{\langle\mathcal{T}\rangle,k}$, where $k\in I-\hat{I}$, of the bundle $L_k^*\mathcal{T}^{\otimes m}\otimes\mathrm{ev}^*T\mathbb{P}^n$ over $\bar{\mathcal{U}}_{\langle\mathcal{T}\rangle}(\mu)$, and describe their behavior with respect to the gluing maps near each space $\mathcal{U}_{\mathcal{T}}(\mu)$. In Section 4, the number of elements of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$ lying near each space $\mathcal{M}_{\mathcal{T}}(\mu)$ will be expressed as the number of zeros of affine maps between certain bundles. These affine maps will involve the sections $\mathcal{D}^{(m)}_{\bar{\mathcal{T}},k}$. Their behavior near various boundary strata is the foundation for the local computations of Section 5.

If $b=\big(S^2,M,I;x,(j,y),u\big)\in\mathcal{B}_{\mathcal{T}}$, $m\geq 1$, and $k\in I$, let

$$\mathcal{D}^{(m)}_{\mathcal{T},k}b=\frac{2}{(m-1)!}\frac{D^{m-1}}{ds^{m-1}}\frac{d}{ds}(u_k\circ q_S)\Big|_{(s,t)=0},$$

where the covariant derivatives are taken with respect to the metric $g_{\mathbb{P}^n,b}$ and $s+it\in\mathbb{C}$. If $\mathcal{T}^*$ is a basic bubble type, the maps $\mathcal{D}^{(m)}_{\mathcal{T},k}$ with $\mathcal{T}<\mathcal{T}^*$ and $k\in I-\hat{I}$ induce a continuous section of $\mathrm{ev}^*T\mathbb{P}^n$ over $\bar{\mathcal{U}}^{(0)}_{\mathcal{T}^*}$ and a continuous section of the bundle $L_k^*\mathcal{T}^{*\otimes m}\otimes\mathrm{ev}^*T\mathbb{P}^n$ over $\bar{\mathcal{U}}_{\mathcal{T}^*}$, described by

$$\mathcal{D}^{(m)}_{\mathcal{T}^*,k}[b,c_k]=c_k^m\mathcal{D}^{(m)}_{\mathcal{T},k}b,\quad \text{if}\ \ b\in\mathcal{U}^{(0)}_{\mathcal{T}},\ \ c_k\in\mathbb{C}.$$

We will often write $\mathcal{D}_{\mathcal{T},k}$ instead of $\mathcal{D}^{(1)}_{\mathcal{T},k}$. If $\mathcal{T}$ is simple, we will abbreviate $\mathcal{D}^{(m)}_{\mathcal{T},k}$ as $\mathcal{D}^{(m)}$. If $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ is a simple bubble type and $k\in\hat{I}$, let $\mathcal{D}^{(m)}_{\mathcal{T},k}$ denote the section $\mathcal{D}^{(m)}_{\bar{\mathcal{T}},k}$.

Theorem 2.8 *If $\mathcal{T}^*=(S^2,M,I^*;j,\underline{d}^*)$ is a basic bubble type and μ is an M-tuple of constraints in general position, the spaces $\bar{\mathcal{U}}^{(0)}_{\mathcal{T}^*}(\mu)$ and $\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$ are oriented topological orbifolds. If $\mathcal{T}<\mathcal{T}^*$, there exist $G_{\mathcal{T}^*}$-invariant functions $\delta,C\in C^\infty\big(\mathcal{U}^{(0)}_{\mathcal{T}}(\mu);\mathbb{R}^+\big)$ and $G_{\mathcal{T}^*}$-equivariant continuous map*

$$\tilde{\gamma}^{\mu}_{\mathcal{T}}\colon F\mathcal{T}_\delta\big|_{\mathcal{U}^{(0)}_{\mathcal{T}}(\mu)}\longrightarrow\bar{\mathcal{U}}^{(0)}_{\mathcal{T}^*}(\mu),$$

which is an orientation-preserving homeomorphism onto an open neighborhood of $\mathcal{U}^{(0)}_{\mathcal{T}}(\mu)$ in $\bar{\mathcal{U}}^{(0)}_{\mathcal{T}^*}(\mu)$ and is identity on $\mathcal{U}^{(0)}_{\mathcal{T}}(\mu)$. This map is smooth on $F^\emptyset\mathcal{T}_\delta$. Furthermore, for any

$$\upsilon=\big[(b,v_h)_{h\in\hat{I}}\big]=\big[\big(S^2,M,I;x,(j,y),u\big),(v_h)_{v\in\hat{I}}\big]\in F\mathcal{T}_\delta\big|_{\mathcal{U}^{(0)}_{\mathcal{T}}(\mu)},$$

$$\Big|\Pi^{-1}_{b_\upsilon,ev(\tilde{\gamma}^{\mu}_{\mathcal{T}}(\upsilon))}\big(\mathcal{D}_{\mathcal{T}^*,k}\tilde{\gamma}^{\mu}_{\mathcal{T}}(\upsilon)\big)-2\sum_{h\in I_k,\chi_{\mathcal{T}}h=1}\Big(\prod_{i\in\hat{I},i\leq h}v_i\Big)\big(du_h|_\infty e_\infty\big)\Big|$$
$$\leq C(b_\upsilon)|\upsilon|^{\frac{1}{p}}\sum_{h\in I_k,\chi_{\mathcal{T}}h=1}\Big(\prod_{i\in\hat{I},i\leq h}|v_i|\Big),$$

where $I_k\subset I$ is the rooted tree containing k.

Remark: This theorem states that there exists an identification $\gamma^{\mu}_{\mathcal{T}}\colon\mathcal{F}\mathcal{T}_\delta\longrightarrow\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$ of neighborhoods of $\mathcal{U}_{\mathcal{T}}(\mu)$ in $\mathcal{FT}$ and in $\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$. Furthermore, with appropriate identifications,

$$\Big|\mathcal{D}_{\mathcal{T}^*,k}\gamma^{\mu}_{\mathcal{T}}(\upsilon)-\alpha_{\mathcal{T}}\big(\rho_{\mathcal{T}}(\upsilon)\big)\Big|\le C(b_\upsilon)|\upsilon|^{\frac{1}{p}}\big|\rho_{\mathcal{T}}(\upsilon)\big|, \qquad \text{where} \tag{2.21}$$

$$\rho_{\mathcal{T}}(\upsilon)=\big(b,(\tilde{\upsilon}_h)_{\chi_{\mathcal{T}}h=1}\big)\in\tilde{\mathcal{F}}\mathcal{T}\equiv\bigoplus_{\chi_{\mathcal{T}}h=1}L_h\mathcal{T}\otimes L^*_{\tilde{\iota}_h}\mathcal{T};\quad \tilde{\upsilon}_h=\prod_{i\in\hat{I},i\le h}\upsilon_i;\quad \tilde{\iota}_h=\min\{i\in I: i<h\};$$

$$\alpha_{\mathcal{T}}\big(b,(\tilde{\upsilon}_h)_{\chi_{\mathcal{T}}h=1}\big)=\sum_{h\in I_k,\chi_{\mathcal{T}}h=1}\mathcal{D}_{\mathcal{T},h}\tilde{\upsilon}_h.$$

This estimate is used frequently in Section 5. Note that if $\mathcal{T}$ is a semiprimitive bubble type, the bundle $\mathcal{FT}$ is defined over $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$. However, $\mathcal{FT}$ is *not* the normal bundle of $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ in $\bar{\mathcal{U}}_{\langle\mathcal{T}\rangle}(\mu)$ unless $M_{\hat{0}}\mathcal{T}\sqcup H_{\hat{0}}\mathcal{T}$ is a two-element set; see [P2]. The theorem implies only that the restrictions of the normal bundle of $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ in $\bar{\mathcal{U}}_{\langle\mathcal{T}\rangle}(\mu)$ and of $\mathcal{FT}$ to $\mathcal{U}_{\mathcal{T}}(\mu)$ are isomorphic.

Proof: (1) All statements of this theorem, except for the analytic estimate, follow immediately from Theorem 3.33 in [Z1]. We deduce the analytic estimate from (2) of Theorem 3.33. Let

$$\gamma^{\mu}_{\mathcal{T}}(\upsilon)=\big(S^2,M,I(\upsilon);x(\upsilon),(j(\upsilon),y(\upsilon)),\tilde{u}_\upsilon\big).$$

By Theorem 3.33, there exist a holomorphic bubble map

$$b'=\big[S^2,M,I;x',(j,y'),u'\big]$$

such that $d_{C^k}(b,b')\le C(b_\upsilon)|\upsilon|^{\frac{1}{p}}$ and with appropriate identifications, $\tilde{u}_\upsilon=\exp_{b',u_{b'}\circ q_\upsilon}\xi$ for some $\xi\in\Gamma(u_{b'}\circ q_\upsilon)$ with $\|\xi\|_{b,C^0}\le C(b_\upsilon)|\upsilon|^{\frac{1}{p}}$. Thus, for the purposes of proving the analytic estimate, we can assume that $u_\upsilon=\exp_{b,u_b\circ q_\upsilon}\xi_\upsilon$ for $\xi\in\Gamma(u_b\circ q_\upsilon)$ with $\|\xi_\upsilon\|_{b,C^0}\le C(b_\upsilon)|\upsilon|^{\frac{1}{p}}$, i.e. it is enough to prove the estimate for the map $\tilde{\gamma}_{\mathcal{T}}$ as defined in [Z1] with $\mathcal{T}$ a simple bubble type. If $d_k\neq 0$, the claim is immediate from the usual Sobolev and elliptic estimates near (k,∞). Thus, we assume that $d_{\hat{0}}=0$. For future use, we obtain equations describing the behavior of $\mathcal{D}^{(m)}\tilde{\gamma}_{\mathcal{T}}(\upsilon)$ for all $m\ge 1$.
(2) We identify $B_{g_{\mathbb{P}^n},b}\big(\mathrm{ev}(b),\frac{1}{2}r_{\mathbb{P}^n}\big)$ with an open subset of $\mathbb{C}^n$ via the $g_{\mathbb{P}^n,\mathrm{ev}(b)}$-parallel transport along the geodesics from $\mathrm{ev}(b)$. We assume that $\delta\in C^\infty(\mathcal{U}^{(0)}_{\mathcal{T}};\mathbb{R}^+)$ satisfies

$$C(b)\delta(b)^{\frac{1}{2p}}+\delta(b)^{\frac{1}{2}}\Big(\sum_{i\in M}\|du_i\|_{b,C^0}\Big)<\frac{1}{2}r_{\mathbb{P}^n}.$$

Let $q\colon B_1(0;\mathbb{C})\longrightarrow S^2$ be the local stretching map as in Subsection 2.2 of [Z1] with $\upsilon=1$, defined with respect to the standard metric on $\mathbb{C}$. Let $f_\upsilon=u_\upsilon\circ q$ and $\tilde{f}_\upsilon=\tilde{u}_\upsilon\circ q$. We denote the usual complex coordinate on $\mathbb{C}$ by z. For any $z\in B_1(0;\mathbb{C})$, let $i_\upsilon(z)$ be such that $q_\upsilon(q(z))\in\Sigma_{b,i_\upsilon(z)}$. If $X\in T_{\mathrm{ev}(b)}\mathbb{P}^n$ and $m\ge 1$, define $R_\upsilon X\psi^{(m)}\in\Gamma^{0,1}(\tilde{f}_\upsilon)$ by

$$R_\upsilon X\psi^{(m)}\big|_z=\begin{cases}X\bar{z}^{m-1}d\bar{z}, & \text{if } \chi_{\mathcal{T}}i_\upsilon(z)=0;\\ \beta\big(\delta(b_\upsilon)|q_\upsilon(q(z))|\big)X\bar{z}^{m-1}d\bar{z}, & \text{if } \chi_{\mathcal{T}}i_\upsilon(z)=1;\\ 0, & \text{if } \chi_{\mathcal{T}}i_\upsilon(z)=2.\end{cases}$$

Note that if $\chi_{\mathcal{T}}i_\upsilon(z)=0$, or $\chi_{\mathcal{T}}i_\upsilon(z)=1$ and $\beta\big(\delta(b_\upsilon)|q_\upsilon(q(z))|\big)\neq 0$, $\tilde{f}_\upsilon(z)$ lies in $B_{g_{\mathbb{P}^n},b}\big(\mathrm{ev}(b),\frac{1}{2}r_{\mathbb{P}^n}\big)$. Thus, $R_\upsilon X\psi^{(m)}$ is well-defined. We now compute $\langle\!\langle\bar{\partial}\tilde{f}_\upsilon,R_\upsilon X\psi^{(m)}\rangle\!\rangle$ in two ways and compare the results. First, note that the map $\tilde{f}_\upsilon$ is holomorphic outside of the annulus

$$A_{\hat{0}}(\upsilon)\equiv B_1(0;\mathbb{C})-B_{\frac{1}{2}}(0;\mathbb{C}).$$

Thus, by the same computation as in the proof of Lemma 4.3, we see that

$$\langle\!\langle \bar{\partial}\tilde{f}_\upsilon, R_\upsilon X\psi^{(m)}\rangle\!\rangle = -\frac{\pi}{m}\big\langle \mathcal{D}^{(m)}\tilde{\gamma}_{\mathcal{T}}(\upsilon), X\big\rangle. \tag{2.22}$$

(3) Since $\tilde{f}_\upsilon = \exp_{\mathrm{ev}(b),f_\upsilon}(\xi_\upsilon\circ q)$ and f_υ is constant on $A_{\hat{0}}(\upsilon)$,

$$2i\langle\!\langle \bar{\partial}\tilde{f}_\upsilon, R_\upsilon X\psi^{(m)}\rangle\!\rangle = \int_{A_{\hat{0}}(\upsilon)}\Big\langle \frac{\bar{\partial}}{\partial\bar{z}}(\xi_\upsilon\circ q), X\Big\rangle z^{m-1}d\bar{z}\wedge dz \tag{2.23}$$

Denote by $A^+_{\hat{0}}(\upsilon)$ and $A^-_{\hat{0}}(\upsilon)$ the outer and inner boundary of $A_{\hat{0}}(\upsilon)$, respectively. For every $h\in\hat{I}$ with $\chi_{\mathcal{T}}h=1$, let

$$A_h(\upsilon) = q^{-1}_{\upsilon,\iota_h}\Big(\big\{z\in\Sigma_{b_\upsilon,\iota_h}: 4\delta(b_\upsilon)^{-1}|\upsilon_h|\le|\phi^{-1}_{b,h}z|\le|\upsilon_h|^{\frac{1}{2}}\big\}\Big)\subset\Sigma_{b_\upsilon,\hat{0}}.$$

Denote by $A^\pm_h(\upsilon)$ the outer and inner boundary of $A_h(\upsilon)$. Let w be the complex coordinate on $\mathbb{C}\subset\Sigma_{b_\upsilon,\hat{0}}=S^2$. Note that q is holomorphic inside of $A^-_{\hat{0}}(\upsilon)$ and outside of $q^{-1}(A^-_h(\upsilon))$. Furthermore, since u_b and $\tilde{u}_\upsilon$ are both holomorphic, on the image of this set under q

$$\frac{\bar{\partial}}{\partial\bar{w}}\xi_\upsilon = -\frac{\bar{\partial}}{\partial\bar{w}}u_\upsilon.$$

The last quantity vanishes outside of the annuli $A_h(\upsilon)$. Thus by integration by parts,

$$\begin{aligned}
&\int_{A_{\hat{0}}(\upsilon)}\Big\langle\frac{\bar{\partial}}{\partial\bar{z}}(\xi_\upsilon\circ q), X\Big\rangle z^{m-1}d\bar{z}\wedge dz\\
&= \sum_{\chi_{\mathcal{T}}h=1}\Bigg(\int_{q^{-1}(A_h(\upsilon))}\Big\langle\Big(\frac{\bar{\partial}u_\upsilon}{\partial\bar{w}}\Big)\overline{\Big(\frac{\partial q}{\partial z}\Big)}, X\Big\rangle z^{m-1}d\bar{z}\wedge dz + \int_{q^{-1}(A^-_h(\upsilon))}\big\langle\xi_\upsilon\circ q, X\big\rangle z^{m-1}dz\Bigg)\\
&= \sum_{\chi_{\mathcal{T}}h=1}\Bigg(\int_{A_h(\upsilon)}\Big\langle\frac{\bar{\partial}u_\upsilon}{\partial\bar{w}}, X\Big\rangle g\, d\bar{w}\wedge dw + \int_{A^-_h(\upsilon)}\big\langle\xi_\upsilon, X\big\rangle g\, dw\Bigg),
\end{aligned}\tag{2.24}$$

where $g(w)=w^{m-1}$. Since $\xi_\upsilon\circ q$ is constant on $A^+_{\hat{0}}(\upsilon)$, the second boundary term is zero. Note that the radius of $A^-_h(\upsilon)$ in $\mathbb{C}\subset S^2$ is bounded by $C(b_\upsilon)|\tilde{\upsilon}_h|$. Furthermore, $|g|\le C_m(b_\upsilon)$ on $A^-_h(\upsilon)$. It follows that

$$\Big|\int_{A^-_h(\upsilon)}\langle\xi_\upsilon, X\rangle g dw\Big| \le C_m(b_\upsilon)|\upsilon|^{\frac{1}{p}}|\tilde{\upsilon}_h|. \tag{2.25}$$

On the other hand, by the same computation as in the proof of Lemma 4.3,

$$\int_{A_h(\upsilon)}\Big\langle\frac{\bar{\partial}u_\upsilon}{\partial\bar{w}}, X\Big\rangle g\, d\bar{w}\wedge dw = -2i\sum_{m'=1}^{m'=m}\frac{\pi a_{m',h}(\upsilon)}{m'}\binom{m-1}{m'-1}\tilde{\upsilon}_h^{m'}\big(\mathcal{D}^{(m')}_{\mathcal{T},h}b\big). \tag{2.26}$$

Combining equations (2.22)-(2.26), we see that

$$\Big|\big\langle\mathcal{D}^{(m)}\tilde{\gamma}_{\mathcal{T}}(\upsilon), X\big\rangle - 2m\sum_{\chi_{\mathcal{T}}h=1}\tilde{\upsilon}_h\big(du_h\big|_\infty e_\infty\big)\Big| \le C(b_\upsilon)|\upsilon|^{\frac{1}{p}}\Big(\sum_{\chi_{\mathcal{T}}h=1}|\tilde{\upsilon}_h|\Big). \tag{2.27}$$

3 Topology

3.1 Maps Between Vector Bundles

In Section 4, we express the number of zeros of the maps $\psi^{\mu}_{\mathcal{T},t\nu}$ (and $\psi^{\mu}_{\mathcal{S},t\nu}$ for certain submanifolds $\mathcal{S}$ of $\mathcal{M}_{\mathcal{T}}$) of Theorem 2.7 in terms of the number of zeros of affine maps between the same vector bundles. The topological justification for this reduction is discussed in this subsection. Subsections 3.2 and 3.3 are used in the explicit computations of Section 5. For simplicity, we state all the results for smooth vector bundles over smooth manifolds, but similar statements apply in the orbifold category. However, in the cases of $g=2$, $n=2,3,4$, the spaces involved are actually manifolds.

Let $\mathcal{I}$ denote the unit interval $[0,1]$. If $\mathcal{Z}$ is a compact oriented zero-dimensional manifold, we denote the signed cardinality of $\mathcal{Z}$ by ${}^{\pm}|\mathcal{Z}|$. All vector bundles we encounter in this subsection will be assumed to be smooth, complex, and normed.

Definition 3.1 *Suppose $\mathcal{M}$ is a smooth manifold and $F,\mathcal{O}\longrightarrow\mathcal{M}$ are vector bundles.*
(1) If $F=\bigoplus_{i=1}^{i=k}F_i$, bundle map $\alpha\colon F\longrightarrow\mathcal{O}$ is a polynomial of degree $d_{[k]}$ if for each $i\in[k]$ there exists

$$p_i\in\Gamma(\mathcal{M};F_i^{*\otimes d_i}\otimes\mathcal{O})\ \ \textit{for } i\in[k] \qquad \textit{s.t.} \qquad \alpha(\upsilon)=\sum_{i=1}^{i=k}p_i\big(\upsilon_i^{d_i}\big)\quad \forall\upsilon=(\upsilon_i)_{i\in[k]}\in\bigoplus_{i=1}^{i=k}F_i.$$

(2) If $\alpha\colon F\longrightarrow\mathcal{O}$ is a polynomial, the rank of α is the number

$$rk\ \alpha\equiv\max\{rk_b\alpha\colon b\in\mathcal{M}\},\quad \textit{where}\quad rk_b\alpha=\dim_{\mathbb{C}}\big(Im\ \alpha_b\big).$$

Polynomial $\alpha: F\longrightarrow\mathcal{O}$ is of constant rank if $rk_b\alpha = rk\ \alpha$ for all $b\in\mathcal{M}$; α is nondegenerate if $rk_b\alpha=rk\ F$ for all $b\in\mathcal{M}$.
(3) If Ω is an open subset of $\mathcal{I}\times F$, $\mathcal{O}$ is a vector bundle, and

$$\{\phi_t\}=\big\{\phi_t\colon\{\upsilon\in F:(t,\upsilon)\in\Omega\}\longrightarrow\mathcal{O}\big\}$$

is a family of smooth bundle maps, bundle map $\alpha: F\longrightarrow\mathcal{O}$ is a dominant term for $\{\phi_t\}$ if there exists $\varepsilon\in C^0(\mathcal{I}\times F;\mathbb{R})$ such that

$$\big|\phi_t(\upsilon)-\alpha(\upsilon)\big|\le\varepsilon(t,\upsilon)\big(t+|\alpha(\upsilon)|\big)\quad\forall(t,\upsilon)\in\Omega\quad \textit{and}\quad \lim_{(t,\upsilon)\longrightarrow0}\varepsilon(t,\upsilon)=0.$$

Dominant term $\alpha\colon F\longrightarrow\mathcal{O}$ of $\{\phi_t\}$ is the resolvent of $\{\phi_t\}$ if α is a polynomial of constant rank.

In (2) above, by $\dim_{\mathbb{C}}(\mathrm{Im}\ \alpha_b)$ we mean the dimension of the image of α_b as an analytic subvariety of the fiber $\mathcal{O}_b$. Note that if $\bar{\Omega}\subset\mathcal{I}\times F$ contains a neighborhood of $\{0\}\times\mathcal{M}$, the resolvent of $\{\phi_t\}$ is unique (if it exists).

Lemma 3.2 *Suppose $\mathcal{M}$ is a smooth manifold,*
(1) $F\equiv F^-\oplus F^+\longrightarrow\mathcal{M}$ and $\mathcal{O}\equiv\mathcal{O}^-\oplus\mathcal{O}^+\longrightarrow\mathcal{M}$ are vector bundles;
(2) Ω is an open subset of $\mathcal{I}\times F$ and $\big\{\phi_t\colon\{\upsilon\in F\colon(t,\upsilon)\in\Omega\}\longrightarrow\mathcal{O}\big\}$ is a family of smooth maps;
(3) $\alpha\colon F\longrightarrow\mathcal{O}$ is a dominant term for $\{\phi_t\}$ s.t. $\alpha(F^+)\subset\mathcal{O}^+$, $\alpha^-\equiv\pi^-\circ\alpha|_{F^-}$ is a constant-rank polynomial, where $\pi^-:\mathcal{O}^-\oplus\mathcal{O}^+\longrightarrow\mathcal{O}^-$ is the projection map, and $(\dim\mathcal{M}+2rk\ \alpha^-)<2rk\ \mathcal{O}^-$;

(4) $\bar{\nu}=(\bar{\nu}^-,\bar{\nu}^+)\in\Gamma(\mathcal{M};\mathcal{O}^-\oplus\mathcal{O}^+)$ is generic with respect to α^-.
Then for every compact subset K of $\mathcal{M}$, there exist $\delta_K>0$ and a neighborhood $U_F(K)$ of K in F such that the map

$$\psi_t\colon \{\upsilon\in F\colon (t,\upsilon)\in\Omega\} \longrightarrow \mathcal{O}, \quad \psi_t(\upsilon) = t\bar{\nu}_\upsilon + \phi_t(\upsilon),$$

has no zeros on $\{\upsilon\in U_F(K)\colon (t,\upsilon)\in\Omega\}$ for all $t\in(0,\delta_K)$.

Proof: (1) Suppose $\tilde{\upsilon}\in\Omega_{\delta_K}|_K$ and $\psi_t(\tilde{\upsilon})=0$. Then by our assumptions on ϕ_t,

$$|\alpha(\tilde{\upsilon})| \le C_K\big(t+\bar{\varepsilon}_K(\delta_K)|\alpha(\tilde{\upsilon})|\big),$$

where $C_K>0$ depends only on K (and $\bar{\nu}$) and $\bar{\varepsilon}_K$ is a continuous function vanishing at zero. Thus, if $\delta_K>0$ is sufficiently small,

$$|\alpha(\tilde{\upsilon})| \le 2C_K t \quad \forall t<\delta_K,\ \tilde{\upsilon}\in F_{\delta_K}|_K \text{ s.t. } \psi_t(\tilde{\upsilon})=0. \tag{3.1}$$

(2) Let $F^- = \bigoplus\limits_{i=1}^{i=k} F_i \longrightarrow \mathcal{M}$ be the bundles and $p_i\in\Gamma(\mathcal{M};F_i^{*\otimes d_i}\otimes\mathcal{O}^-)$ the sections as in (1) of Definition 3.1 corresponding to α^-. Define

$$\varphi_t\in\Gamma\big(\mathcal{M};\mathrm{End}(F^-)\big) \quad\text{by}\quad \varphi_t(\upsilon_i)=t^{-1/d_i}\upsilon_i \ \text{ if } \upsilon_i\in F_i.$$

Then by our assumption on ϕ_t and equation (3.1),

$$\big|\bar{\nu}^- + \alpha^-(\varphi_t(\tilde{\upsilon}^-))\big| \le \tilde{C}_K\bar{\varepsilon}_K(\delta_K) \quad \forall t<\delta_K,\ \tilde{\upsilon}\in F_{\delta_K}|_K \text{ s.t. } \psi_t(\tilde{\upsilon})=0, \tag{3.2}$$

where $\tilde{C}_K$ is determined by K. Since α^- has constant rank, the image of α^- is closed and is the total space of a bundle of affine analytic varieties of complex dimension $\mathrm{rk}\,\alpha^- < \mathrm{rk}\,\mathcal{O}^- - \frac{1}{2}\dim\mathcal{M}$. Thus, by assumption (4) of the lemma, $\bar{\nu}^-$ does not intersect the image of α^-, and there exists $\epsilon_K>0$ such that

$$\big|\bar{\nu}^- + \alpha^-(\upsilon^-)\big| \ge \epsilon_K \quad \forall \upsilon\in F^-|_K. \tag{3.3}$$

If $\epsilon_K > \tilde{C}_K\bar{\varepsilon}_K(\delta_K)$, by (3.2) and (3.3), $\pi^-\circ\psi_t$ (and thus ψ_t) has no zeros on $F_{\delta_K}|_K$.

We will call family $\big\{\phi_t\colon\{\upsilon\in F\colon(t,\upsilon)\in\Omega\}\longrightarrow\mathcal{O}\big\}$ of smooth maps *hollow* if it admits a dominant term α that satisfies hypothesis (3) of Lemma 3.2.

Definition 3.3 *Suppose $\mathcal{M}$ is a smooth manifold and $F\longrightarrow\mathcal{M}$ is a vector bundle.*
(1) Subset Y of F is small if Y contains no fiber of F and there exists a smooth manifold Z of dimension $(\dim F-1)$ and a smooth map $f\colon Z\longrightarrow F$ such that the image of f is closed in F and contains Y.
(2) If $F,\tilde{F}\longrightarrow\mathcal{M}$ are smooth complex vector bundles, $\rho\in\Gamma(\mathcal{M};F^{*\otimes d}\otimes\tilde{F})$ induces a $\tilde{d}$-to-1 cover $F\longrightarrow\tilde{F}$ if the map

$$F_b\longrightarrow\tilde{F}_b, \quad \upsilon\longrightarrow\rho(\upsilon)\equiv\rho\big(\upsilon^d\big),$$

is $\tilde{d}$-to-1 on a dense open subset of every fiber F_b of F.

Lemma 3.4 *Suppose $\mathcal{M}$ is a smooth manifold, $F=\bigoplus_{i=1}^{i=k} F_i$ and $\mathcal{O}$ are vector bundles over $\mathcal{M}$, and*

$$\alpha=\sum_{i=1}^{i=k} p_i : F \longrightarrow \mathcal{O}, \quad \textit{where} \quad p_i\in\Gamma\big(\mathcal{M}; F_i^{*\otimes d_i}\otimes\mathcal{O}\big),$$

is a nondegenerate polynomial. Then there exists a small subset Y_α of $F=\bigoplus_{i=1}^{i=k} F_i$, which is invariant under scalar multiplication in each component separately, with the following property. If K is a compact subset of $\mathcal{O}-\alpha(Y_\alpha)$, there exists $C_K>0$ such that

$$|\upsilon|\le C_K|\alpha(\upsilon)| \quad \forall\ \upsilon\in F \ \ \textit{s.t.}\ \alpha(\upsilon)\in K.$$

Proof: (1) Let $Y_\alpha\subset F$ be the closed subset on which the differential of the fiberwise map $\upsilon\longrightarrow\alpha(\upsilon)$ does not have full rank, i.e. its rank is less than rk F. Since α is nondegenerate, Y_α contains no fiber of F. By our assumptions on α,

$$D(\alpha|_{F_b})\big|_\upsilon=\big(D(p_1|_{F_{1,b}})\big|_{\upsilon_1},\ldots,D(p_k|_{F_{k,b}})\big|_{\upsilon_k}\big): F_1\oplus\ldots\oplus F_k\longrightarrow\mathcal{O}, \quad \forall\ b\in\mathcal{M},\ \upsilon=\upsilon_{[k]}\in\bigoplus_{i=1}^{i=k}F_i.$$

Since $p_i|_{F_{i,b}}$ is a homogeneous polynomial of degree d_i, its derivative is a homogeneous polynomial of degree (d_i-1). Thus, Y_α is preserved under scalar multiplication in each component separately. It also clearly satisfies the second condition of (1) of Definition 3.3.
(2) On $F-Y_\alpha$, α is a covering map onto its image with the number of leaves bounded by some number N_α. Thus, if K is any compact subset of $\mathcal{O}-\alpha(Y_\alpha)$, $\alpha^{-1}(K)$ is a compact subset of F. Therefore, there exists C_K such that

$$|\upsilon|\le C_K|\alpha(\upsilon)| \quad \forall\upsilon\in F \ \ \text{s.t.}\ \alpha(\upsilon)\in K.$$

Note that if $0\notin\alpha(Y_\alpha)$, then α is a linear injection on every fiber, and the above inequality holds on all of F.

Lemma 3.5 *Suppose $\mathcal{M}$ is a smooth manifold,*
(1) $F=\bigoplus_{i=1}^{i=k} F_i$ and $\mathcal{O}$ are vector bundles over $\mathcal{M}$ with $rk\,F+\frac{1}{2}\dim\mathcal{M}=rk\,\mathcal{O}$;
(2) Y is a small subset of $F=\bigoplus_{i=1}^{i=k} F_i$, which is invariant under the scalar multiplication in each component separately;
(3) Ω is an open subset of $\mathcal{I}\times F$ such that $\Omega\cup(\{0\}\times X)$ is a neighborhood of $\{0\}\times X$ in $\mathcal{I}\times\big(F-(Y-X)\big)$;
(4) $\big\{\phi_t\colon\{\upsilon\in F\colon(t,\upsilon)\in\Omega\}\longrightarrow\mathcal{O}\big\}$ is a family of smooth maps;
(5) nondegenerate polynomial $\alpha\colon F\longrightarrow\mathcal{O}$ is the resolvent of $\{\phi_t\}$;
(6) $\bar\nu\in\Gamma(\mathcal{M};\mathcal{O})$ is generic with respect to (Y,α), and the map

$$F\longrightarrow\mathcal{O}, \quad \upsilon\longrightarrow\bar\nu_\upsilon+\alpha(\upsilon), \tag{3.4}$$

has a finite number of (transverse) zeros.
If ψ_t is transversal to zero for all t, there exists a compact subset $K_{\alpha,\bar\nu}$ of $\mathcal{M}$ with the following

property. If K is a precompact open subset of $\mathcal{M}$ containing $K_{\alpha,\bar{\nu}}$, there exist $\delta_K, \epsilon_K > 0$ such that for all $t \in (0, \epsilon_K)$,

$$^{\pm}\big|\big\{\upsilon \in F_{\delta_K}|_K : (t,\upsilon) \in \Omega,\ \psi_t(\upsilon)=0\big\}\big| = ^{\pm}\big|\big\{\upsilon \in F : \bar{\nu}_\upsilon + \alpha(\upsilon) = 0\big\}\big|,$$

where $\psi_t(\upsilon) = t\bar{\nu}_\upsilon + \phi_t(\upsilon)$ as before. Furthermore, all the zeros of $\psi_t\big|_{F_{\delta_K}|K}$ lie over $K_{\alpha,\bar{\nu}}$.

Proof: (1) Since the map in (3.4) has a finite number of zeros, all of them lie in the interior of $F_{C_{\alpha,\bar{\nu}}}|_{K_{\alpha,\bar{\nu}}}$ for some compact subset $K_{\alpha,\bar{\nu}}$ of $\mathcal{M}$ and number $C_{\alpha,\bar{\nu}} > 0$. Suppose $K \subset \mathcal{M}$ is a precompact open subset containing $K_{\alpha,\bar{\nu}}$, $\delta_K > 0$ is such that $F_{\delta_K}|_{K-Y} \subset \Omega$, and $\tilde{\upsilon} \in \Omega_{\delta_K}|_K$ is such that $\psi_t(\tilde{\upsilon}) = 0$. By the same argument as in the proof of Lemma 3.2, if $\delta_K > 0$ is sufficiently small,

$$\big|\alpha(\tilde{\upsilon})\big| \le C_K t \quad \text{and} \quad \big|t\bar{\nu}_{\tilde{\upsilon}} + \alpha(\tilde{\upsilon})\big| \le \bar{\varepsilon}_K(\delta_K) t \quad \forall t < \delta_K,\ \tilde{\upsilon} \in F_{\delta_K}|_K \text{ s.t. } \psi_t(\tilde{\upsilon}) = 0, \tag{3.5}$$

where C_K and $\bar{\varepsilon}_K = \bar{\varepsilon}_K(\delta_K)$ depend only on K, and $\bar{\varepsilon}_K(\delta_K)$ tends to zero with δ_K. Let $\phi_t : F \longrightarrow F$ be the map defined in (2) of the proof of Lemma 3.2, with F^- replaced by F. By (3.5),

$$\begin{aligned}\alpha(\phi_t(\tilde{\upsilon})) \in \mathcal{K}_{\bar{\nu}}\big(K; C_K, \bar{\varepsilon}_K(\delta_K)\big) \equiv \big\{\varpi \in \mathcal{O}_{C_K} : |\bar{\nu}_\varpi + \varpi| \le \bar{\varepsilon}_K(\delta_K)\big\} \\ \forall t < \delta_K,\ \tilde{\upsilon} \in F_{\delta_K}|_K \text{ s.t. } \psi_t(\tilde{\upsilon}) = 0.\end{aligned} \tag{3.6}$$

(2) If $\bar{\nu}$ is generic, the map in (3.4) does not vanish on Y_α, where Y_α is as in Lemma 3.4. Since $\alpha(Y_\alpha)$ is a closed subset of $\mathcal{O}$, there exists $\epsilon_K > 0$ such that

$$\big|\bar{\nu}_\upsilon + \alpha(\upsilon)\big| > \epsilon_K \quad \forall\ \upsilon \in Y_\alpha|_K.$$

Thus, if $\bar{\varepsilon}_K(\delta_K) < \epsilon_K$, $\mathcal{K}_{\bar{\nu}}\big(K; C_K, \bar{\varepsilon}_K(\delta_K)\big)$ is a compact subset of $\mathcal{O}$ disjoint from $\alpha(Y_\alpha)$. Then by (3.6) and Lemma 3.4,

$$\big|\phi_t(\tilde{\upsilon})\big| \le C_K^* \quad \forall t < \delta_K,\ \tilde{\upsilon} \in F_{\delta_K}|_K \text{ s.t. } \psi_t(\tilde{\upsilon}) = 0, \tag{3.7}$$

where C_K^* depends only on K.

(3) There is a one-to-one sign-preserving correspondence between the zeros of ψ_t on $\Omega_{\delta_K}|_K$ and the zeros of

$$\tilde{\psi}_t : \Omega_{\delta_K}(K,t) \equiv \big\{\upsilon \in F : (t, \phi_t^{-1}(\upsilon)) \in \Omega_{\delta_K}|_K\big\} \longrightarrow \mathcal{O}, \quad \tilde{\psi}_t(\upsilon) = t^{-1}\psi_t\big(\phi_t^{-1}(\upsilon)\big).$$

By (3.7), all the zeros of $\tilde{\psi}_t$ on $\Omega_{\delta_K}(K,t)$ are in fact contained in $F_{C_K^*}|_K$. We can assume that $C_K^* > C_{\alpha,\bar{\nu}}$. By our assumptions on ϕ_t,

$$\big|\tilde{\psi}_t(\upsilon) - (\bar{\nu}_\upsilon + \alpha(\upsilon))\big| \le C_K \bar{\epsilon}_K(\delta_K) \quad \forall \upsilon \in \Omega_{\delta_K}(K,t) \cap F_{C_K^*}|_K, \tag{3.8}$$

where $C_K > 0$ depends only on K. We define a cobordism between the zeros of $\tilde{\psi}_t$ and the zeros of $\bar{\nu} + \alpha$ on $\Omega_{\delta_K}(K,t) \cap F_{C_K^*}|_K$ by

$$\Psi : \mathcal{I} \times \Omega_{\delta_K}(K,t) \cap F_{C_K^*}|_K \longrightarrow \mathcal{O}, \quad \Psi_\tau(\upsilon) = \tau\tilde{\psi}_t(\upsilon) + (1-\tau)\big(\bar{\nu}_\upsilon + \alpha(\upsilon)\big) + \eta_\tau(\upsilon),$$

where $\eta : \mathcal{I} \times \Omega_{\delta_K}(K,t) \longrightarrow \mathcal{O}$ is any smooth function with very small C^0-norm such that $\eta_0 = \eta_1 = 0$ and Ψ is transversal to zero. It remains to see that $\Psi^{-1}(0)$ is compact. Suppose $\Psi_{\tau_r}(\upsilon_r) = 0$

and (τ_r, υ_r) converges $(\tilde{\tau}, \tilde{\upsilon}) \in \mathcal{I} \times F_{2C_K^*}|_{\bar{K}}$; we need to show that $\tilde{\upsilon} \in \Omega_{\delta_K}(K,t) \cap F_{C_K^*}|_K$. By equation (3.8),

$$\big|\bar{\nu}_{\upsilon_r} + \alpha(\upsilon_r)\big| \le C_K \bar{\epsilon}_K(\delta_K) + \|\eta\|_{C^0} \quad \forall r \quad \Longrightarrow \quad \big|\bar{\nu}_{\tilde{\upsilon}} + \alpha(\tilde{\upsilon})\big| \le C_K \bar{\epsilon}_K(\delta_K) + \|\eta\|_{C^0}. \tag{3.9}$$

On the other hand, since $\bar{\nu}$ is generic, the map in (3.4) does not vanish on Y. Furthermore, all the zeros of this map are contained in the interior of $F_{C_{\alpha,\bar{\nu}}}|_{K_{\alpha,\bar{\nu}}}$. Thus, by compactness,

$$\tilde{\epsilon}_K \equiv \inf \Big\{ \big|\bar{\nu}_\upsilon + \alpha(\upsilon)\big| : \upsilon \in \big(Y \cap F_{2C_K^*}\big) \cup \big(F_{C_K^*}|_K - F_{C_{\alpha,\bar{\nu}}}|_{K_{\alpha,\bar{\nu}}}\big) \Big\} > 0, \tag{3.10}$$

where $\tilde{\epsilon}_K$ depends only on K. If $\tilde{\epsilon}_K > C_K \bar{\epsilon}_K(\delta_K) + \|\eta\|_{C^0}$, by (3.9) and (3.10),

$$\tilde{\upsilon} \in F_{C_{\alpha,\bar{\nu}}}|_{K_{\alpha,\bar{\nu}}} \subset F_{C_K^*}|_K - Y \subset \Omega_{\delta_K}(K,t).$$

The last inclusion follows from the very first assumption on δ_K above. We conclude that $\Psi^{-1}(0)$ is compact.

Corollary 3.6 *Suppose $\mathcal{M}$ is a smooth oriented manifold,*
(1) $F \equiv F^- \oplus F^+$, $\tilde{F}^-$, and $\mathcal{O} \equiv \mathcal{O}^- \oplus \mathcal{O}^+$ are vector bundles over $\mathcal{M}$ with

$$rk\, F^- = rk\, \tilde{F}^- = rk\, \mathcal{O}^- - \frac{1}{2}\dim \mathcal{M} \quad \text{and} \quad rk\, F^+ = rk\, \mathcal{O}^+;$$

(2) $\rho \in \Gamma(\mathcal{M}; F^{-*\otimes k} \otimes \tilde{F}^-)$ induces a $\tilde{d}$-to-1 cover $F \longrightarrow \tilde{F}$, and $\alpha^- \in \Gamma(\mathcal{M}; \tilde{F}^{-*} \otimes \mathcal{O}^-)$;
(3) $\alpha : F \longrightarrow \mathcal{O}$ is a nondegenerate polynomial such that $\alpha^+ \equiv \alpha|_{F^+} : F^+ \longrightarrow \mathcal{O}^+$ is linear and $\pi^- \circ \alpha = \alpha^- \circ \rho$;
(4) Y is a small subset of F, which is invariant under the scalar multiplication in each component separately;
(5) Ω is an open subset of $\mathcal{I} \times F$ such that $\Omega \cup X$ is a neighborhood of $\{0\} \times X$ in $\mathcal{I} \times \big(F - (Y - X)\big)$;
(6) $\big\{\phi_t : \{\upsilon \in F : (t,\upsilon) \in \Omega\} \longrightarrow \mathcal{O}\big\}$ is a family of smooth maps with resolvent α;
(7) $\bar{\nu} = (\bar{\nu}^-, \bar{\nu}^+) \in \Gamma(\mathcal{M}; \mathcal{O}^- \oplus \mathcal{O}^+)$ is generic with respect to $(\alpha^+, \alpha^-, \rho, Y)$, and the map

$$\tilde{F}^- \longrightarrow \mathcal{O}^-, \quad \varpi \longrightarrow \bar{\nu}^-_{\varpi} + \alpha^-(\varpi), \tag{3.11}$$

has a finite number of (transverse) zeros.
If ψ_t is transversal to zero for all t, there exists a compact subset $K_{\alpha,\bar{\nu}}$ of $\mathcal{M}$ with the following property. If K is precompact open subset of $\mathcal{M}$ containing $K_{\alpha,\bar{\nu}}$, there exist $\delta_K, \epsilon_K > 0$ such that for all $t \in (0, \epsilon_K)$,

$$^{\pm}\big|\big\{\upsilon \in F_{\delta_K}|_K : (t,\upsilon) \in \Omega,\ \psi_t(\upsilon) = 0\big\}\big| = \tilde{d} \cdot {}^{\pm}\big|\big\{\varpi \in \tilde{F}^- : \bar{\nu}^-_{\varpi} + \alpha^-(\varpi) = 0\big\}\big|,$$

where $\psi_t(\upsilon) = t\bar{\nu}_\upsilon + \phi_t(\upsilon)$. Furthermore, all the zeros of $\psi_t\big|_{F_{\delta_K}|K}$ lie over $K_{\alpha,\bar{\nu}}$.

Proof: Let $K_{\alpha,\bar{\nu}}$ and $\delta_K > 0$ be as in Lemma 3.5. Then if K is a precompact open subset of $\mathcal{M}$, for all $t \in (0, \epsilon_K)$ the signed number of zeros of ψ_t on $\Omega_{\delta_K}|_K$ is the same as the signed number of solutions of

$$F|_K \longrightarrow \mathcal{O}, \quad \begin{cases} \bar{\nu}^-_\upsilon + \alpha^-(\rho(\upsilon^-)) = 0 \in \mathcal{O}^-; \\ \bar{\nu}^+_\upsilon + \alpha^+(\upsilon^+) + \pi^+(\alpha(\upsilon^-)) = 0 \in \mathcal{O}^+. \end{cases} \tag{3.12}$$

For every solution of the first equation, there is a unique solution of the second equation. Since α^+ is complex-linear on the fibers, the signed number of solutions of (3.12) is the same as the signed number of solutions of the first equation. Since the first equation has no solutions on Y_{α^-} if $\bar{\nu}$ is generic and ρ is $\tilde{d}$-to-1 outside of Y_{α^-}, ρ induces a $\tilde{d}$-to-1 sign-preserving map from the set of zeros of (3.11) to the set of solutions of the first equation.

3.2 Contributions to the Euler Class

If $\bar{\mathcal{M}}$ is a smooth oriented compact n-manifold and $V \longrightarrow \bar{\mathcal{M}}$ is an oriented vector bundle of rank n, the euler class of V is the number of zeros of any section $s\colon \bar{\mathcal{M}} \longrightarrow V$ which is transverse to the zero set. In this subsection, under slightly more topological assumptions on $\bar{\mathcal{M}}$ and V, we discuss a relationship between subsets of the zero set of a non-transverse section and the euler class of V.

Definition 3.7 *(1) Compact oriented topological manifold* $\bar{\mathcal{M}} = \mathcal{M}_n \sqcup \bigsqcup_{i=0}^{i=n-2} \mathcal{M}_i$ *of dimension* n *is* *mostly smooth, or ms, if*
(1a) each $\mathcal{M}_i$ *is a smooth manifold of dimension* i*, and* $\mathcal{M} \equiv \mathcal{M}_n$ *is a dense open subset of* $\bar{\mathcal{M}}$*;*
(1b) for each $i \in [n-2]$, $\bar{\mathcal{M}}_i - \mathcal{M}_i \subset \bigcup_{j=0}^{j-2} \mathcal{M}_j$*;*
(2) If $\bar{\mathcal{Z}} = \mathcal{Z} \sqcup \bigsqcup \mathcal{Z}_j$ *and* $\bar{\mathcal{M}} = \mathcal{M} \sqcup \bigsqcup \mathcal{M}_i$ *are ms-manifolds, continuous map* $\pi\colon \mathcal{Z} \longrightarrow \mathcal{M}$ *is an ms-map if for each* j *there exists* i *such that* $\pi\colon \mathcal{Z}_j \longrightarrow \mathcal{M}_i$ *is a smooth map.*
(3) If $\bar{\mathcal{M}}$ *is an ms-manifold, topological vector bundle* $V \longrightarrow \bar{\mathcal{M}}$ *is an ms-bundle if* $V|_{\mathcal{M}_i}$ *is a smooth vector bundle for* $i = n$ *and all* $i \in [n-2]$.
(4) If $V \longrightarrow \bar{\mathcal{M}}$ *is an ms-bundle, continuous section* $s : \mathcal{M} \longrightarrow V$ *is an ms-section if* $s|_{\mathcal{M}_i}$ *is* C^2*-smooth for* $i = n$ *and all* $i \in [n-2]$.

The dense open submanifold $\mathcal{M}$ of $\bar{\mathcal{M}}$ will be called the *smooth base* of $\bar{\mathcal{M}}$. Note that if $E \longrightarrow \bar{\mathcal{M}}$ is an ms-bundle, then the (complex) projectivization $\mathbb{P}E$ of E is an ms-manifold. Furthermore, the projection map $\pi_E\colon \mathbb{P}E \longrightarrow \bar{\mathcal{M}}$ is an ms-map, and the tautological line bundle $\gamma_E \longrightarrow \mathbb{P}E$ is an ms-bundle.

If $V \longrightarrow \bar{\mathcal{M}}$ is an ms-bundle, we denote the space of ms-sections of V by $\Gamma(\bar{\mathcal{M}}; V)$. Using (4) of Definition 3.7, we define an ms-polynomial map between two ms-bundles analogously to (1) of Definition 3.1. We topologize $\Gamma(\bar{\mathcal{M}}; V)$ as follows. If $s_k, s \in \Gamma(\bar{\mathcal{M}}; V)$, the sequence $\{s_k\}$ *converges to* s if s_k converges to s in the C^0-norm on all of $\bar{\mathcal{M}}$ and in the C^2-norm on compact subsets of $\mathcal{M}_i$ for $i = n$ and all $i \in [n-2]$. The C^0-norm is defined with respect to the norm on $V \longrightarrow \bar{\mathcal{M}}$. In order to define the C^2-norm on compact subsets of $\mathcal{M}_i$, we fix a connection in each smooth bundle in $V \longrightarrow \mathcal{M}_i$.

Definition 3.8 *Let* $\bar{\mathcal{M}}$ *be an ms-manifold as in Definition 3.7.*
(1) If $\mathcal{Z} \subset \mathcal{M}_i$ *is a smooth oriented submanifold, a normal-bundle model for* $\mathcal{Z}$ *is a tuple* (F, Y, ϑ)*, where*
(1a) $F \longrightarrow \mathcal{Z}$ *is a smooth complex normed vector bundle and* Y *is a small subset of* F*;*
(1b) for some $\delta \in C^\infty(\mathcal{Z}; \mathbb{R}^+)$, $\vartheta\colon F_\delta - (Y - \mathcal{Z}) \longrightarrow \bar{\mathcal{M}}$ *is a continuous map such that*
(1b-i) $\vartheta\colon F_\delta - (Y - \mathcal{Z}) \longrightarrow \bar{\mathcal{M}}$ *is a homeomorphism onto an open neighborhood of* $\mathcal{Z}$ *in* $\mathcal{M} \cup \mathcal{Z}$*;*
(1b-ii) $\vartheta|_{\mathcal{Z}}$ *is the identity map, and* $\vartheta\colon F_\delta - (Y - \mathcal{Z}) \longrightarrow \mathcal{M}$ *is an orientation preserving diffeomorphism on an open subset of* $\mathcal{M}$.
(2) A closure of normal-bundle model (F, Y, ϑ) *is a tuple* $(\bar{\mathcal{Z}}, \tilde{F}, \pi)$*, where*
(2a) $\bar{\mathcal{Z}}$ *is an ms-manifold with smooth base* $\mathcal{Z}$*;*
(2b) $\pi\colon \bar{\mathcal{Z}} \longrightarrow \bar{\mathcal{M}}$ *is an ms-map such that* $\pi|_{\mathcal{Z}}$ *is the identity;*
(2c) $\tilde{F} \longrightarrow \bar{\mathcal{Z}}$ *is an ms-bundle such that* $\tilde{F}|_{\mathcal{Z}} = F$.

If $\mathcal{Z}$ is a smooth submanifold of $\mathcal{M}$, an identification of the normal bundle $\mathcal{N}\mathcal{Z}$ of $\mathcal{Z}$ in $\mathcal{M}$ with a neighborhood of $\mathcal{Z}$ in $\mathcal{M}$ induces a normal bundle model for $\mathcal{Z}$. Definition 3.8 extends this standard construction to the ms-category.

Definition 3.9 *Suppose $E,\mathcal{O}\longrightarrow\bar{\mathcal{M}}$ are ms-bundles and $\alpha\colon E\longrightarrow\mathcal{O}$ is an ms-polynomial.*
(1) Subset $\mathcal{Z}$ of $\mathcal{M}$ is α-regular if there exist a normal bundle model (F,Y,ϑ) for $\mathcal{Z}$, constant-rank polynomial $p: F\oplus E\longrightarrow\mathcal{O}$ over $\mathcal{Z}$, smooth bundle isomorphisms $\vartheta_E: \vartheta^*E\longrightarrow\pi_F^*E$ and $\vartheta_{\mathcal{O}}\colon\vartheta^*\mathcal{O}\longrightarrow\pi_F^*\mathcal{O}$ covering the identity on $F_\delta-(Y-\mathcal{Z})$, and $\varepsilon\in C(F;\mathbb{R})$ such that
(1a) ϑ_E and $\vartheta_{\mathcal{O}}$ are smooth on $F_\delta-Y-\mathcal{Z}$ and restrict to the identity over $\mathcal{Z}$;
(1b) $\lim_{w\longrightarrow0}\varepsilon(w)=0$;
(1c) $|\vartheta_{\mathcal{O}}\alpha(\vartheta_E^{-1}(w,\upsilon))-p(w,\upsilon)|\le\varepsilon(w)|p(w,\upsilon)|$ for all $w\in F_\delta-(Y-X)$, $\upsilon\in E$.
(2) α is a regular polynomial if $\bar{\mathcal{M}}$ is a union of finitely many α-regular subsets.

Lemma 3.10 *Suppose $E,\mathcal{O}\longrightarrow\bar{\mathcal{M}}$ are ms-bundles, such that $\mathrm{rk}\,E+\frac{1}{2}\dim\bar{\mathcal{M}}=\mathrm{rk}\,\mathcal{O}$, and $\alpha\colon E\longrightarrow\mathcal{O}$ is a regular polynomial, such that α is nondegenerate on $\mathcal{M}$. Let $\nu\in\Gamma(\bar{\mathcal{M}};\mathcal{O})$ be an ms-section such that the map*

$$\psi_{\alpha,\nu}\colon E\longrightarrow\mathcal{O},\qquad \psi_{\alpha,\nu}(\upsilon)=\nu_\upsilon+\alpha(\upsilon),$$

does not vanish on $E|_{\bar{\mathcal{M}}-\mathcal{M}}$ and is transversal to the zero set in $\mathcal{O}|_{\mathcal{M}}$. Then $\psi_{\alpha,\nu}^{-1}(0)$ is finite, and $N(\alpha)\equiv^{\pm}|\psi_{\alpha,\nu}^{-1}(0)|$ is independent of the choice of ν as above.

Proof: (1) We first show that for every $x\in\bar{\mathcal{M}}-\mathcal{M}$ there exists a neighborhood U of x in $\bar{\mathcal{M}}$ such that $\psi_{\alpha,\nu}$ does not vanish on $E|_U$. By (2) of Definition 3.9, there exists an α-regular subset $\mathcal{Z}$ of $\bar{\mathcal{M}}$ containing x. Let (F,Y,ϑ), δ, p, ϑ_E, $\vartheta_{\mathcal{O}}$, and ε be as in (1) of Definition 3.9. It can be assumed that δ is such that

$$\varepsilon(w)<\frac{1}{2}\quad\text{and}\quad\big|\nu_{\vartheta(w)}\big|\le2\big|\nu_w\big|\equiv2\big|\nu_{b_w}\big|\qquad\forall w\in F_\delta-(Y-\mathcal{Z}).$$

Then, if $\psi_{\alpha,\nu}(\vartheta_E^{-1}(w,\upsilon))=0$ for some $(w,\upsilon)\in F\oplus E$ with $w\in F_\delta-(Y-\mathcal{Z})$, $|\alpha(w,\upsilon)|\le4|\nu_w|$ by (1c) of Definition 3.9. Thus, if $\{(w_k,\upsilon_k)\}\subset F\oplus E$ is such that $\psi_{\alpha,\nu}(\vartheta_E^{-1}(w_k,\upsilon_k)=0$ and $w_k\longrightarrow x\in F$, a subsequence of $\{\alpha(w_k,\upsilon_k)\}$ converges to an element $\varpi\in\mathcal{O}_x$. Since α is a polynomial map of constant rank, there exists $(0,\upsilon)\in F\oplus E$ such that $\alpha(0,\upsilon)=\varpi$. Since $\alpha(0,\upsilon)=p(0,\upsilon)$, it follows that $\psi_{\alpha,\nu}(\upsilon)=0$ contrary to the assumption.
(2) By (1), there exists a compact subset $K_{\alpha,\nu}$ of $\mathcal{M}$ such that $\psi_{\alpha,\nu}^{-1}(0)\subset E|_{K_{\alpha,\nu}}$. Since $\psi_{\alpha,\nu}$ is transversal to zero, $\nu(\mathcal{M})\cap\alpha(Y_\alpha)=\emptyset$, where $Y_\alpha\subset E|_{\mathcal{M}}$ is as in Lemma 3.4. It follows that $\psi_{\alpha,\nu}^{-1}(0)$ is a finite subset of $E|_{\mathcal{M}}$.
(3) The final claim of the lemma is obtained by constructing a cobordism between $\psi_{\alpha,\nu}$ and $\psi_{\alpha,\nu'}$. More precisely, we take a smooth family $\{\nu_\tau\colon\tau\in\mathcal{I}\}$ of ms-sections of $\mathcal{O}$ such that $\nu_0=\nu$, $\nu_1=\nu'$, $\psi_{\alpha,\nu_\tau}^{-1}(0)\subset E|_{\mathcal{M}}$, and the section

$$\Psi_\alpha\colon\mathcal{I}\times E\longrightarrow\mathcal{O},\qquad\Psi_\alpha(\tau,\upsilon)=\psi_{\alpha,\nu_\tau}(\upsilon),$$

is transversal to the zero set in $\mathcal{O}$. Such a family can always be chosen, since $\bar{\mathcal{M}}$–$\mathcal{M}$ has codimension two in $\bar{\mathcal{M}}$. Then, by the same argument as in (1) and (2), $\Psi_\alpha^{-1}(0)$ is a smooth compact oriented submanifold of $E|_{\mathcal{M}}$ with boundary $\psi_{\alpha,\nu_1}^{-1}(0)-\psi_{\alpha,\nu_0}^{-1}(0)$.

Definition 3.11 *Suppose $\bar{\mathcal{M}}$ is an ms-manifold of dimension $2n$, $V\longrightarrow\bar{\mathcal{M}}$ is an ms-bundle of rank n, $s\in\Gamma(\bar{\mathcal{M}};V)$, and $\mathcal{Z}\subset\mathcal{M}_i\cap s^{-1}(0)$.*
(1) $\mathcal{Z}$ is s-hollow if there exist a normal bundle model (F,Y,ϑ) for $\mathcal{Z}$ and a bundle isomorphism $\vartheta_V\colon\vartheta^*V\longrightarrow\pi_F^*V$, covering the identity on $F_\delta-(Y-\mathcal{Z})$, such that

(1a) $\vartheta_V|_{F_\delta - Y - \mathcal{Z}}$ *is smooth and* $\vartheta_V|_{\mathcal{Z}}$ *is the identity;*
(1b) $\phi_0 \equiv \vartheta_V \circ \vartheta^* s\colon F_\delta - (Y - \mathcal{Z}) \longrightarrow V$ *is hollow.*
(2) $\mathcal{Z}$ *is* *s-regular* *if there exist a normal bundle model* (F, Y, ϑ) *for* $\mathcal{Z}$ *with closure* $(\bar{\mathcal{Z}}, \tilde{F}, \pi)$*, regular polynomial* $\alpha : \tilde{F} \longrightarrow \pi^* V$*, and a bundle isomorphism* $\vartheta_V : \vartheta^* V \longrightarrow \pi_F^* V$ *covering the identity on* $F_\delta - (Y - \mathcal{Z})$*, such that*
(2a) $\vartheta_V|_{F_\delta - Y - \mathcal{Z}}$ *is smooth and* $\vartheta_V|_{\mathcal{Z}}$ *is the identity;*
(2b) $\alpha|_{\mathcal{Z}}$ *is nondegenerate and is the resolvent for* $\phi_0 \equiv \vartheta_V \circ \vartheta^* s : F_\delta - (Y - \mathcal{Z}) \longrightarrow V$*, and* Y *is preserved under scalar multiplication in each of the components of* F *for the splitting corresponding to* α *as in (1) of Definition 3.1.*

Lemma 3.12 *If* $(\bar{\mathcal{M}}, V, s)$ *and* $(\mathcal{Z}, F, Y, \vartheta)$ *are as in Definition 3.11, there exist a number* $\mathcal{C}_{\mathcal{Z}}(s) \in \mathbb{Z}$*, which equals zero if* $\mathcal{Z}$ *is s-hollow, and a dense open subset* $\Gamma_{\mathcal{Z}}(s) \subset \Gamma(\bar{\mathcal{M}}; V)$ *with the following properties. For every* $\nu \in \Gamma_{\mathcal{Z}}(s)$*,*
(1) there exists $\epsilon_\nu > 0$ *such that for all* $t \in (0, \epsilon_\nu)$*, all the zeros of* $t\nu + s$ *are contained in* $\mathcal{M}$ *and* $(t\nu + s)\big|_{\mathcal{M}}$ *is transversal to the zero set in* V*;*
(2) there exist a compact subset $K_\nu \subset \mathcal{Z}$*, open neighborhood* $U_\nu(K)$ *of* K *in* $\bar{\mathcal{M}}$ *for each compact subset* $K \subset \mathcal{Z}$*, and* $\epsilon_\nu(U) \in (0, \epsilon_\nu)$ *for each open subset* U *of* $\bar{\mathcal{M}}$ *such that*

$$^{\pm}\big|\{b \in U : t\nu(b) + s(b) = 0\}\big| = \mathcal{C}_{\mathcal{Z}}(s) \qquad \text{if } t \in (0, \epsilon_\nu(U)),\ K_\nu \subset K \subset U \subset U_\nu(K).$$

Proof: It is clear that we can choose a dense open subset $\Gamma'_{\mathcal{Z}}(s) \subset \Gamma(\bar{\mathcal{M}}; V)$ such that every $\nu \in \Gamma'_{\mathcal{Z}}(s)$ satisfies requirement (1) of the lemma. If $\mathcal{Z}$ is s-hollow, we also need that $\bar{\nu} \equiv \nu|_{\mathcal{Z}}$ is generic with respect to the corresponding polynomial α^- in the sense of the proof of Lemma 3.2. We can then take $K_\nu = \emptyset$. If $\mathcal{Z}$ is s-regular, let $\bar{\nu} = \pi^* \nu \in \Gamma(\bar{\mathcal{Z}}; \pi^* V)$. By Lemma 3.10, the second part of (6) of Lemma 3.5 is satisfied, as long as $t\nu + s$ is transversal to the zero set on each smooth strata. The other requirements on $\bar{\nu}$ in Lemma 3.5 are finitely many transversality properties. We then take

$$\mathcal{C}_{\mathcal{Z}}(s) = {}^{\pm}\big|\big\{\upsilon \in F : \bar{\nu}_\upsilon + \alpha(\upsilon) = 0\big\}\big|.$$

By Lemma 3.10, this number is well-defined.

The total number of zeros of a section $t\nu + s$ satisfying condition (1) of Lemma 3.12 is precisely the euler class $e(V)$ of the bundle $V \longrightarrow \bar{\mathcal{M}}$. Thus, due to (2) of Lemma 3.12, we call $\mathcal{C}_{\mathcal{Z}}(s)$ the *s-contribution* (or simply contribution) of $\mathcal{Z}$ to $e(V)$. If $\mathcal{Z}$ is any subset of $\bar{\mathcal{M}}$ such that $\mathcal{Z} \cap s^{-1}(0)$ satisfies the requirements of Definition 3.11, let $\mathcal{C}_{\mathcal{Z}}(s) = \mathcal{C}_{\mathcal{Z} \cap s^{-1}(0)}(s)$. In addition, if $\mathcal{Z}$ is a closed subset of $\bar{\mathcal{M}}$ such that $s^{-1}(0) - \mathcal{Z}$ is also closed, we can easily define $\mathcal{C}_{\mathcal{Z}}(s)$ by Lemma 3.12.

Corollary 3.13 *Let* $V \longrightarrow \bar{\mathcal{M}}$ *be an ms-bundle of rank* n *over an ms-manifold of dimension* $2n$*. Suppose* $\mathcal{U}$ *is an open subset of* $\mathcal{M}$ *and* $s \in \Gamma(\bar{\mathcal{M}}; V)$ *is such that* $s|_{\mathcal{U}}$ *is transversal to the zero set.*
(1) If $s^{-1}(0) \cap \mathcal{U}$ *is a finite set,* $^{\pm}|s^{-1}(0) \cap \mathcal{U}| = \langle e(V), [\bar{\mathcal{M}}] \rangle - \mathcal{C}_{\bar{\mathcal{M}} - \mathcal{U}}(s)$*.*
(2) If $\bar{\mathcal{M}} - \mathcal{U} = \bigsqcup_{i=1}^{i=k} \mathcal{Z}_i$*, where each* $\mathcal{Z}_i$ *is s-hollow or s-regular, then* $s^{-1}(0) \cap \mathcal{U}$ *is finite, and*

$$^{\pm}|s^{-1}(0) \cap \mathcal{U}| = \big\langle e(V), [\bar{\mathcal{M}}] \big\rangle - \mathcal{C}_{\bar{\mathcal{M}} - \mathcal{U}}(s) = \big\langle e(V), [\bar{\mathcal{M}}] \big\rangle - \sum_{i=1}^{i=k} \mathcal{C}_{\mathcal{Z}_i}(s).$$

If $\mathcal{Z}_i$ *is s-hollow,* $\mathcal{C}_{\mathcal{Z}_i}(s) = 0$*. If* $\mathcal{Z}_i$ *is s-regular and* $\alpha_i : \tilde{F}_i \longrightarrow V$ *is the corresponding polynomial,*

$$\mathcal{C}_{\mathcal{Z}_i}(s) = {}^{\pm}\big|\{\upsilon \in \tilde{F}_i :\ \bar{\nu}_\upsilon + \alpha_i(\upsilon) = 0\}\big| \equiv N(\alpha_i),$$

where $\bar{\nu}\in\Gamma(\bar{\mathcal{Z}}_i;V)$ *is a generic section. Finally, if* $\alpha_i\in\Gamma(\bar{\mathcal{Z}}_i;\tilde{F}_i^{*\otimes k}\otimes\pi^*V)$ *has constant rank over* $\bar{\mathcal{Z}}_i$ *and factors through a* $\tilde{k}$*-to-*1 *cover* $\rho_i\colon \tilde{F}_i\longrightarrow\tilde{F}_i^{\otimes k}$,

$$\mathcal{C}_{\mathcal{Z}_i}(s)=\tilde{k}\big\langle e\big(\pi^*V/\alpha_i(\tilde{F}_i)\big),[\bar{\mathcal{Z}}_i]\big\rangle.$$

All statements of this corollary have already been proved. A splitting of the zero set as in (2) of Corollary 3.13 always exists in the complex-analytic category. It should be possible to generalize the constructions of this subsection to an arbitrary compact oriented topological manifold. However, Lemma 3.10 will no longer be valid, and another approach will be needed to deal with the zeros of $\psi_{\alpha,\nu}$ that tend to infinity. For the cases that we encounter in Section 5, the version of s-regularity of Definition 3.11 suffices.

3.3 Zeros of Polynomial Maps

We now present a procedure for computing the number of zeros of a polynomial map between two complex vector bundles over a compact oriented manifold. All the polynomials we encounter in Section 5 are of degree-one. Thus, we focus on the degree-one case, but discuss the general case at the end for the sake of completeness.

Suppose $\bar{\mathcal{M}}$ is an ms-manifold, $E,\mathcal{O}\longrightarrow\bar{\mathcal{M}}$ are ms-bundles such that $\text{rk }E+\frac{1}{2}\dim\bar{\mathcal{M}}=\text{rk }\mathcal{O}$, and $\alpha\in\Gamma(\bar{\mathcal{M}};E^*\otimes\mathcal{O})$ is an ms-section. Let $\bar{\nu}\in\Gamma(\bar{\mathcal{M}};\mathcal{O})$ be such that $\bar{\nu}$ has no zeros, the map

$$\psi_{\alpha,\bar{\nu}}\equiv\bar{\nu}+\alpha\colon E\longrightarrow\mathcal{O}$$

is transversal to the zero set in $\mathcal{O}$ on $E|_{\mathcal{M}}$, and all its zeros are contained in $E|_{\mathcal{M}}$. The first step in our procedure of determining the number of zeros of $\psi_{\alpha,\bar{\nu}}$ reduces this issue to the case E is a line bundle. Let $\mathbb{P}E$ be the projectivization of E (over $\mathbb{C}$) and let $\gamma_E\longrightarrow\mathbb{P}E$ be the tautological line bundle. Then α induces an ms-section $\alpha_E\in\Gamma(\mathbb{P}E;\gamma_E^*\otimes\pi_E^*\mathcal{O})$, where $\pi_E:\mathbb{P}E\longrightarrow\bar{\mathcal{M}}$ is the bundle projection map. The number of zeros of $\psi_{\alpha,\bar{\nu}}$ is the same as the number of zeros of the induced map

$$\psi^E_{\alpha,\bar{\nu}}\equiv\pi_E^*\bar{\nu}+\alpha_E\colon\gamma_E\longrightarrow\pi_E^*\mathcal{O}.$$

Thus, we can always reduce the computation to the case E is a line bundle.

The second step describes the number of zeros of $\psi_{\alpha,\bar{\nu}}$ topologically in the case E is a line bundle. Since $\bar{\nu}$ has no zeros, it spans a trivial subbundle $\mathbb{C}\bar{\nu}$ of $\mathcal{O}$. Let $\mathcal{O}^\perp$ be the quotient of $\mathcal{O}$ by this trivial subbundle. Denote the $\mathbb{C}\bar{\nu}$- and $\mathcal{O}^\perp$-components of α by α^t and $\alpha^\perp$, respectively. Then the zeros of $\psi_{\alpha,\bar{\nu}}$ are described by

$$\begin{cases}\bar{\nu}_b+\alpha_b^t(v)=0\in\mathbb{C}\bar{\nu};\\ \alpha_b^\perp(v)=0\in\mathcal{O}^\perp;\end{cases}\qquad b\in bar\mathcal{M},\ v\in E_b. \tag{3.13}$$

Since $\bar{\nu}$ does not vanish, all solutions of the first equations (3.13) are nonzero. The solution of the second equation with nonzero v is $(E-\bar{\mathcal{M}})|_{\alpha^{\perp-1}(0)}$. Furthermore, if $b\in\alpha^{\perp-1}(0)$ and $\alpha(b)\neq0$, $\alpha^t\colon E\longrightarrow(\mathbb{C}\bar{\nu})_b$ is an isomorphism. Thus, for every $b\in\alpha^{\perp-1}(0)-\alpha^{-1}(0)$, there exists a unique $v\in E_b$ solving the first equation in (3.13), and the sign of (b,v) as a zero of $\psi_{\alpha,\bar{\nu}}$ agrees with the sign of b as a zero of $\alpha^\perp$. On the other hand, (3.13) has no solutions on $E|_{\alpha^{-1}(0)}$. It follows that the number of zeros of $\psi_{\alpha,\bar{\nu}}$ is the number of zeros of $\alpha^\perp$ on $\bar{\mathcal{M}}-\alpha^{-1}(0)$, i.e.

$${}^\pm|\psi^{-1}(0)|=\big\langle e(E^*\otimes\mathcal{O}^\perp),[\bar{\mathcal{M}}]\big\rangle-\mathcal{C}_{\alpha^{-1}(0)}(\alpha^\perp); \tag{3.14}$$

see Corollary 3.13.

As discussed in the previous subsection, computing $\mathcal{C}_{\mathcal{Z}}(s)$ in reasonably good cases reduces to counting the number of zeros of polynomial maps between vector bundles over ms-manifolds, but with the rank of the target bundle one less than the rank of the bundle $\mathcal{O}$ we started with. Thus, this process will eventually terminate. The lemma below summarizes the last two paragraphs. Let $\lambda_E = c_1(\gamma_E^*)$.

Lemma 3.14 *Suppose $\bar{\mathcal{M}}$ is an ms-manifold and $E, \mathcal{O} \longrightarrow \bar{\mathcal{M}}$ are ms-bundles such that*

$$rk\, E + \frac{1}{2}\dim \bar{\mathcal{M}} = rk\, \mathcal{O}.$$

If $\alpha \in \Gamma(\bar{\mathcal{M}}; E^* \otimes \mathcal{O})$ and $\bar{\nu} \in \Gamma(\bar{\mathcal{M}}; \mathcal{O})$ are such that α is regular, $\bar{\nu}$ has no zeros, the map

$$\psi_{\alpha,\bar{\nu}} \equiv \bar{\nu} + \alpha \colon E \longrightarrow \mathcal{O}$$

is transversal to the zero set on $E|_{\mathcal{M}}$, and all its zeros are contained in $E|_{\mathcal{M}}$, then $\psi_{\alpha,\bar{\nu}}^{-1}(0)$ is a finite set, ${}^{\pm}|\psi_{\alpha,\bar{\nu}}^{-1}(0)|$ depends only on α, and

$$N(\alpha) \equiv {}^{\pm}|\psi_{\alpha,\bar{\nu}}^{-1}(0)| = \big\langle c(\mathcal{O})c(E)^{-1}, [\bar{\mathcal{M}}]\big\rangle - \mathcal{C}_{\alpha_E^{-1}(0)}(\alpha_E^{\perp}).$$

Proof: Let $n = \mathrm{rk}\, E$, $m = \mathrm{rk}\, \mathcal{O}$, and $\lambda_E = c_1(\gamma_E^*)$. From Lemma 3.10, equation (3.14), and the construction above, we obtain the first two claims of the lemma along with

$$\begin{aligned} N(\alpha) &= \sum_{k=0}^{k=m-1} \big\langle c_k(\mathcal{O}^{\perp})\lambda_E^{m-1-k}, [\mathbb{P}E]\big\rangle - \mathcal{C}_{\alpha_E^{-1}(0)}(\alpha_E^{\perp}) \\ &= \sum_{k=0}^{k=m-1} \big\langle c_k(\mathcal{O})\lambda_E^{m-1-k}, [\mathbb{P}E]\big\rangle - \mathcal{C}_{\alpha_E^{-1}(0)}(\alpha_E^{\perp}). \end{aligned} \tag{3.15}$$

On the other hand,

$$\begin{aligned} &\lambda_E^n + \sum_{k=1}^{k=n} c_k(E)\lambda_E^{n-k} = 0 \in H^{2n}(\mathbb{P}E) \qquad \text{and} \\ &\big\langle \mu\lambda_E^{n-1}, [\mathbb{P}E]\big\rangle = \big\langle \mu, [\bar{\mathcal{M}}]\big\rangle \quad \forall \mu \in H^{2m-2n}(\bar{\mathcal{M}}); \end{aligned} \tag{3.16}$$

see [BT] for example. The last statement of the lemma follows from (3.15) and (3.16).

Remark: If $\alpha : E \longrightarrow \mathcal{O}$ is a polynomial, and not just a linear map, the first step in computing the number of zeros of the map $\psi_{\alpha,\bar{\nu}} = \bar{\nu} + \alpha$ would be to reduce to the case α is a linear map via a projectivization construction similar to the one in the second paragraph of this subsection. For example, suppose $\alpha = p_1 + p_2$, where $p_i \in \Gamma(\bar{\mathcal{M}}; E_i^{*\otimes d_i} \otimes \mathcal{O})$ and $E = E_1 \oplus E_2$. Then the number of zeros of $\psi_{\alpha,\bar{\nu}}$ is the same as the number of zeros of

$$\psi_{\alpha,\bar{\nu}}^{E_1} \equiv \pi_{E_1}^*\bar{\nu} + p_{1,E_1} + \pi_{E_1}^* p_2 \colon \gamma_{E_1} \oplus \pi_{E_1}^* E_2 \longrightarrow \pi_{E_1}^*\mathcal{O}$$

over $\mathbb{P}E_1$, where $p_{1,E_1}\in\Gamma(\mathbb{P}E_1;\gamma_{E_1}^{*\otimes d_1})$ is the section induced by p_1. If $\bar{\nu}$ is generic, this number is d_1-times the number of zeros of the map

$$\tilde{\psi}_{\alpha,\bar{\nu}}^{E_1}\equiv\pi_{E_1}^*\bar{\nu}+p_{1,E_1}+\pi_{E_1}^*p_2\colon\gamma_{E_1}^{\otimes d_1}\oplus\pi_{E_1}^*E_2\longrightarrow\pi_{E_1}^*\mathcal{O}.$$

Note that p_{1,E_1} is *linear* on $\gamma_{E_1}^{\otimes d_1}$. Taking the projection of $\pi_{E_1}^*E_2$ over $\mathbb{P}E_1$ and repeating the above procedure, we obtain an affine map

$$\psi_{\alpha,\bar{\nu}}^{E_1,E_2}\colon\pi_{E_2}^*\gamma_{E_1}^{\otimes d_1}\oplus\gamma_{\pi_{E_1}^*E_2}^{\otimes d_2}\longrightarrow\pi_{\pi_{E_1}^*E_2}^*\pi_{E_1}^*\mathcal{O}.$$

4 Resolvents for $\{\psi_{\mathcal{T},t\nu}^{\mu}\}$ and $\{\psi_{\mathcal{S},t\nu}^{\mu}\}$

4.1 A Power Series Expansion for $\pi_{\upsilon,-}^{0,1}\bar{\partial}u_{\upsilon}$

Throughout this section, we assume that $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ is a simple bubble type, with $d_{\hat{0}}=0$ and $\sum_{i\in I}d_i=d$, and μ is an N-tuple of constraints in general position of total codimension

$$\mathrm{codim}_{\mathbb{C}}\mu=d(n+1)-n(g-1)+N.$$

Our goal is to extract leading-order terms from the bundle map $\psi_{\mathcal{T},t\nu}^{\mu}$ of Theorem 2.7 and to describe the zero set of $\psi_{\mathcal{T},t\nu}^{\mu}$ as the union of the zero sets of affine maps between finite-rank vector bundles. The main topological tool is Subsection 3.1.

Nearly all of this subsection is devoted to obtaining the power series expansion for $\pi_{\upsilon,-}^{0,1}\bar{\partial}u_{\upsilon}$ of Proposition 4.4. However, we first state an estimate for $\pi_{\upsilon,-}^{0,1}\nu_{\upsilon,t}$, which is immediate from Theorem 2.7.

Let $\{\psi_j\}$ denote an orthonormal basis for $\mathcal{H}_{\Sigma}^{0,1}$. Given $q\in\mathbb{P}^n$ and an orthonormal basis $\{X_i\}$ for $T_q\mathbb{P}^n$, put

$$\bar{\nu}_q=\sum_{i=1,j=1}^{i=n,j=g}\bigg(\int_{z\in\Sigma}\big\langle\nu(z,q),X_i\psi_j\big\rangle_z\bigg)X_i\psi_j\equiv\pi_{\mathcal{H}_{\Sigma}^{0,1}}\nu(\cdot,q)\in\mathcal{H}_{\Sigma}^{0,1}\otimes T_q\mathbb{P}^n.$$

Note that $\bar{\nu}$ is well-defined.

Lemma 4.1 *There exist $\delta,C\in C^{\infty}(\mathcal{M}_{\mathcal{T}}^{(0)};\mathbb{R}^+)$ such that for all $\upsilon\in F^{(\emptyset)}\mathcal{T}_{\delta}$ and $t\in(0,\delta(b_{\upsilon}))$,*

$$\big\|\pi_{\upsilon,-}^{0,1}\nu_{\upsilon,t}-\tilde{R}_{\upsilon}\bar{\nu}_{ev(b_{\upsilon})}\big\|_{\upsilon,2}\le C(b_{\upsilon})\big(t+|\upsilon|^{\frac{1}{p}}\big).$$

Suppose $\upsilon=\big((\Sigma,[N],I;x,(j,y),u),(v_h)_{h\in\hat{I}}\big)\in F^{(\emptyset)}\mathcal{T}$ is such that q_{υ} is defined. For any $h\in\hat{I}$, let $\tilde{h}(\mathcal{T})=\min\{i\in\hat{I}\colon i\le h\}$. By the basic gluing construction of Subsection 2.2 in [Z1],

$$\tilde{v}_h=d\phi_{b_{\upsilon},\tilde{h}(\mathcal{T})}\big|_{\tilde{x}_h(\upsilon)}\big(dq_{\upsilon,\iota_h}^{-1}\big|_{\tilde{x}_h(\upsilon)}d\phi_{b_{\upsilon},h}^{-1}\big|_0 v_h\big)=\prod_{i\in\hat{I},i\le h}v_i\in T_{x_{\tilde{h}(\mathcal{T})}}\Sigma,$$

where $\phi_{b_{\upsilon},h}$ is a holomorphic identification of neighborhoods of x_h in $\Sigma_{b_{\upsilon},\iota_h}$ and in $F_{b_{\upsilon},h}^{(0)}\equiv T_{x_h}\Sigma_{b_{\upsilon},h}$. If $\Sigma_{b_{\upsilon},h}=S^2$, we also identify $T_{x_h}\Sigma_{b_{\upsilon},h}$ with $\mathbb{C}$ with the map q_N.

Lemma 4.2 *For all $\upsilon\in F^{(\emptyset)}\mathcal{T}$ such that q_υ is defined, $\bar\partial u_\upsilon$ vanishes outside of the annuli $A^-_{\upsilon,h}$ with $\chi_{\mathcal{T}}h=1$ and $A^{\pm}_{\upsilon,h}$ with $\chi_{\mathcal{T}}h=2$. Furthermore, there exists $\delta\in C^\infty(\mathcal{M}_{\mathcal{T}};\mathbb{R}^+)$ such that for all $\upsilon\in F^{(\emptyset)}\mathcal{T}_\delta$ and $h\in\hat I$ with $\chi_{\mathcal{T}}h=1$, on $\tilde A^-_{\upsilon,h}\equiv\big\{z\in F^{(0)}_{h,b_\upsilon}:\frac{1}{2}|\upsilon_h|^{\frac{1}{2}}\le|z|_{b_\upsilon}\le|\upsilon_h|^{\frac{1}{2}}\big\}$,*

$$\Pi^{-1}_{b_\upsilon,\bar u_\upsilon(z)}\bar\partial\big(u_\upsilon\circ q^{-1}_{\upsilon,\iota_h}\big)\circ d\phi^{-1}_{b_\upsilon,h}\Big|_z = -|\upsilon_h|^{-\frac{1}{2}}\bigg(\sum_{m\ge1}\big(1-\beta_{|\upsilon_h|}(2|z|)\big)^{(m-1)}\mathcal{D}^{(m)}_{\mathcal{T},h}\Big(\Big[b_\upsilon,\Big(\frac{\upsilon_h}{z}\Big)\Big]\Big)\bigg)\bar\partial\beta\Big|_{2|\upsilon_h|^{-\frac{1}{2}}z},$$

where $\bar u_\upsilon(z)\in T_{ev(b_\upsilon)}\mathbb{P}^n$ is given by

$$\exp_{b_\upsilon,ev(b_\upsilon)}\bar u_\upsilon(z)=u_\upsilon\big(q^{-1}_{\upsilon,\iota_h}\phi^{-1}_{b_\upsilon,h}(z)\big)=u_h\big(q_{h,(x_h,\upsilon_h)}\phi^{-1}_{b_\upsilon,h}(z)\big),\quad |\bar u_\upsilon(z)|_{b_\upsilon}<r_{\mathbb{P}^n}.$$

This sum converges uniformly on $\tilde A^-_{\upsilon,h}$.

Remark: By construction, $q_\upsilon=q_{\upsilon,(x_h,\upsilon_h)}\circ q_{\upsilon,\iota_h}$ on $A^-_{\upsilon,h}$, and on $q_{\upsilon,\iota_h}(A^-_{\upsilon,h})$

$$q_{\upsilon,(x_h,\upsilon_h)}(z)=\big(h,q_Sp_{h,(x_h,\upsilon_h)}(z)\big),\quad \textit{where}\quad p_{h,(x_h,\upsilon_h)}(z)=\big(1-\beta_{|\upsilon_h|}(2|\phi_{b_\upsilon,h}z|)\big)\overline{\Big(\frac{\upsilon_h}{\phi_{b_\upsilon,h}z}\Big)}.$$

Proof: The first claim follows from (G3); see Subsection 2.1. If $y\in\Sigma_{b_\upsilon,h}$ and $|q_S^{-1}(y)|\le2\delta_{\mathcal{T}}(b_\upsilon)$, define $\bar u_h(y)\in T_{\mathrm{ev}(b_\upsilon)}\mathbb{P}^n$ by

$$\exp_{b_\upsilon,\mathrm{ev}(b_\upsilon)}\bar u_h(y)=u_h(y),\quad |\bar u_h(y)|_{b_\upsilon}<r_{\mathbb{P}^n}.$$

By construction, $u_\upsilon\circ q^{-1}_{\upsilon,\iota_h}=u_h\circ q_\upsilon\circ q^{-1}_{\upsilon,\iota_h}$ on $q_{\upsilon,\iota_h}(A^-_{\upsilon,h})$. Since $\Pi^{-1}_{b_\upsilon,\bar u_\upsilon}\circ du_h$ is $\mathbb{C}$-linear on $q_\upsilon(A^-_{\upsilon,h})$, for any $z\in\tilde A^-_{\upsilon,h}$

$$\begin{aligned}\Pi^{-1}_{b_\upsilon,\bar u_\upsilon(\cdot)}\bar\partial(u_\upsilon\circ q^{-1}_{\upsilon,\iota_h})\circ d\phi^{-1}_{b_\upsilon,h}\Big|_z&=\Pi^{-1}_{b_\upsilon,\bar u_\upsilon(\cdot)}du_h\circ\bar\partial(q_\upsilon\circ q^{-1}_{\upsilon,\iota_h})\circ d\phi^{-1}_{b_\upsilon,h}\Big|_z\\&=-2|\upsilon_h|^{-\frac{1}{2}}\Big(\frac{\upsilon_h}{z}\Big)\Pi^{-1}_{b_\upsilon,\bar u_\upsilon(\cdot)}(du_h\circ dq_S)\Big|_{p_{h,(x_h,\upsilon_h)}\phi^{-1}_{\upsilon,h}(z)}\circ\partial\beta\Big|_{2|\upsilon_h|^{-\frac{1}{2}}z};\end{aligned}\tag{4.1}$$

see Lemma 2.2 in [Z1]. Since $g_{\mathbb{P}^n,b_\upsilon}$ is flat on $u_\upsilon(A^-_{\upsilon,h})$ by our choice of metrics,

$$\Pi^{-1}_{b_\upsilon,\bar u_\upsilon}(du_h\circ dq_S)=d(\bar u_h\circ q_S)\tag{4.2}$$

on $q_S^{-1}q_\upsilon(A^-_{\upsilon,h})$. Since $\bar u_h\circ q_S$ is antiholomorphic and the metric $g_{\mathbb{P}^n,b_\upsilon}$ is flat near $\mathrm{ev}(b_\upsilon)$,

$$\begin{aligned}d(\bar u_h\circ q_S)\big|_x\Big(\frac{\partial}{\partial s}\Big)&=d(\bar u_h\circ q_S)\big|_x\Big(\frac{\partial}{\partial\bar y}\Big)=\sum_{m\ge1}\frac{\overline{x}^{m-1}}{(m-1)!}\frac{d^m}{d\bar y^m}\big(\bar u_h\circ q_S\big)\big|_{(s,t)=0}\\&=\sum_{m\ge1}\frac{\overline{x}^{m-1}}{(m-1)!}\frac{D^{m-1}}{ds^{m-1}}\frac{d}{ds}(u_h\circ q_S)\big|_{(s,t)=0},\end{aligned}\tag{4.3}$$

for any $x\in q_\upsilon(A^-_{\upsilon,h})$, where $y=s+it\in\mathbb{C}$ is the complex coordinate. The second claim follows from equations (4.1)-(4.3). For the last claim, note that the sum converges uniformly on $\tilde{A}^-_{\upsilon,h}$ as long as $q_\upsilon(A^-_{\upsilon,h})$ is contained in the ball of convergence for the power series expansion for $\bar{u}_h$ at 0.

If $\psi\in\mathcal{H}^{0,1}_\Sigma$, $b\in\mathcal{M}^{(0)}_{\mathcal{T}}$, $m\geq 1$, and the metric $g_{b,\hat{0}}$ is flat near x, we define $D^{(m)}_{b,x}\psi\in T^{0,1}_x\Sigma^{\otimes m}$ as follows. If (s,t) are conformal coordinates centered at x such that s^2+t^2 is the square of the $g_{b,\hat{0}}$-distance to x, let

$$\{D^{(m)}_{b,x}\psi\}\Big(\frac{\partial}{\partial s}\Big)\equiv\{D^{(m)}_{b,x}\psi\}\bigg(\underbrace{\frac{\partial}{\partial s},\ldots,\frac{\partial}{\partial s}}_{m}\bigg)=\frac{\pi}{m!}\Big\{\frac{D^{m-1}}{ds^{m-1}}\psi_j\Big|_{(s,t)=0}\Big\}\Big(\frac{\partial}{\partial s}\Big),$$

where the covariant derivatives are taken with respect to the metric $g_{b,\hat{0}}$. Since $\psi_j\in\mathcal{H}^{0,1}_\Sigma$, $\psi_j=f(ds-idt)$ for some anti-holomorphic function f. Since $g_{b,\hat{0}}$ is flat near x, it follows that $D^{(m)}_{b,x}\psi\in T^{0,1}_x\Sigma^{\otimes m}$. If $\{\psi_j\}$ is an orthonormal basis for $\mathcal{H}^{0,1}_\Sigma$, let $s^{(m)}_{b,x}\in T^*_x\Sigma^{\otimes m}\otimes\mathcal{H}^{0,1}_\Sigma$ be given by

$$s^{(m)}_{b,x}(v)\equiv s^{(m)}_{b,x}(\underbrace{v,\ldots,v}_{m})=\sum_{j\in[g]}\overline{\Big\{D^{(m)}_{b,x}\psi_j\Big\}(v)}\psi_j.$$

The section $s^{(m)}_{b,x}$ is always independent of the choice of a basis for $\mathcal{H}^{0,1}_\Sigma$, but is dependent on the choice of the metric $g_{b,\hat{0}}$ if $m>1$. However, $s^{(1)}_{b,x}$ depends only on (Σ,j); we denote this section by $s_{\Sigma,x}$. By [GH, p246], $s_{\Sigma,x}$ does not vanish and thus spans a subbundle of $\Sigma\times\mathcal{H}^{0,1}_\Sigma\longrightarrow\Sigma$. We denote this subbundle by $\mathcal{H}^+_\Sigma$ and its orthogonal complement by $\mathcal{H}^-_\Sigma$. A slightly different description of these bundles is given in Subsection 2.3. Let

$$\pi^+,\pi^-\in\Gamma\big(\Sigma;(\Sigma\times\mathcal{H}^{0,1}_\Sigma)^*\otimes\mathcal{H}^\pm_\Sigma\big)$$

be the corresponding orthogonal projection maps. Denote by $s^{(m,\pm)}_{b,x}$ the composition $\pi^\pm_x\circ s^{(m,\pm)}_{b,x}$.

Lemma 4.3 *There exists $\delta\in C^\infty(\mathcal{M}_{\mathcal{T}};\mathbb{R}^+)$ such that for all $\upsilon\in F^{(\emptyset)}\mathcal{T}_\delta$, $X\in T_{ev(b_\upsilon)}\mathbb{P}^n$, and $\psi\in\mathcal{H}^{0,1}_\Sigma$,*

$$\big\langle\!\big\langle\pi^{0,1}_{\upsilon,-}\bar\partial u_\upsilon,R_\upsilon X\psi\big\rangle\!\big\rangle_{\upsilon,2}=-\sum_{m\geq1}\sum_{\chi_{\mathcal{T}}h=1}\big\langle\mathcal{D}^{(m)}_{\mathcal{T},h}b_\upsilon,X\big\rangle\overline{\Big(\big\{D^{(m)}_{b_\upsilon,\tilde{x}_h(\upsilon)}\psi\big\}\big((d\phi_{b_\upsilon,x_{\tilde{h}(\mathcal{T})}}|_{\tilde{x}_h(\upsilon)})^{-1}\tilde{\upsilon}_h\big)\Big)}.$$

Furthermore, the sum is absolutely convergent.

Proof: Since $\big\langle\bar\partial u_\upsilon,R_\upsilon X\psi\big\rangle=0$ outside of the annuli $A^-_{\upsilon,h}$ with $\chi_{\mathcal{T}}h=1$,

$$\big\langle\!\big\langle\pi^{0,1}_{\upsilon,-}\bar\partial u_\upsilon,R_\upsilon X\psi\big\rangle\!\big\rangle=\big\langle\!\big\langle\bar\partial u_\upsilon,R_\upsilon X\psi\big\rangle\!\big\rangle=\sum_{\chi_{\mathcal{T}}h=1}\int_{A^-_{\upsilon,h}}\big\langle\bar\partial u_\upsilon,R_\upsilon X\psi\big\rangle. \tag{4.4}$$

Since $q^{-1}_{\upsilon,\iota_h}\circ\phi^{-1}_{b_\upsilon,h}$ is holomorphic on $\tilde{A}^-_{\upsilon,h}$, $\Pi^{-1}_{b_\upsilon,\bar{u}_\upsilon}$ is unitary on $u_\upsilon(A^-_{\upsilon,h})$, and the inner-product of one-forms is conformally invariant,

$$\begin{aligned}\int_{A^-_{\upsilon,h}}\big\langle\bar\partial u_\upsilon,R_\upsilon X\psi\big\rangle&=\int_{\tilde{A}^-_{\upsilon,h}}\big\langle\bar\partial(u_\upsilon\circ q^{-1}_{\upsilon,\iota_h})\circ d\phi^{-1}_{b_\upsilon,h},R_\upsilon X\psi\circ dq^{-1}_{\upsilon,\iota_h}\circ d\phi^{-1}_{b_\upsilon,h}\big\rangle\\&=\int_{\tilde{A}^-_{\upsilon,h}}\big\langle\Pi^{-1}_{b_\upsilon,\bar{u}_\upsilon}\bar\partial(u_\upsilon\circ q^{-1}_{\upsilon,\iota_h})\circ d\phi^{-1}_{b_\upsilon,h},X\psi\circ dq^{-1}_{\upsilon,i_h}\circ\phi^{-1}_{b_\upsilon,h}\big\rangle,\end{aligned} \tag{4.5}$$

since $\Pi^{-1}_{b_\upsilon,\bar{u}_\upsilon} R_\upsilon X\psi = X\psi$ on $A^-_{\upsilon,h}$. If $\iota_h=\hat{0}$, we identify $F^{(0)}_{h,b_\upsilon}=T_{x_h}\Sigma$ with $\mathbb{C}$ in a $g_{b_\upsilon,\hat{0}}$-unitary way. In all cases, we can then write

$$\psi\circ dq^{-1}_{\upsilon,\iota_h}\circ d\phi^{-1}_{b_\upsilon,h}=f\,d\bar{z}.$$

Since ψ is harmonic and $q^{-1}_{\upsilon,\iota_h}\circ\phi^{-1}_{b_\upsilon,h}$ is holomorphic on $\tilde{A}_{\upsilon,h}$, f is anti-holomorphic. Using the change of variables $2|v_h|^{-\frac{1}{2}}z=re^{i\theta}$, we obtain

$$\begin{aligned}
&\int_{\tilde{A}_{\upsilon,h}}\Big\langle |v_h|^{-\frac{1}{2}}\big(1-\beta_{|v_h|}(2|z|)\big)^{m-1}\mathcal{D}^{(m)}_{\mathcal{T},h}\Big(\Big[b_\upsilon,\frac{v_h}{z}\Big]\Big)\bar{\partial}\beta\big|_{2|v_h|^{-\frac{1}{2}}z},\,X\psi\circ dq^{-1}_{\upsilon,\iota_h}\circ d\phi^{-1}_{b_\upsilon,h}\Big\rangle\\
&\quad=\big\langle\mathcal{D}^{(m)}_{\mathcal{T},h}b_\upsilon,X\big\rangle v_h^m\int_{\tilde{A}_{\upsilon,h}}\Big\{\big(1-\beta(2|v_h|^{-\frac{1}{2}}|z|)\big)^{m-1}\beta'\big|_{2|v_h|^{-\frac{1}{2}}|z|}\Big\}|v_h|^{-\frac{1}{2}}z^{-m}\frac{z}{|z|}\bar{f} \qquad (4.6)\\
&\quad=\big\langle\mathcal{D}^{(m)}_{\mathcal{T},h}b_\upsilon,X\big\rangle v_h^m|v_h|^{-\frac{m-1}{2}}2^{m-2}\frac{1}{m}\int_1^2\int_0^{2\pi}\Big\{\big(1-\beta(r)\big)^m\Big\}'(re^{i\theta})^{-(m-1)}\bar{f}\big(\frac{1}{2}|v_h|^{\frac{1}{2}}re^{i\theta}\big)d\theta dr.
\end{aligned}$$

Since $\bar{f}$ is holomorphic, for any $r>0$,

$$\begin{aligned}
\int_0^{2\pi}(re^{i\theta})^{-(m-1)}\bar{f}\big(\frac{1}{2}|v_h|^{\frac{1}{2}}re^{i\theta}\big)d\theta&=-i\int_{|z|=r}z^{-m}\bar{f}\big(\frac{1}{2}|v_h|^{\frac{1}{2}}z\big)dz\\
&=\frac{2\pi}{(m-1)!}\frac{d^{(m-1)}}{dz^{(m-1)}}\bar{f}\big(\frac{1}{2}|v_h|^{\frac{1}{2}}z\big)\Big|_{z=0}=\frac{2\pi}{(m-1)!}2^{-(m-1)}|v_h|^{\frac{m-1}{2}}\bar{f}^{\langle m-1\rangle}(0).
\end{aligned} \qquad (4.7)$$

Since the metric $g_{b,\hat{0}}$ is flat near $\tilde{x}_h$,

$$\begin{aligned}
\frac{\pi}{m!}v_h^m\bar{f}^{\langle m-1\rangle}(0)&=\overline{\big\{D^{(m)}_{b_\upsilon,\tilde{x}_h(\upsilon)}\psi\big\}\big(dq^{-1}_{\upsilon,\iota_h}\big|_{x_h}d\phi^{-1}_{b_\upsilon,h}\big|_0 v_h\big)}\\
&=\overline{\big\{D^{(m)}_{b_\upsilon,\tilde{x}_h(\upsilon)}\psi\big\}\big((d\phi_{b_\upsilon,x_{\tilde{h}(\mathcal{T})}}|_{\tilde{x}_h(\upsilon)})^{-1}\tilde{v}_h\big)}.
\end{aligned} \qquad (4.8)$$

The claim follows from equations (4.4)-(4.8) and Lemma 4.2.

Proposition 4.4 *If $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ is a simple bubble type with $d_{\hat{0}}=0$, there exists $\delta\in C^\infty(\mathcal{M}_\mathcal{T};\mathbb{R}^+)$ such that*

$$\pi^{0,1}_{\upsilon,-}\bar{\partial}u_\upsilon=-\tilde{R}_\upsilon\sum_{m\ge1}\sum_{\chi_\mathcal{T}h=1}\big(\mathcal{D}^{(m)}_{\mathcal{T},h}b\big)\Big(s^{(m)}_{b,\tilde{x}_h(\upsilon)}\big(d\phi_{b,x_{\tilde{h}(\mathcal{T})}}|^{-1}_{\tilde{x}_h(\upsilon)}\tilde{v}_h\big)\Big)\quad\forall\ \upsilon=\big[b,(v_h)_{h\in\hat{I}}\big]\in F^\emptyset\mathcal{T}_\delta.$$

Furthermore, the sum is absolutely convergent.

Proof: This proposition follows from Lemma 4.3 and equation (2.14).

4.2 First-Order Estimate for $\psi^\mu_{\mathcal{T},t\nu}$

If $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ is a bubble type as before, we denote by $\chi(\mathcal{T})$ the subset of elements h of I such that $\chi_\mathcal{T}h=1$. For any $\upsilon\in F\mathcal{T}$ and $h\in\chi(\mathcal{T})$, let

$$\alpha^{(k)}_{\mathcal{T},h}(\upsilon)=\big(\mathcal{D}^{(k)}_{\mathcal{T},h}b_\upsilon\big)s^{(k)}_{b_\upsilon,x_{\tilde{h}(\mathcal{T})}}(\tilde{v}_h),\qquad \alpha^{(k)}_\mathcal{T}(\upsilon)=\sum_{h\in\chi(\mathcal{T})}\alpha^{(k)}_{\mathcal{T},h}(\upsilon),$$
$$\text{if}\qquad \upsilon=\big[(\Sigma,[N],I;x,(j,y),u),(v_h)_{h\in\hat{I}}\big].$$

We denote $\alpha^{(1)}_{\mathcal{T},h}$ and $\alpha^{(1)}_\mathcal{T}$ by $\alpha_{\mathcal{T},h}$ and $\alpha_\mathcal{T}$, respectively.

Lemma 4.5 *There exist $\delta, C \in C^\infty(\mathcal{M}_\mathcal{T};\mathbb{R}^+)$ such that for all $\upsilon \in F^\emptyset\mathcal{T}_\delta$,*

$$\big\|\pi^{0,1}_{\upsilon,-}\bar{\partial}u_\upsilon + \tilde{R}_\upsilon\alpha_\mathcal{T}(\upsilon)\big\|_2 \le C(b_\upsilon)|\upsilon|\sum_{h\in\chi(\mathcal{T})}|\upsilon|_h.$$

Proof: This is immediate from Proposition 4.4, since

$$\big\|s_{\tilde{x}_h(\upsilon)}\big(d\phi_{b_\upsilon,x_{\tilde{h}(\mathcal{T})}}|^{-1}_{\tilde{x}_h(\upsilon)}\tilde{\upsilon}_h\big) - s_{x_{\tilde{h}(\mathcal{T})}}(\tilde{\upsilon}_h)\big\|_2 \le C(b_\upsilon)\big|\phi_{b_\upsilon,x_{\tilde{h}(\mathcal{T})}}\tilde{x}_h(\upsilon)\big|_{b_\upsilon}\big|\tilde{\upsilon}_h\big| \le C'(b_\upsilon)|\upsilon|\big|\tilde{\upsilon}_h\big|_b;$$
$$\sum_{m\ge 2}\big|\mathcal{D}^{(m)}_{\mathcal{T},h}b_\upsilon\big||\tilde{\upsilon}_h|^m \le C(b_\upsilon)|\tilde{\upsilon}_h|^2,$$

for all $h\in\hat{I}$ with $\chi_\mathcal{T}h=1$ and $\upsilon\in F\mathcal{T}_\delta$ with $\delta\in C^\infty(\mathcal{M}_\mathcal{T};\mathbb{R}^+)$ sufficiently small.

Lemma 4.6 *There exist $\delta, C\in C^\infty(\mathcal{M}_\mathcal{T}(\mu);\mathbb{R}^+)$ such that for all $\upsilon\in F^\emptyset\mathcal{T}_\delta$,*

$$\Big\|\psi^\mu_{\mathcal{T},t\nu}(\upsilon) - \big(t\bar{\nu}_{ev(b_\upsilon)}+\alpha_\mathcal{T}(\upsilon)\big)\Big\|_2 \le C(b_\upsilon)\big(t+|\upsilon|^{\frac{1}{p}}\big)\Big(t+\sum_{h\in\chi(\mathcal{T})}|\upsilon|_h\Big),$$

where $\psi^\mu_{\mathcal{T},t\nu}$ denotes $\psi^\mu_{\mathcal{M}_\mathcal{T},t\nu}$.

Proof: By Lemma 2.2 and Theorem 2.7,

$$\big\|\pi^{0,1}_{\upsilon,-}D_\upsilon\xi_{\upsilon,t\nu}\big\|_2 \le C(b_\upsilon)\Big(\sum_{h\in\chi(\mathcal{T})}|\upsilon|_h\Big)\|D_\upsilon\xi_{\upsilon,t\nu}\|_{\upsilon,p,1} \le C'(b_\upsilon)\big(t+|\upsilon|^{\frac{1}{p}}\big)\sum_{h\in\chi(\mathcal{T})}|\upsilon|_h.$$

Combining this estimate with Lemmas 4.1 and 4.5, we obtain

$$\Big\|\psi_{\mathcal{T},t\nu}(\upsilon) - \big(t\bar{\nu}_{\mathrm{ev}(b_\upsilon)}+\alpha_\mathcal{T}(\upsilon)\big)\Big\|_2 \le C(b_\upsilon)\big(t+|\upsilon|^{\frac{1}{p}}\big)\Big(t+\sum_{h\in\chi(\mathcal{T})}|\upsilon|_h\Big) \tag{4.9}$$

for all $\upsilon\in F^\emptyset\mathcal{T}_\delta$, provided $\delta\in C^\infty(\mathcal{M}_\mathcal{T};\mathbb{R}^+)$ is sufficiently small. On the other hand, if $b_\upsilon\in\mathcal{M}_\mathcal{T}(\mu)$,

$$\begin{aligned}&\big\|\varphi^\mu_{\mathcal{T},t\nu}(\upsilon)\big\|_{b_\upsilon} \le C(b_\upsilon)\big(t+|\upsilon|^{\frac{1}{p}}\big) \Longrightarrow\\ &\Big\|\big(t\bar{\nu}_{\mathrm{ev}(\phi^\mu_\mathcal{T}\varphi^\mu_{\mathcal{T},t\nu}(\upsilon))}+\alpha_\mathcal{T}(\Phi^\mu_\mathcal{T}\varphi^\mu_{\mathcal{T},t\nu}(\upsilon))\big) - \Pi_{b_\upsilon,\phi^\mu_\mathcal{T}\varphi^\mu_{\mathcal{T},t\nu}(\upsilon)}\big(t\bar{\nu}_{\mathrm{ev}(b_\upsilon)}+\alpha_\mathcal{T}(\upsilon)\big)\Big\|_2\\ &\qquad\qquad \le C(b_\upsilon)\big(t+|\upsilon|^{\frac{1}{p}}\big)\Big(t+\sum_{h\in\chi(\mathcal{T})}|\upsilon|_h\Big),\end{aligned} \tag{4.10}$$

where $\varphi^\mu_{\mathcal{T},t\nu}=\varphi^\mu_{\mathcal{M}_\mathcal{T},t\nu}$ is the section of Theorem 2.7 for any fixed regularization $\big(\Phi_\mathcal{T}\equiv Id,\Phi^\mu_\mathcal{T}\big)$ of $\mathcal{M}_\mathcal{T}(\mu)$. The claim follows from (4.9) and (4.10).

Our next step is to apply Lemma 3.2 or Corollary 3.6 to the map $\psi^\mu_{\mathcal{T},t\nu}$ whenever possible. In terms of notation of Subsection 3.1, we take

$$F^+=\mathcal{O}^+=\{0\},\qquad F^-=F\mathcal{T},\qquad \mathcal{O}^-=\mathcal{H}^{0,1}_\Sigma\otimes\mathrm{ev}^*T\mathbb{P}^n,\qquad \tilde{F}^-=\bigoplus_{h\in\chi(\mathcal{T})}\bigotimes_{i\in\hat{I},i\le h}F_i\mathcal{T};$$
$$\phi_h\big([b,\upsilon_{\hat{I}}]\big)=\Big[b,\bigotimes_{i\in\hat{I},i\le h}\upsilon_i\Big]=\big[b,\tilde{\upsilon}_h\big],\quad \alpha^-(\phi(\upsilon))\equiv\alpha_\mathcal{T}(\upsilon),$$

where ϕ_h denotes the hth component of $\phi: F^- \longrightarrow \tilde{F}^-$. Note that $\alpha^- \in \Gamma(\mathcal{M}_\mathcal{T}; \tilde{F}^{-*} \otimes \mathcal{O}^-)$ is well-defined. A priori, α^- may not have full rank on every fiber over $\mathcal{M}_\mathcal{T}(\mu)$. We will call a subset $K \subset \mathcal{M}_\mathcal{T}(\mu)$ *$\mathcal{T}$-regular* if α^- has full rank over K. From Theorem 2.7, Lemma 3.2, and Corollary 3.6, we then obtain

Corollary 4.7 *Suppose d is a positive integer, $\mathcal{T} = (\Sigma, [N], I; j_{[N]}, \underline{d})$ is a simple bubble type, with $d_{\hat{0}} = 0$ and $\sum_{i \in I} d_i = d$, and μ is an N-tuple of constraints in general position such that*

$$\mathit{codim}_\mathbb{C}\mu = d(n+1) - n(g-1) + N.$$

Let $\nu \in \Gamma(\Sigma \times \mathbb{P}^n; \Lambda^{0,1}\pi_\Sigma^* T^*\Sigma \otimes \pi_{\mathbb{P}^n}^* T\mathbb{P}^n)$ be a generic section. If $\iota_h \neq \hat{0}$ for some $h \in \hat{I}$, for every regular compact subset K of $\mathcal{M}_\mathcal{T}(\mu)$, there exist a neighborhood U_K of K in $\bar{C}^\infty_{(d;[N])}(\Sigma; \mu)$ and $\epsilon_K > 0$ such that for any $t \in (0, \epsilon_K)$, $U_K \cap \mathcal{M}_{\Sigma,d,t\nu}(\mu) = \emptyset$. If $\iota_h = \hat{0}$ for all $h \in \hat{I}$, there exists a compact regular subset $K_\mathcal{T}$ of $\mathcal{M}_\mathcal{T}(\mu)$ with the following property. If K is a compact regular subset of $\mathcal{M}_\mathcal{T}(\mu)$ containing $K_\mathcal{T}$, there exist a neighborhood U_K of K in $\bar{C}^\infty_{(d;[N])}(\Sigma; \mu)$ and $\epsilon_K > 0$ such that for all $t \in (0, \epsilon_K)$, the signed cardinality of $U_K \cap \mathcal{M}_{\Sigma,d,t\nu}(\mu)$ equals to the signed number of zeros of the map

$$F\mathcal{T}\big|_{\mathcal{M}_\mathcal{T}(\mu)} \longrightarrow \mathcal{H}^{0,1}_\Sigma \otimes ev^* T\mathbb{P}^n, \quad \upsilon \longrightarrow \bar{\nu}_{ev(b_\upsilon)} + \alpha_\mathcal{T}(\upsilon). \tag{4.11}$$

Proof: In either case, by Theorem 2.7, there exist a neighborhood U_K of K in $\bar{C}^\infty_{(d;[N])}(\Sigma; \mu)$ and $\delta_K, \epsilon_K > 0$ such that for any $t \in (0, \epsilon_K)$, there exists a sign-preserving bijection between $U_K \cap \mathcal{M}_{\Sigma,d,t\nu}(\mu)$ and the zeros of $\psi^\mu_{\mathcal{T},t\nu}$ on $F^\emptyset\mathcal{T}_{\delta_K}|_{U_K \cap \mathcal{M}_\mathcal{T}(\mu)}$, provided $U_K \cap \mathcal{M}_\mathcal{T}(\mu)$ is precompact in $\mathcal{M}_\mathcal{T}(\mu)$. Furthermore, δ_K can be required to be arbitrarily small. If K is regular, U_K can be chosen so that the closure of $U_K \cap \mathcal{M}_\mathcal{T}(\mu)$ in $\mathcal{M}_\mathcal{T}(\mu)$ is also regular. Then by Lemma 4.6,

$$\left\| \psi^\mu_{\mathcal{T},t\nu}(\upsilon) - \big(t\bar{\nu}_{\mathrm{ev}(b_\upsilon)} + \alpha_\mathcal{T}(\upsilon)\big) \right\|_2 \le C_K\big(t + |\upsilon|^{\frac{1}{p}}\big)\big(t + |\alpha_\mathcal{T}(\upsilon)|\big) \quad \forall \upsilon \in F^\emptyset\mathcal{T}_{\delta_K}\big|_K,$$

where $C_K > 0$ depends only on K. Thus, the first claim follows from Lemma 3.2. The second follows from Corollary 3.6, provided that for a generic ν the set of zeros of the map in (4.11) is $\mathcal{T}$-regular and finite; see below.

The affine maps of Corollaries 4.7, 4.14, 4.18, and 4.22 extend over the natural compactifications of the spaces $\mathcal{M}_\mathcal{T}(\mu)$ and $\mathcal{S}_{\mathcal{T};k}(\mu)$ described in Subsection 4.9. Along with counting the zeros of these affine maps in Section 5, we also show that the linear part of each of the affine maps is regular in the sense of Definition 3.9. Thus, by Lemma 3.10 these affine maps have a finite numbers of transverse zeros, which must lie over the subspace of the base where the linear part of the affine map has full rank.

4.3 Consequences of the First-Order Estimate for $\psi^\mu_{\mathcal{T},t\nu}$

In this subsection, we show that $\mathcal{M}_\mathcal{T}(\mu)$ is $\mathcal{T}$-regular for most bubble types $\mathcal{T}$ under consideration, and nearly all of them fall under the first case of Corollary 4.7. We call $\mathcal{T}$ *effective*, if for some generic choice of ν and of the constraints $\mu_1, \ldots, \mu_N$, $\overline{\bigcup_{t<1} \mathcal{M}_{\Sigma,t\nu,d}(\mu)}$ intersects $\bar{\mathcal{M}}_\mathcal{T}(\mu)$. If K is a compact subset of $\bar{\mathcal{M}}_\mathcal{T}(\mu)$, we call K *effective* if $\overline{\bigcup_{t<1} \mathcal{M}_{\Sigma,t\nu,d}(\mu)}$ intersects K.

Lemma 4.8 *Let $\mathcal{T} = (\Sigma, [N], I; j, \underline{d})$ be a simple bubble type. If $j_l = \hat{0}$ for some $l \in [N]$ and K is a $\mathcal{T}$-regular subset of $\mathcal{M}_\mathcal{T}(\mu)$, then K is not effective.*

Proof: By Corollary 4.7, it is sufficient to show that the map

$$\bar{\nu}+\alpha_{\mathcal{T}}\colon F\mathcal{T} \longrightarrow \mathcal{H}_{\Sigma}^{0,1}\otimes \mathrm{ev}^*T\mathbb{P}^n$$

has no zeros for a generic ν. For a generic ν, the zero set of this section is zero-dimensional. However, if $j_l=\hat{0}$ for some $l\in[N]$, we can move $y_l\in\Sigma$ freely, without changing the value of $\bar{\nu}+\alpha_{\mathcal{T}}$. Thus, if the zero-set of the section is nonempty, it must be at least one-dimensional, which is not the case for a generic ν.

Lemma 4.9 *Let $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ be a bubble type with $d_{\hat{0}}=0$. If*

$$ng-|\hat{I}|-\Big(|H_{\hat{0}}\mathcal{T}|+|M_{\hat{0}}\mathcal{T}|+\sum_{i\in\hat{I},d_i=0}\big(|H_i\mathcal{T}|+|M_i\mathcal{T}|-2\big)\Big)\le n-|\chi(\mathcal{T})|,$$

$\mathcal{M}_{\mathcal{T}}(\mu)$ is $\mathcal{T}$-regular. Furthermore, if the number on the left-hand side above is negative, then $\mathcal{M}_{\mathcal{T}}(\mu)$ is empty.

Proof: (1) The dimension of $\mathcal{M}(\mu)$ is given by

$$\dim\mathcal{M}_{\mathcal{T}}(\mu)=\big(d(n+1)+n+N-|\hat{I}|\big)-\big(d(n+1)-n(g-1)+N\big)=ng-|\hat{I}|.$$

However, given $b=\big(\Sigma,[N],I;x,(j,y),u\big)\in\mathcal{M}_{\mathcal{T}}(\mu)$, we are free to vary x_h if $\iota_h=\hat{0}$ (i.e. $x_h\in\Sigma$) and y_l if $j_l=\hat{0}$. Similarly, if $i\in\hat{I}$, $d_i=0$, and $|H_i\mathcal{T}|+|M_i\mathcal{T}|>2$, we can vary $|H_i\mathcal{T}|+|M_i\mathcal{T}|-2$ marked and singular points on $\Sigma_{b,i}$. Thus, the space $\mathcal{M}_{\mathcal{T}}(\mu)$ must have dimension at least

$$d_{\min}(\mathcal{T})\equiv|H_{\hat{0}}\mathcal{T}|+|M_{\hat{0}}\mathcal{T}|+\sum_{i\in\hat{I},d_i=0}\big(|H_i\mathcal{T}|+|M_i\mathcal{T}|-2\big),$$

if $\mathcal{M}_{\mathcal{T}}(\mu)$ is nonempty. Therefore, we can assume $|\chi(\mathcal{T})|\le n$.
(2) Let $h_1,\ldots,h_{|\chi(\mathcal{T})|}$ be the elements of $\chi(\mathcal{T})$. The section $s_{\Sigma}\in\Gamma(\Sigma;T^*\Sigma\otimes\mathcal{H}_{\Sigma}^{0,1})$ does not vanish; see [GH, p246]. Thus, the section α^- defined above has rank at least k if the section

$$\bar{\mathcal{D}}_{\mathcal{T};k}\in\Gamma\Big(\mathcal{M}_{\mathcal{T}}(\mu);\Big(\bigoplus_{m\le k}L_{h_m}^*\mathcal{T}\Big)\otimes\mathrm{ev}^*T\mathbb{P}^n\Big),\quad \bar{\mathcal{D}}_{\mathcal{T};k}\left([b,c_{\{h_m:m\le k\}}]\right)=\sum_{m\le k}\mathcal{D}_{\mathcal{T},h_m}\left([b,c_{h_m}]\right),$$

has rank k. We prove inductively that under the assumptions of the lemma this is the case for all $k\le|\chi(\mathcal{T})|$. If $k=0$, there is nothing to prove. So we can assume that $k>0$ and that the statement has been shown to be true for $k-1$. The $k-1$ statement shows that the image of $\bar{\mathcal{D}}_{\mathcal{T};k-1}$ is a rank $k-1$ subbundle of $\mathrm{ev}^*T\mathbb{P}^n$. Let $\pi_{k-1}^{\perp}$ denote the orthogonal projection onto the orthogonal complement of this rank $(k-1)$-subbundle in $\mathrm{ev}^*T\mathbb{P}^n$ with respect to the standard metric in $\mathbb{P}^n$. We need to show that the section

$$\pi_{k-1}^{\perp}\circ\mathcal{D}_{\mathcal{T};k}\in\Gamma\big(\mathcal{M}_{\mathcal{T}}(\mu);L_{h_k}\mathcal{T}^*\otimes\pi_{k-1}^{\perp}(\mathrm{ev}_{\mathcal{T}}^*T\mathbb{P}^n)\big)$$

does not vanish. By Corollary 6.3, $\pi_{k-1}^{\perp}\circ\mathcal{D}_{\mathcal{T};k}$ is transverse to zero for a generic choice of the constraints $\mu_1,\ldots,\mu_N$. Its zero set must have dimension at least $d_{\min}(\mathcal{T})$, if nonempty, since the movements of points described in (1) do not effect $\pi_{k-1}^{\perp}\circ\mathcal{D}_{\mathcal{T};k}$. Thus, $\pi_{k-1}^{\perp}\circ\mathcal{D}_{\mathcal{T},k}$ does not vanish if

$$\dim(\mathcal{M}_{\mathcal{T}}(\mu))-d_{\min}(\mathcal{T})<n-(k-1).$$

By the assumption of the lemma, this is the case as long as $k\le|\chi(\mathcal{T})|$.

Figure 1: The Two Possibilities for $\mathcal{T}$ of Corollary 4.10

Corollary 4.10 *Let* $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ *be an effective bubble type with* $d_{\hat{0}}=0$*. If* $g=2$ *and* $n=2$*, then either*
(1) $|\hat{I}|=1$ *and* $j_l\neq\hat{0}$ *for all* $l\in[N]$*, or*
(2) $|\hat{I}|=2$*,* $H_{\hat{0}}\mathcal{T}=\hat{I}$*, and* $j_l\neq\hat{0}$ *for all* $l\in[N]$*.*
Furthermore, in Case (2) $\alpha_{\mathcal{T}}$ *has full rank over all of* $\mathcal{M}_{\mathcal{T}}(\mu)$*.*

We illustrate the statement of Corollary 4.10 in Figure 1. We represent each of the potentially effective bubble types $\mathcal{T}$ by the domain of any stable map in the space $\mathcal{M}_{\mathcal{T}}(\mu)$. Each disk represents a sphere. We shade the component(s) of the domain on which any (or every) map in $\mathcal{M}_{\mathcal{T}}(\mu)$ is nonconstant. The labels d, $d_{\hat{1}}$, and $d_{\hat{2}}$ indicate the degree of the map on each of the bubble components; we must have $d_{\hat{1}}+d_{\hat{2}}=d$. In the case of Figure 1, all marked points must be distributed between the shaded components of the domain.

Due to Corollary 4.10, Corollary 4.7 describes topologically the number of elements of the set $\mathcal{M}_{\Sigma,d,t\nu}(\mu)$ that lie near a compact subset K of $\mathcal{M}_{\Sigma,d,0}(\mu)$, provided K is disjoint from the space

$$\mathcal{S}_{\mathcal{T},1}(\mu)\equiv\alpha_{\mathcal{T}}^{-1}(0)\subset\mathcal{M}_{\mathcal{T}}(\mu),$$

where $\mathcal{T}$ is the bubble type specified by (1) in Corollary 4.10 and by the first diagram in Figure 1. By definition of $\alpha_{\mathcal{T}}$, the set $\mathcal{S}_{\mathcal{T},1}(\mu)$ consists of the elements of $\mathcal{M}_{\mathcal{T}}(\mu)$ such that the differential of the bubble map at the attaching node is zero, i.e. the corresponding rational curve in $\mathbb{P}^2$ has a cusp at the image of Σ. Determining the number of elements of $\mathcal{M}_{\Sigma,d,t\nu}(\mu)$ that lie near $\mathcal{S}_{\mathcal{T},1}(\mu)$ requires higher-order estimates. In Subsection 4.4, we determine the number of elements of $\mathcal{M}_{\Sigma,d,t\nu}(\mu)$ that lie near a compact subset K of $\mathcal{S}_{\mathcal{T},1}(\mu)$ such that for no element of K the corresponding singular point on Σ is one of the six hyperelliptic points of Σ. Finally, in Subsection 4.5, we determine the number of elements of $\mathcal{M}_{\Sigma,d,t\nu}(\mu)$ that lie near the subset K of $\mathcal{S}_{\mathcal{T},1}(\mu)$ such that for every element of K the corresponding singular point on Σ is a hyperelliptic point of Σ.

Proof of Corollary 4.10: (1) By Lemma 4.9, $\mathcal{M}_{\mathcal{T}}(\mu)$ is empty, unless $ng-|\hat{I}|\geq 1$, i.e. $|\hat{I}|\leq 3$. Suppose $|\hat{I}|=3$. If $|H_{\hat{0}}\mathcal{T}|\geq 2$,

$$ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|\leq 4-3-2<0,$$

and thus $\mathcal{M}_{\mathcal{T}}(\mu)$ is empty by Lemma 4.9. If $|H_{\hat{0}}\mathcal{T}|=1$,

$$n-|\chi(\mathcal{T})|\geq 2-(|\hat{I}|-1)=0=ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|,$$

and by Lemma 4.9 the space $\mathcal{M}_{\mathcal{T}}(\mu)$ is $\mathcal{T}$-regular. The space $\mathcal{M}_{\mathcal{T}}(\mu)$ is compact, since by the above $\mathcal{M}_{\mathcal{T}'}(\mu)=\emptyset$ if $\mathcal{T}'<\mathcal{T}$. Corollary 4.7 then implies that $\mathcal{M}_{\mathcal{T}}(\mu)$ is not effective, i.e. $\mathcal{T}$ is not

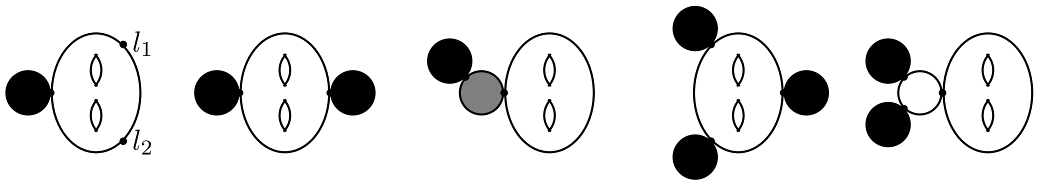

Figure 2: The Five Possibilities for $\mathcal{T}$ of Corollary 4.11

effective.
(2) Suppose $|\hat{I}|=2$. If $|H_{\hat{0}}\mathcal{T}|=2$ and $j_l=\hat{0}$ for some $l\in[N]$,

$$ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|-|M_{\hat{0}}\mathcal{T}|\leq 4-2-2-1<0,$$

and thus $\mathcal{M}_{\mathcal{T}}(\mu)$ is empty by Lemma 4.9. If $|H_{\hat{0}}\mathcal{T}|=1$,

$$n-|\chi(\mathcal{T})|=2-1=ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|,$$

and it follows from Lemma 4.9 and Corollary 4.7, that every compact subset of $\mathcal{M}_{\mathcal{T}}(\mu)$ is not effective. Furthermore, $\bar{\mathcal{M}}_{\mathcal{T}}(\mu)-\mathcal{M}_{\mathcal{T}}(\mu)$ consists of three-bubble strata, all of which are not effective by (1) above. Thus, $\mathcal{T}$ is not effective, unless $\iota_h=\hat{0}$ for all $h\in\hat{I}$ and $j_l\neq\hat{0}$ for all $l\in[N]$. The second statement about the $|\hat{I}|=2$ case is immediate from Lemma 4.9.
(3) Finally, suppose $|\hat{I}|=1$ and $j_l=\hat{0}$ for some $l\in[N]$. Then,

$$n-|\chi(\mathcal{T})|=2-1\geq ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|-|M_{\hat{0}}\mathcal{T}|,$$

and thus by Lemmas 4.8 and 4.9, every compact subset of $\mathcal{M}_{\mathcal{T}}(\mu)$ is not effective. Furthermore, $\bar{\mathcal{M}}_{\mathcal{T}}(\mu)-\mathcal{M}_{\mathcal{T}}(\mu)$ consists of two- and three-bubble strata that by (1) and (2) are not effective. It follows that $\mathcal{T}$ is not effective.

Corollary 4.11 *Let $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ be an effective bubble type with $d_{\hat{0}}=0$. If $g=2$ and $n=3$, then either*
(1) $|\hat{I}|=1$, or
(2a) $|\hat{I}|=2$, $H_{\hat{0}}\mathcal{T}=\hat{I}$, and $j_l\neq\hat{0}$ for all $l\in[N]$, or
(2b) $|\hat{I}|=2$, $H_{\hat{0}}\mathcal{T}\neq\hat{I}$, and $j_l\neq\hat{0}$ for all $l\in[N]$, or
(3a) $|\hat{I}|=3$, $H_{\hat{0}}\mathcal{T}=\hat{I}$, and $j_l\neq\hat{0}$ for all $l\in[N]$, or
(3b) $|\hat{I}|=3$, $\iota_h=\hat{1}$ for some $\hat{1}\in\hat{I}$ and all $h\in\hat{I}-\{\hat{1}\}$, $d_{\hat{1}}=0$, and $j_l\neq\hat{0},\hat{1}$ for all $l\in[N]$.
Furthermore, in Case (3a) $\alpha_{\mathcal{T}}$ has full rank on all of $\mathcal{M}_{\mathcal{T}}(\mu)$.

We illustrate the statement of Corollary 4.11 in Figure 2, using the same conventions as in Figure 1. In the first case, the genus-two Riemann surface Σ may carry some of the marked points. In the remaining four cases, all of the marked points are distributed between the shaded components. In the third diagram, the lightly shaded disk indicates that the restriction of the maps in $\mathcal{M}_{\mathcal{T}}(\mu)$ to the corresponding bubble component may or may not be constant. In the former case, this component must carry at least one marked point.

By the last remark of Corollary 4.11, Corollary 4.7 describes topologically the number of elements of the set $\mathcal{M}_{\Sigma,d,t\nu}(\mu)$ that lie near a compact subset K of $\mathcal{M}_{\mathcal{T}}(\mu)$ for any bubble type $\mathcal{T}$ as in (3a) of Corollary 4.11 and in the fourth diagram in Figure 2. If $\mathcal{T}$ is a bubble type as in (1) or (2b) of Corollary 4.11 and in the first or third diagram of Figure 2, respectively, Corollary 4.7 describes the number of elements of $\mathcal{M}_{\Sigma,d,t\nu}(\mu)$ that lie near a compact subset K of $\mathcal{M}_{\mathcal{T}}(\mu)$, provided K is disjoint from the space

$$\mathcal{S}_{\mathcal{T},1}(\mu) \equiv \alpha_{\mathcal{T}}^{-1}(0) \subset \mathcal{M}_{\mathcal{T}}(\mu).$$

The elements of $\mathcal{S}_{\mathcal{T},1}(\mu)$ are characterized geometrically in exactly the same way as in the $n=2$ case above. As in the $n=2$ case, we give a topological description for the number of elements of $\mathcal{M}_{\Sigma,d,t\nu}(\mu)$ that lie near a compact subset K of $\mathcal{S}_{\mathcal{T},1}(\mu)$ in Subsections 4.4 and 4.5.

If $\mathcal{T}$ is a bubble type as in (2a) or (3b) of Corollary 4.11 and in the second or last diagram of Figure 2, respectively, Corollary 4.7 describes the number of elements of $\mathcal{M}_{\Sigma,d,t\nu}(\mu)$ that lie near a compact subset K of $\mathcal{M}_{\mathcal{T}}(\mu)$, provided K is disjoint from the space

$$\mathcal{S}_{\mathcal{T},2}(\mu) \equiv \alpha_{\mathcal{T}}^{-1}(0) \subset \mathcal{M}_{\mathcal{T}}(\mu).$$

As discussed in the first paragraph of Subsection 4.6, in the first case $\mathcal{S}_{\mathcal{T},2}(\mu)$ consists of the stable maps in $\mathcal{M}_{\mathcal{T}}(\mu)$ such that the image of the differentials at the attaching nodes of the two bubble components is the same complex line and the two singular points on Σ are conjugates. The first condition means that the two rational curves form a tacnode at the image of Σ in $\mathbb{P}^3$. For $\mathcal{T}$ as in (2a) of Corollary 4.11 and in the second diagram of Figure 2, we determine the number of elements of $\mathcal{M}_{\Sigma,d,t\nu}(\mu)$ that lie near a compact subset K of $\mathcal{S}_{\mathcal{T},2}(\mu)$ in Subsection 4.6. Finally, if $\mathcal{T}$ is as in (3b) of Corollary 4.11 and in the last diagram of Figure 2, $\mathcal{S}_{\mathcal{T},2}(\mu)$ consists of the stable maps in $\mathcal{M}_{\mathcal{T}}(\mu)$ such that the image of the differentials at the attaching nodes of the two shaded bubble components is the same complex line. In Subsection 4.7, we determine the number of elements of $\mathcal{M}_{\Sigma,d,t\nu}(\mu)$ that lie near a compact subset K of $\mathcal{S}_{\mathcal{T},2}(\mu)$ such that for no element of K the corresponding singular point on Σ is one of the six hyperelliptic points of Σ. In Subsection 4.8, we determine the number of elements of $\mathcal{M}_{\Sigma,d,t\nu}(\mu)$ that lie near the subset K of $\mathcal{S}_{\mathcal{T},2}(\mu)$ such that for every element of K the corresponding singular point on Σ is a hyperelliptic point of Σ. We eventually find that only the simplest possible bubble types are effective: that in the first diagram of Figure 2 with no marked points on Σ and those in the second and fourth diagrams in Figure 2; see Subsection 4.9.

Proof of Corollary 4.11: (1) Similarly to the proof of Corollary 4.10, $\mathcal{M}_{\mathcal{T}}(\mu)$ is empty unless $|\hat{I}|\le 5$. If $|\hat{I}|=5$, $\mathcal{M}_{\mathcal{T}}(\mu)$ is compact and $|H_{\hat{0}}\mathcal{T}|=1$. Let $\hat{1}\in\hat{I}$ be such that $\iota_{\hat{1}}=\hat{0}$. If $d_{\hat{1}}>0$,

$$n-|\chi(\mathcal{T})| = 3-1 > 0 = ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|,$$

and $\mathcal{M}_{\mathcal{T}}(\mu)$ is not effective by Lemma 4.9 and Corollary 4.7. Suppose $d_{\hat{1}}=0$. Then $|H_{\hat{1}}\mathcal{T}|\ge 2$; otherwise $\mathcal{M}_{\mathcal{T}}(\mu)$ is empty by Lemma 4.9. It follows that

$$n-|\chi(\mathcal{T})| \ge 3-(|\hat{I}|-2) = 0 = ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|.$$

Thus, by Lemma 4.9 and Corollary 4.7, $\mathcal{T}$ is not effective.
(2) Suppose $|\hat{I}|=4$. If $|H_{\hat{0}}\mathcal{T}|\ge 3$, $\mathcal{M}_{\mathcal{T}}(\mu)$ is empty by Lemma 4.9. Let $\hat{1}\in\hat{I}$ be as above. If $|H_{\hat{0}}\mathcal{T}|=2$,

$$n-|\chi(\mathcal{T})| \ge 3-(|\hat{I}|-1) = 0 = ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|.$$

If $|H_{\hat{0}}\mathcal{T}|=1$ and $d_{\hat{1}}>0$,

$$n-|\chi(\mathcal{T})| = 3-1 > 1 = ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|.$$

If $|H_{\hat{0}}\mathcal{T}|=1$, $d_{\hat{1}}=0$, and $|H_{\hat{1}}\mathcal{T}|=3$,

$$n-|\chi(\mathcal{T})|\geq 3-(|\hat{I}|-1)=0=ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|-(|H_{\hat{1}}\mathcal{T}|-2).$$

Finally, if $|H_{\hat{0}}\mathcal{T}|=1$, $d_{\hat{1}}=0$, and $|H_{\hat{1}}\mathcal{T}|=2$,

$$n-|\chi(\mathcal{T})|\geq 3-(|\hat{I}|-2)=1=ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|.$$

Thus, by Corollary 4.7 and Lemma 4.9, in all four cases, no compact subset of $\mathcal{M}_{\mathcal{T}}(\mu)$ is effective. Since $\bar{\mathcal{M}}_{\mathcal{T}}(\mu)-\mathcal{M}_{\mathcal{T}}(\mu)$ consists of five-bubble strata that are not effective by (1) above, it follows that $\mathcal{T}$ is not effective.
(3) Suppose $|\hat{I}|=3$. If $H_{\hat{0}}\mathcal{T}=\hat{I}$ and $j_l=\hat{0}$ for some $l\in[N]$,

$$ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|-|M_{\hat{0}}\mathcal{T}|=6-3-3-1<0,$$

and thus $\mathcal{M}_{\mathcal{T}}(\mu)$ is empty by Lemma 4.9. If $|H_{\hat{0}}\mathcal{T}|=2$,

$$n-|\chi(\mathcal{T})|\geq 3-(|\hat{I}|-1)=1\geq ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|.$$

If $|H_{\hat{0}}\mathcal{T}|=1$ and $d_{\hat{1}}>0$,

$$n-|\chi(\mathcal{T})|=2=ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|.$$

If $|H_{\hat{0}}\mathcal{T}|=1$ and $|H_{\hat{1}}\mathcal{T}|=1$,

$$n-|\chi(\mathcal{T})|=2=ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|.$$

Thus, in all three cases, by Lemma 4.9 and Corollary 4.7, no compact subset of $\mathcal{M}_{\mathcal{T}}(\mu)$ is effective. Since $\bar{\mathcal{M}}_{\mathcal{T}}(\mu)-\mathcal{M}_{\mathcal{T}}(\mu)$ consists of four- and five-bubble strata that are not effective by (1) and (2) above, $\mathcal{T}$ is not effective in these three cases. On the other hand, if $|H_{\hat{0}}\mathcal{T}|=2$, $j_l=\hat{0}$ or $j_l=\hat{1}$ for some $l\in[N]$, and $d_{\hat{1}}=0$,

$$n-|\chi(\mathcal{T})|\geq 1\geq ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|-|M_{\hat{0}}\mathcal{T}|-\big(|H_{\hat{1}}\mathcal{T}|+|M_{\hat{1}}\mathcal{T}|-2\big).$$

Thus, by Lemmas 4.8 and 4.9, no compact subset of $\mathcal{M}_{\mathcal{T}}(\mu)$ is effective. Similarly to the above, it follows that $\mathcal{T}$ is not effective.
(4) Suppose $|\hat{I}|=2$ and $j_l=\hat{0}$ for some $l\in[N]$. If $|H_{\hat{0}}\mathcal{T}|=2$,

$$n-|\chi(\mathcal{T})|\geq 1\geq ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|-|M_{\hat{0}}\mathcal{T}|.$$

If $|H_{\hat{0}}\mathcal{T}|=1$,

$$n-|\chi(\mathcal{T})|=2\geq ng-|\hat{I}|-|H_{\hat{0}}\mathcal{T}|-|M_{\hat{0}}\mathcal{T}|.$$

Thus, in either case, no compact subset of $\mathcal{M}_{\mathcal{T}}(\mu)$ is effective by Lemmas 4.8 and 4.9. Furthermore,

$$\bar{\mathcal{M}}_{\mathcal{T}}(\mu)-\mathcal{M}_{\mathcal{T}}(\mu)=\bigcup_{\mathcal{T}'<\mathcal{T}}\mathcal{M}_{\mathcal{T}'}(\mu),$$

where $\mathcal{T}'$ is either a four- or five-bubble strata, or a three bubble-strata $\mathcal{T}'=(\Sigma,[N],I';j',\underline{d}')$ such that either $|H_{\hat{0}}\mathcal{T}|=1$, or $d'_{\hat{1}'}=0$ and $j'_l=\hat{0}$ or $\hat{1}'$. By (1)-(3) above, none of such bubble types is effective, and thus $\mathcal{T}$ is not effective.

4.4 Second-Order Estimate for $\psi^{\mu}_{\mathcal{T},t\nu}$, Case 1

We now refine the first-order estimate for $\psi^{\mu}_{\mathcal{T},t\nu}$ along the sets on which the section α^- defined above does not have full rank. These are precisely the sets on which the section $\bar{\mathcal{D}}_{\mathcal{T},|\chi(\mathcal{T}|}$ defined in the proof of Lemma 4.9 does not have full rank.

One set on which $\bar{\mathcal{D}}_{\mathcal{T},|\chi(\mathcal{T})|}$ fails to have full rank is the zero set of $\mathcal{D}_{\mathcal{T},h_1}$. If $n=2,3$, by Lemma 4.9, $\mathcal{D}_{\mathcal{T},h_1}$ does not vanish unless h_1 is the only element of the set $\chi(\mathcal{T})$. Thus, we assume that this is the case. We denote the zero-locus of $\mathcal{D}_{\mathcal{T},h_1}$ by $\mathcal{S}_{\mathcal{T},1}\subset\mathcal{M}_{\mathcal{T}}$, which will be abbreviated as $\mathcal{S}$ in this subsection. Since $\mathcal{D}_{\mathcal{T},h_1}$ is transversal to zero by Corollary 6.3, $\mathcal{S}$ is a complex submanifold of $\mathcal{M}_{\mathcal{T}}$ of codimension n. Its normal bundle $\mathcal{N}\mathcal{S}$ in $\mathcal{M}_{\mathcal{T}}$ is the restriction of $L^*_{k_1}\mathcal{T}\otimes\mathrm{ev}^*T\mathbb{P}^n$ to $\mathcal{S}_{\mathcal{T},1}$. Let $(\Phi_{\mathcal{S}},\Phi^{\mu}_{\mathcal{S}})$ be a regularization of $\mathcal{S}_{\mathcal{T},1}(\mu)\equiv\mathcal{S}\cap\mathcal{M}_{\mathcal{T}}(\mu)$. This regularization can be chosen so that

$$\mathcal{D}_{\mathcal{T},h_1}\tilde{\phi}_{\mathcal{S}}(b,X)=\Pi_{b,\tilde{\phi}_{\mathcal{S}}(b,X)}X\quad\forall\ (b,X)\in\mathcal{N}\tilde{\mathcal{S}}=\mathrm{ev}^*T\mathbb{P}^n,\tag{4.12}$$

where $\tilde{\phi}_{\mathcal{S}}$ is the lift of $\phi_{\mathcal{S}}$ to the preimage $\tilde{\mathcal{S}}$ of $\mathcal{S}$ and its normal bundle $\mathcal{N}\tilde{\mathcal{S}}$ in $\mathcal{M}^{(0)}_{\mathcal{T}}$; see Subsection 3.8 in [Z1]. The bundle $\mathcal{N}\mathcal{S}$ carries a natural norm induced by the $g_{\mathbb{P}^n,\mathrm{ev}}$-metric on $\mathbb{P}^n$. Denote by $F\mathcal{S}$ and $F^{\emptyset}\mathcal{S}$ the bundles described in Subsection 2.4 corresponding to the submanifold $\mathcal{S}_{\mathcal{T},1}$. Let $\hat{1}\in H_{\hat{0}}\mathcal{T}$ be the unique element such that $\hat{1}\le h_1$. If $[b;X,\upsilon]\in F\mathcal{S}=\mathcal{N}\mathcal{S}\oplus F\mathcal{T}$, put

$${}^{(2)}\alpha_{\mathcal{T};1}(X,\upsilon)=X(b_{\upsilon})s_{\Sigma,x_{\hat{1}}}\tilde{\upsilon}_{h_1}+\alpha^{(2)}_{\mathcal{T},h_1}(\upsilon).$$

Lemma 4.12 *There exist $\delta,C\in C^{\infty}(\mathcal{S};\mathbb{R}^+)$ such that for all $\varpi=[(b;X,\upsilon)]\in F^{\emptyset}\mathcal{S}_{\delta}$,*

$$\Big\|\pi^{0,1}_{\Phi_{\mathcal{S}}(\varpi),-}\bar{\partial}u_{\Phi_{\mathcal{S}}(\varpi)}+\tilde{R}_{\Phi_{\mathcal{S}}(\varpi)}\Pi_{b,\phi_{\mathcal{S}}(X)}{}^{(2)}\alpha_{\mathcal{T};1}(X,\upsilon)\Big\|_2\le C(b)|\upsilon|\big(|\upsilon|^2_{h_1}+|X||\upsilon|_{h_1}\big).$$

Proof: The proof is almost identical to the proof of Lemma 4.5. The only difference is that we use two terms of the power series of Proposition 4.4. We then make use of the assumption (4.12) on $\phi_{\mathcal{S}}$ and smooth dependence of $\mathcal{D}^{(2)}_{\mathcal{T},h_1}$ on X.

Lemma 4.13 *There exist $\delta,C\in C^{\infty}(\mathcal{S}_{\mathcal{T},1}(\mu);\mathbb{R}^+)$ such that for all $\varpi=[(b;X,\upsilon)]\in F^{\emptyset}\mathcal{S}_{\delta}$,*

$$\Big\|\psi^{\mu}_{\mathcal{S},t\nu}(\varpi)-\big(t\bar{\nu}_{ev(b)}+{}^{(2)}\alpha_{\mathcal{T};1}(X,\upsilon)\big)\Big\|_2\le C(b)\big(t+|\upsilon|^{\frac{1}{p}}\big)\big(t+|\upsilon|^2_{h_1}+|X||\upsilon|_{h_1}\big).$$

Proof: This claim follows from Lemmas 4.1 and 4.12 in a way analogous to the proof of Lemma 4.6. The only difference is that we need to improve the estimate on $\pi^{0,1}_{\upsilon,-}D_{\upsilon}\xi_{\upsilon,t\nu}$ made in the proof of Lemma 4.6. Let $\{\psi_j\}$ be an orthonormal basis for $\mathcal{H}^{0,1}_{\Sigma}$, such that $\psi_1\in\mathcal{H}^{+}_{\Sigma}(\tilde{x}_{h_1}(\upsilon))$, and $\{X_i\}$ an orthonormal basis for $T_{\mathrm{ev}(\phi_{\mathcal{S}}(X))}\mathbb{P}^n$. By Theorem 2.7, with $\upsilon(X)=\Phi_{\mathcal{S}}(\varpi)$,

$$\begin{aligned}\Big|\big\langle\!\big\langle\pi^{0,1}_{\upsilon(X),-}D_{\upsilon(X)}\xi_{\upsilon(X),t\nu},R_{\upsilon(X)}X_i\psi_j\big\rangle\!\big\rangle\Big|&=\Big|\big\langle\!\big\langle\xi_{\upsilon(X),t\nu},D^*_{\upsilon(X)}R_{\upsilon(X)}X_i\psi_j\big\rangle\!\big\rangle\Big|\\&\le C(b)(t+|\upsilon|^{\frac{1}{p}})\|D^*_{\upsilon(X)}R_{\upsilon(X)}X_j\psi_j\|_{C^0}.\end{aligned}\tag{4.13}$$

Since $\xi\in\tilde{\Gamma}_+(\upsilon)$, by construction in Subsection 2.3,

$$\big\langle\!\big\langle\xi_{\upsilon(X),t\nu},D^*_{\upsilon(X)}R_{\upsilon(X)}X_i\psi_1\big\rangle\!\big\rangle=0.\tag{4.14}$$

On the other hand, since $\psi_2|_{\tilde{x}_{h_1}(\upsilon)}=0$ and $\|\nabla\psi_2\|_{g_{\phi_{\mathcal{S}}(X),\hat{0}},C^0}\le C(b)$, by equation (2.12)

$$\big\|D^*_{\upsilon(X)}R_{\upsilon(X)}X_i\psi_2\big\|_{C^0(\tilde{A}_{\upsilon(X),h_1})}\le C(b)|\upsilon|^2_{h_1}, \tag{4.15}$$

where $\tilde{A}_{\upsilon(X),h_1}$ is the annulus defined in Lemma 2.2. By equations (4.13)-(4.15),

$$\Big|\pi^{0,1}_{\upsilon(X),-}D_\upsilon\xi_{\upsilon(X),t\nu}\Big|\le C(b)(t+|\upsilon|^{\frac{1}{p}})|\upsilon|^2_{h_1}.$$

The next step is to apply Lemma 3.2 or Corollary 3.6 whenever possible. Let

$$F^+=\text{ev}^*T\mathbb{P}^n\otimes\bigotimes_{i\in\hat{I},i\le h_1}F_i\mathcal{T},\quad F^-=F\mathcal{T},\quad \tilde{F}^-=\Big(\bigotimes_{i\in\hat{I},i\le h_1}F_i\mathcal{T}\Big)^{\otimes2},\quad \mathcal{O}^\pm=\mathcal{H}^\pm_\Sigma\otimes\text{ev}^*T\mathbb{P}^n;$$

$$\alpha^+\big([X,\upsilon]\big)=Xs_{\Sigma,x_{\hat{1}}}\tilde{\upsilon}_{h_1},\quad \phi\big([b,\upsilon_{\hat{I}}]\big)=\big[b,\tilde{\upsilon}_{h_1}\otimes\tilde{\upsilon}_{h_1}\big],\quad \alpha^-\big(\phi(\upsilon)\big)\equiv\pi^-_{x_{\hat{1}}(b_\upsilon)}\alpha^{(2)}_{\mathcal{T}}(\upsilon).$$

Note that $\alpha^+\in\Gamma(\mathcal{S};F^{+*}\otimes\mathcal{O}^+)$, since $\pi^-\circ Xs_\Sigma=0$. Since the map $(X,\upsilon)\longrightarrow(X\otimes\tilde{\upsilon}_{h_1},\upsilon)$ is injective on $F^\emptyset\mathcal{T}$, we can view $\psi^\mu_{\mathcal{S},t\nu}$ as a map on an open subset of $F^-\oplus F^+$. Analogously to the first-order case of Subsection 4.2, subset $K\subset\mathcal{S}_{\mathcal{T},1}(\mu)$ will be called *second-order regular* if α^- has full rank over K.

Corollary 4.14 *Suppose d is a positive integer, $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ is a simple bubble type, with $d_{\hat{0}}=0$ and $\sum_{i\in I}d_i=d$, and μ is an N-tuple of constraints in general position such that*

$$\mathit{codim}_{\mathbb{C}}\mu=d(n+1)-n(g-1)+N.$$

Let $\nu\in\Gamma(\Sigma\times\mathbb{P}^n;\Lambda^{0,1}\pi^*_\Sigma T^*\Sigma\otimes\pi^*_{\mathbb{P}^n}T\mathbb{P}^n)$ be a generic section. If $|\hat{I}|>1$, for every second-order regular compact subset K of $\mathcal{S}_{\mathcal{T},1}(\mu)$, there exist a neighborhood U_K of K in $\bar{C}^\infty_{(d;[N])}(\Sigma;\mu)$ and $\epsilon_K>0$ such that for any $t\in(0,\epsilon_K)$, $U_K\cap\mathcal{M}_{\Sigma,d,t\nu}(\mu)=\emptyset$. If $|\hat{I}|=1$, there exists a compact regular subset $K_{\mathcal{T},1}$ of $\mathcal{S}_{\mathcal{T},1}(\mu)$ with the following property. If K is a compact subset of $\mathcal{S}_{\mathcal{T},1}(\mu)$ containing $K_{\mathcal{T},1}$, there exist a neighborhood U_K of K in $\bar{C}^\infty_{(d;[N])}(\Sigma;\mu)$ and $\epsilon_K>0$ such that for all $t\in(0,\epsilon_K)$, the signed cardinality of $U_K\cap\mathcal{M}_{\Sigma,d,t\nu}(\mu)$ equals to twice the signed number of zeros of the map

$$T\Sigma^{\otimes2}\otimes L_{\hat{1}}\mathcal{T}^{\otimes2}\big|_{\mathcal{S}_{\mathcal{T},1}(\mu)}\longrightarrow\mathcal{H}^-_\Sigma\otimes ev^*T\mathbb{P}^n,\quad [b,v]\longrightarrow\bar{\nu}^-_b+\alpha^{(2,-)}\big([b,v]\big). \tag{4.16}$$

Proof: In either case, by Theorem 2.7, there exist a neighborhood U_K of K in $\bar{C}^\infty_{(d;[N])}(\Sigma;\mu)$ and $\delta_K,\epsilon_K>0$ such that for any $t\in(0,\epsilon_K)$, there exists a sign-preserving bijection between $U_K\cap\mathcal{M}_{\Sigma,d,t\nu}(\mu)$ and the zeros of $\psi^\mu_{\mathcal{S},t\nu}$ on $F^\emptyset\mathcal{S}_{\delta_K}\big|_{U_K\cap\mathcal{S}_{\mathcal{T},1}(\mu)}$, provided $U_K\cap\mathcal{S}_{\mathcal{T},1}(\mu)$ is precompact in $\mathcal{S}_{\mathcal{T},1}(\mu)$. If K is second-order regular, U_K can be chosen so that the closure of $U_K\cap\mathcal{S}_{\mathcal{T},1}(\mu)$ in $\mathcal{S}_{\mathcal{T},1}(\mu)$ is also second-order regular. Since K is regular and α^+ is injective on all fibers,

$$|\upsilon|^2_{h_1}=|\phi(\upsilon)|\le C_K\big|\alpha^-(\phi(\upsilon))\big|\implies|\upsilon|^2_{h_1}+|X||\upsilon|_{h_1}\le C'_K\big|{}^{(2)}\alpha_{\mathcal{T};1}(X,\upsilon)\big|\quad\forall(X,\upsilon)\in F^\emptyset\mathcal{S}_{\delta_K}\big|_K,$$

where $C_K,C'_K>0$ depend only on K. Thus, by Lemma 4.13,

$$\Big\|\psi^\mu_{\mathcal{S},t\nu}(\varpi)-\big(t\bar{\nu}_{\text{ev}(b_\varpi)}+{}^{(2)}\alpha_{\mathcal{T};1}(\varpi)\big)\Big\|_2\le C_K\big(t+|\varpi|^{\frac{1}{p}}\big)\big(t+\big|{}^{(2)}\alpha_{\mathcal{T};1}(\varpi)\big|\big)\quad\forall\varpi\in F^\emptyset\mathcal{S}_{\delta_K}\big|_K,$$

where $C_K>0$ depends only on K. The first claim now follows from Lemma 3.2. The second follows from Corollary 3.6, provided that for a generic ν the set of zeros of the map in (4.16) is second-order regular and finite; see the last paragraph of Subsection 4.2.

4.5 Third-Order Estimate for $\psi^{\mu}_{\mathcal{T},t\nu}$, Case 1

We continue with the case of Subsection 4.4. Then

$$\alpha^-([b,\tilde{v}_{h_1}])=(\mathcal{D}^{(2)}_{\mathcal{T},h_1}b)s^{(2,-)}_{b,x_{\hat{1}}}(\tilde{v}_{h_1}).$$

By Corollary 6.3, for a generic choice of the constraints $\mu_1,\ldots,\mu_N$, $\mathcal{D}^{(2)}_{\mathcal{T},h_1}$ is transversal to zero along $\mathcal{S}_{\mathcal{T},1}(\mu)$ if $d_{h_1}\geq 2$. Since the zero set of $\mathcal{D}^{(2)}_{\mathcal{T},h_1}$ must have dimension at least $d_{\min}(\mathcal{T})\geq 1$ by the same argument as in the proof of Lemma 4.9, $\mathcal{D}^{(2)}_{\mathcal{T},h_1}$ does not vanish along $\mathcal{S}_{\mathcal{T},1}(\mu)$ if $d_{h_1}\geq 2$. On the other hand, if $d_{h_1}=1$, $\mathcal{S}_{\mathcal{T},1}=\emptyset$, since the differential of any degree-one holomorphic map from S^2 to $\mathbb{P}^n$ is nowhere zero. In fact, $\mathcal{S}_{\mathcal{T},1}(\mu)=\emptyset$ even for $d_{h_1}=2$, since the image of any degree-two map with a somewhere vanishing differential is a line, and no line intersects $\mu_1,\ldots,\mu_N$ if $n=2,3$. Thus, we can assume $d_{h_1}\geq 3$. It follows that the only way the above homomorphism α^- can fail to have full rank on $\tilde{F}^-$ is if $s^{(2,-)}_{b,x_{\hat{1}}}=0$. While $s^{(2)}_{b,x_{\hat{1}}}$ depends on the choice of the metric $g_{b,\hat{0}}$ on Σ, the section $s^{(2,-)}\in\Gamma\big(\Sigma;T^*\Sigma^{\otimes 2}\otimes\mathcal{H}^-_{\Sigma}\big)$ is independent of the metric and is globally defined on Σ. This can be seen by a direct computation. It has transverse zeros at the six branch points of the double cover $\Sigma\longrightarrow\mathbb{P}^1$ induced by s_{Σ}; see [GH, p246]. Denote by $z_1,\ldots,z_6$ these six points. Then the set on which α^- fails to have full rank is $\bigcup\limits_{m\in[6]}\mathcal{S}^{(m)}_{\mathcal{T},1}(\mu)$, where

$$\mathcal{S}^{(m)}_{\mathcal{T},1}=\big\{b\in\mathcal{S}_{\mathcal{T},1}: x_{\hat{1}}(b)=z_m\big\},\quad \mathcal{S}^{(m)}_{\mathcal{T},1}(\mu)=\mathcal{S}^{(m)}_{\mathcal{T},1}\cap\mathcal{M}_{\mathcal{T}}(\mu).$$

The sets $\mathcal{S}^{(m)}_{\mathcal{T},1}$ are obviously disjoint.

Since the normal bundle of $\mathcal{S}^{(m)}_{\mathcal{T},1}$ in $\mathcal{S}_{\mathcal{T},1}$ is $T_{z_m}\Sigma$, the normal bundle $\mathcal{N}\mathcal{S}$ of $\mathcal{S}^{(m)}_{\mathcal{T},1}$ in $\mathcal{M}_{\mathcal{T}}(\mu)$ is $T_{z_m}\Sigma\oplus\mathcal{N}\mathcal{S}_1$, where $\mathcal{N}\mathcal{S}_1$ is the normal bundle of $\mathcal{S}_{\mathcal{T},1}$ in $\mathcal{M}_{\mathcal{T}}(\mu)$, as described in the previous subsection. Let $\big(\Phi_{\mathcal{S}},\Phi^{\mu}_{\mathcal{S}}\big)$ be a regularization of $\mathcal{S}^{(m)}_{\mathcal{T},1}(\mu)$ induced by the regularization of $\mathcal{S}_{\mathcal{T},1}(\mu)$ described in Subsection 4.4. In particular,

$$\mathcal{D}_{\mathcal{T},h_1}\tilde{\phi}_{\mathcal{S}}(b,w,X)=\Pi_{b,\tilde{\phi}_{\mathcal{S}}(b,w,X)}X\quad\forall\ (b,w,X)\in T_{z_m}\Sigma\oplus\mathcal{N}\tilde{\mathcal{S}}_1=T_{z_m}\Sigma\oplus\mathrm{ev}^*T\mathbb{P}^n,\tag{4.17}$$

where $\tilde{\phi}_{\mathcal{S}}$ is the lift of $\phi_{\mathcal{S}}$to $\mathcal{M}^{(0)}_{\mathcal{T}}$. We can also assume that $\Phi^{\mu}_{\mathcal{S}}$ is given by the $g_{\mathbb{P}^n,b}$-parallel transport on $\mathcal{N}_b\mathcal{S}_1$. The bundle $\mathcal{N}\mathcal{S}$ carries a natural norm induced by the $g_{\mathbb{P}^n,\mathrm{ev}}$-metric on $\mathbb{P}^n$ and $g_{\cdot,\hat{0}}$-metric on Σ. Denote by $F\mathcal{S}$ and $F^{\emptyset}\mathcal{S}$ the bundles described in Subsection 2.4 corresponding to the submanifold $\mathcal{S}^{(m)}_{\mathcal{T},1}$. If $(b,w,X,\upsilon)\in F^{\emptyset}\mathcal{S}$ is sufficiently small, let

$$\tilde{x}_{\hat{1}}(w,\upsilon)=\tilde{x}_{\hat{1}}\big(\phi_{\mathcal{S}}(w,X,\upsilon)\big)=\tilde{x}_{\hat{1}}\big(\phi_{\mathcal{S}}(w,0,\upsilon)\big)\in\Sigma.$$

We identify a small neighborhood of z_m in Σ with a neighborhood of 0 in $T_{z_m}\Sigma$ via the $g_{b,\hat{0}}$-exponential map. Put

$$\tilde{\alpha}(w,X,\upsilon)=(Xb)s_{\Sigma,\tilde{x}_{\hat{1}}(w,\upsilon)}(\tilde{v}_{h_1})+\Pi^{-1}_{b,\phi_{\mathcal{S}}(b,X)}\big(\mathcal{D}^{(2)}_{\mathcal{T},h_1}\phi_{\mathcal{S}}(b,X)\big)s^{(2)}_{b,\tilde{x}_{\hat{1}}(w,\upsilon)}(\tilde{v}_{h_1})+\big(\mathcal{D}^{(3)}_{\mathcal{T},h_1}b\big)s^{(3)}_{b,z_m}(\tilde{v}_{h_1}).$$

If $(b,w,X,\upsilon)\in F^{\emptyset}\mathcal{S}|_{\mathcal{S}^{(m)}_{\mathcal{T},1}(\mu)}$ is sufficiently small, let

$$\tilde{\alpha}^{\mu}(w,X,\mu)=(Xb)s_{\Sigma,\tilde{x}_{\hat{1}}(w,\upsilon)}(\tilde{v}_{h_1})+\big(\mathcal{D}^{\mu,(2)}_{\mathcal{S},t\nu}(w,X,\upsilon)\big)s^{(2)}_{b,\tilde{x}_{\hat{1}}(w,\upsilon)}(\tilde{v}_{h_1})+\big(\mathcal{D}^{(3)}_{\mathcal{T},h_1}b\big)s^{(3)}_{b,z_m}(\tilde{v}_{h_1}),$$

where, with $\varphi^{\mu}_{\mathcal{S},t\nu}$ as in Theorem 2.7,

$$\mathcal{D}^{\mu,(2)}_{\mathcal{S},t\nu}(w,X,\upsilon) = \Pi^{-1}_{\phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,\upsilon),\phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,\upsilon)}\Pi^{-1}_{b,\phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,\upsilon)}\big(\mathcal{D}^{(2)}_{\mathcal{T},h_1}\phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,\upsilon)\big).$$

Lemma 4.15 *There exist $\delta, C \in C^{\infty}(\mathcal{S}^{(m)}_{\mathcal{T},1};\mathbb{R}^+)$ such that for all $\varpi = [(b,w,X,\upsilon)] \in F^{\emptyset}\mathcal{S}_{\delta}$,*

$$\Big\|\pi^{0,1}_{\Phi_{\mathcal{S}}(\varpi),-}\bar{\partial}u_{\Phi_{\mathcal{S}}(\varpi)} + \tilde{R}_{\Phi_{\mathcal{S}}(\varpi)}\Pi_{b,\phi_{\mathcal{S}}(X)}\tilde{\alpha}(w,X,\upsilon)\Big\|_2 \le C(b)|\varpi||\upsilon|^3_{h_1}.$$

Proof: The proof is the same as that of Lemma 4.12, except here we use the first three terms of the expansion of Proposition 4.4. Note that $|\tilde{x}_{\hat{1}}(w,\upsilon)| \le C(b)(|w|+|\upsilon|)$.

Lemma 4.16 *There exist $\delta, C \in C^{\infty}(\mathcal{S}^{(m)}_{\mathcal{T},1}(\mu);\mathbb{R}^+)$ such that for all $\varpi = (b,w,X,\upsilon) \in F^{\emptyset}\mathcal{S}_{\delta}$*

$$\big\|\psi^{\mu}_{\mathcal{S},t\nu}(\varpi) - (t\bar{\nu}_{ev(b)} + \tilde{\alpha}^{\mu}(w,X,\upsilon))\big\|_2 \le C(b)(t+|\varpi|^{\frac{1}{p}})\big(t+|\upsilon|^3_{h_1}+|\tilde{x}_1(w,\upsilon)||\upsilon|^2_{h_1}\big).$$

Proof: The proof is similar to the proofs of Lemmas 4.6 and 4.13, but we need to obtain an even stronger bound on

$$\big\|\pi^{0,1}_{\Phi_{\mathcal{S}}(\varpi),-}D_{\Phi_{\mathcal{S}}(\varpi)}\xi_{\Phi_{\mathcal{S}}(\varpi),t\nu}\big\|_2.$$

Let $\{\psi_j\}$ be an orthonormal basis for $\mathcal{H}^{0,1}_{\Sigma}$ such that $\psi_1 \in \mathcal{H}^{+}_{\Sigma}\big(\tilde{x}_{h_1}(w,\upsilon)\big)$, and $\{X_i\}$ an orthonormal basis for $T_{\mathrm{ev}(\phi_{\mathcal{S}}(X,\upsilon))}\mathbb{P}^n$. Then, as in the proof of Lemma 4.13, with $\upsilon(\varpi) = \Phi_{\mathcal{S}}(\varpi)$,

$$\big\langle\!\big\langle D_{\Phi_{\mathcal{S}}(\varpi)}\xi_{\upsilon(\varpi),t\nu}, R_{\upsilon(\varpi)}X_i\psi_1\big\rangle\!\big\rangle = 0; \tag{4.18}$$

$$\Big|\big\langle\!\big\langle \pi^{0,1}_{\upsilon(\varpi),-}D_{\Phi_{\mathcal{S}}(\varpi)}\xi_{\upsilon(\varpi),t\nu}, R_{\upsilon(\varpi)}X\big\rangle\!\big\rangle\Big| \le C(b)\big(t+|\upsilon|^{\frac{1}{p}}\big)\big\|D^*_{\upsilon(\varpi)}R_{\upsilon(\varpi)}X_i\psi_2\big\|_{\upsilon(\varpi),1}. \tag{4.19}$$

The one-form ψ_2 vanishes at $\tilde{x}_{h_1}(w,\upsilon)$ by definition and $\|\nabla\psi_2\|_{g_{b,\hat{0}},C^0} \le C|\tilde{x}_{h_1}(w,\upsilon)|$, since the derivative of the corresponding one-form for z_m vanishes. Thus, by equation (2.12)

$$\big\|D^*_{\upsilon(\varpi)}R_{\upsilon(\varpi)}X_i\psi_2\big\|_{g_{\upsilon(\varpi)},L^1(\tilde{A}_{\upsilon(\varpi),h})} \le C(b)(|\tilde{x}_{h_1}(w,\upsilon)||\upsilon|_{h_1}+|\upsilon|^2_{h_1})|\upsilon|_{h_1}, \tag{4.20}$$

as needed for our bound. Finally, we use our assumption that $\Phi^{\mu}_{\mathcal{S}}$ is given by the $g_{b,\hat{0}}$-parallel transport on $\mathcal{N}_b\mathcal{S}_1$.

For any $(w,X,\upsilon) \in F^{\emptyset}_b\mathcal{S}|_{\mathcal{S}^{(m)}_{\mathcal{T},1}(\mu)}$ sufficiently small, let

$$Y(w,X,\upsilon) = (Xb)s_{\Sigma,\tilde{x}_1(w,\upsilon)}(\tilde{v}_{h_1}) + \big(\mathcal{D}^{\mu,(2)}_{\mathcal{S},t\nu}(w,X,\upsilon)\big)s^{(2,+)}_{b,\tilde{x}_{\hat{1}}(w,\upsilon)}(\tilde{v}_{h_1},\tilde{v}_{h_1});$$

$${}^{(3)}\alpha^{(m),-}_{\mathcal{T};1}(w,\upsilon) = \big(\mathcal{D}^{(2)}_{\mathcal{T},h_1}b\big)s^{(3,-)}_{b,z_m}\big(\tilde{x}_{\hat{1}}(w,\upsilon),\tilde{v}_{h_1},\tilde{v}_{h_1}\big) + \big(\mathcal{D}^{(3)}_{\mathcal{T},h_1}b\big)s^{(3,-)}_{b,z_m}(\tilde{v}_{h_1});$$

$$r^{+}_{\mathcal{T};1}(\upsilon) = \big(\mathcal{D}^{(3)}_{\mathcal{T},h_1}b\big)s^{(3,+)}_{b,z_m}(\tilde{v}_{h_1}), \quad \bar{\nu}^{\pm}_b = \pi_{z_m}\bar{\nu}_b.$$

Corollary 4.17 *There exist $\delta, C \in C^{\infty}(\mathcal{S}^{(m)}_{\mathcal{T},1}(\mu);\mathbb{R}^+)$ such that for all $\varpi = [(b,w,X,\upsilon)] \in F^{\emptyset}\mathcal{S}_{\delta}$*

$$\begin{aligned}\big\|\pi^{+}_{x_{\hat{1}}(w,\upsilon)}\psi^{\mu}_{\mathcal{S},t\nu}(w,X,\upsilon) - (t\pi^{+}_{x_{\hat{1}}(w,\upsilon)}\bar{\nu}_b + Y(w,X,\upsilon) + r^{+}_{\mathcal{T};1}(\upsilon))\big\|_2 \\ \le C(b)(t+|\varpi|^{\frac{1}{p}})\big(t+|\upsilon|^3_{h_1}+|\tilde{x}_1(w,\upsilon)||\upsilon|^2_{h_1}\big);\\ \big\|\pi^{-}_{x_{\hat{1}}(w,\upsilon)}\psi^{\mu}_{\mathcal{S},t\nu}(w,X,\upsilon) - (t\pi^{-}_{z_m}\bar{\nu}_b + {}^{(3)}\alpha^{(m),-}_{\mathcal{T};1}(w,\upsilon))\big\|_2 \\ \le C(b)(t+|\varpi|^{\frac{1}{p}})\big(t+|\upsilon|^3_{h_1}+|\tilde{x}_1(w,\upsilon)||\upsilon|^2_{h_1}\big).\end{aligned}$$

Proof: The first estimate is clear from Lemma 4.16. For the second, note that since $s^{(2,-)}_{b,z_m}=0$, $|\pi^-_{\tilde{x}_{\hat{1}}(w,\upsilon)}-\pi^-_{z_m}|\le C|\tilde{x}_{\hat{1}}(w,\upsilon)|^2$, and thus

$$\big|s^{(2,-)}_{b,\tilde{x}_{\hat{1}}(w,\upsilon)}(\tilde{\upsilon}_{h_1})-s^{(3,-)}_{b,z_m}(\tilde{x}_{\hat{1}}(w,\upsilon),\tilde{\upsilon}_{h_1},\tilde{\upsilon}_{h_1})\big|\le C|\tilde{x}_{\hat{1}}(w,\upsilon)|^2|\tilde{\upsilon}_{h_1}|^2 \Longrightarrow$$
$$\big|\pi^-_{x_{\hat{1}}(w,\upsilon)}\tilde{\alpha}^{\mu}(w,X,\upsilon)-{}^{(3)}\alpha^-_{\mathcal{T};1}(w,\upsilon)\big|\le C(b)|(t,w,X,\upsilon)|^{\frac{1}{p}}\big(|x_{\hat{1}}(w,\upsilon)||\tilde{\upsilon}_{h_1}|^2+|\tilde{\upsilon}_{h_1}|^3\big)$$

Furthermore, $|\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,\upsilon)|_b\le C(b)(t+|\varpi|^{\frac{1}{p}})$.

The next step is to apply Lemma 3.2 and Corollary 3.6. Let

$$F^+=\mathcal{H}^+_{\Sigma}\otimes \mathrm{ev}^*T\mathbb{P}^n,\quad F^-=T_{z_m}\Sigma\oplus F\mathcal{T},\quad \mathcal{O}^{\pm}=\mathcal{H}^{\pm}_{\Sigma}\otimes\mathrm{ev}^*T\mathbb{P}^n;$$
$$\tilde{F}^-=T_{z_m}\Sigma\otimes\Big(\bigotimes_{i\in\hat{I},i\le h_1}F_i\mathcal{T}\Big)^{\otimes 2}\oplus\Big(\bigotimes_{i\in\hat{I},i\le h_1}F_i\mathcal{T}\Big)^{\otimes 3};$$
$$\phi\big([b;w,\upsilon_{\hat{I}}]\big)=\big[b,x_{\hat{1}(w,\upsilon)}\otimes\tilde{\upsilon}_{h_1}\otimes\tilde{\upsilon}_{h_1},\tilde{\upsilon}_{h_1}\otimes\tilde{\upsilon}_{h_1}\otimes\tilde{\upsilon}_{h_1}\big];$$
$$\pi^+\alpha(w,\upsilon)=r^+_{\mathcal{T};1}(\upsilon),\quad \alpha^+(Y)=\pi^+_{z_m}Y,\quad \alpha^-(\phi(w,\upsilon))\equiv{}^{(3)}\alpha^{(m),-}_{\mathcal{T},1}(w,\upsilon).$$

Note that $\alpha^-\in\Gamma(\mathcal{S};\tilde{F}^{-*}\otimes\mathcal{O}^-)$ is well-defined. Since the map

$$(w,X,\upsilon)\longrightarrow\big(Y(w,X,\upsilon),w,\upsilon\big)$$

is injective on $F^{\emptyset}\mathcal{S}$, we can view $\psi^{\mu}_{\mathcal{S},t\nu}$ as a map on an open subset of $F^-\oplus F^+$.

Corollary 4.18 *Suppose d is a positive integer, $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ is a simple bubble type, with $d_{\hat{0}}=0$ and $\sum_{i\in I}d_i=d$, and μ is an N-tuple of constraints in general position such that*

$$\mathrm{codim}_{\mathbb{C}}\mu=d(n+1)-n(g-1)+N.$$

Let $\nu\in\Gamma(\Sigma\times\mathbb{P}^n;\Lambda^{0,1}\pi^*_{\Sigma}T^*\Sigma\otimes\pi^*_{\mathbb{P}^n}T\mathbb{P}^n)$ be a generic section. If $|\hat{I}|>1$, for every compact subset K of $\mathcal{S}^{(m)}_{\mathcal{T},1}(\mu)$, there exist a neighborhood U_K of K in $\bar{C}^{\infty}_{(d;[N])}(\Sigma;\mu)$ and $\epsilon_K>0$ such that for any $t\in(0,\epsilon_K)$, $U_K\cap\mathcal{M}_{\Sigma,d,t\nu}(\mu)=\emptyset$. If $|\hat{I}|=1$, there exists a compact subset $\tilde{K}^{(m)}_{\mathcal{T},1}$ of $\mathcal{S}^{(m)}_{\mathcal{T},1}(\mu)$ with the following property. If K is a compact subset of $\mathcal{S}^{(m)}_{\mathcal{T},1}(\mu)$ containing $\tilde{K}^{(m)}_{\mathcal{T},1}$, there exist a neighborhood U_K of K in $\bar{C}^{\infty}_{(d;[N])}(\Sigma;\mu)$ and $\epsilon_K>0$ such that for all $t\in(0,\epsilon_K)$, the signed cardinality of $U_K\cap\mathcal{M}_{\Sigma,d,t\nu}(\mu)$ equals to three times the signed number of zeros of the map

$$T_{z_m}\Sigma^{\otimes 3}\otimes\big(L_{\hat{1}}\mathcal{T}^{\otimes 2}\oplus L_{\hat{1}}\mathcal{T}^{\otimes 3}\big)\big|_{\mathcal{S}^{(m)}_{\mathcal{T},1}(\mu)}\longrightarrow\mathcal{H}^-_{\Sigma}\otimes ev^*T\mathbb{P}^n,$$
$$[b,w,\upsilon_{\hat{1}}]\longrightarrow\bar{\nu}^-_b+\big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}b\big)s^{(3,-)}_{b,z_m}(w)+\big(\mathcal{D}^{(3)}_{\mathcal{T},\hat{1}}b\big)s^{(3,-)}_{b,z_m}(\upsilon).\tag{4.21}$$

Proof: The proof is similar to the proofs of Corollaries 4.7 and 4.14, but two modifications are needed to be mentioned. First, we need to show that α^- always has full rank. Since we are assuming that $d_{h_1}\ge 3$, the sections $\mathcal{D}^{(1)}_{\mathcal{T},h_1}$, $\mathcal{D}^{(2)}_{\mathcal{T},h_1}$, and $\mathcal{D}^{(3)}_{\mathcal{T},h_1}$ over $\mathcal{M}_{\mathcal{T}}$ have transverse images in $T\mathbb{P}^n$. Thus, the sections of $\mathbb{P}(\mathrm{ev}^*T\mathbb{P}^n)\longrightarrow S^{(m)}_{\mathcal{T},1}(\mu)$ induced by $\mathcal{D}^{(2)}_{\mathcal{T},h_1}$ and $\mathcal{D}^{(3)}_{\mathcal{T},h_1}$ are mutually transversal. However, the fiber dimension of $\mathbb{P}(\mathrm{ev}^*T\mathbb{P}^n)$ is $n-1$, while the dimension of $S^{(m)}_{\mathcal{T},1}(\mu)$ is

$n-2$. Thus, the two sections do not intersect and α^- has full rank on all fibers over $S^{(m)}_{\mathcal{T},1}(\mu)$. The second difference with the proofs of Corollaries 4.7 and 4.14 is that we replace the section $\psi^{\mu}_{\mathcal{S},t\nu}$ by the map

$$(w,\upsilon,X) \longrightarrow \pi^+_{z_m}\pi^+_{x_{\hat{1}}(w,\upsilon)}\psi^{\mu}_{\mathcal{S},t\nu}(w,\upsilon,X) + \pi^-_{z_m}\pi^-_{x_{\hat{1}}(w,\upsilon)}\psi^{\mu}_{\mathcal{S},t\nu}(w,\upsilon,X),$$

which has exactly the same zeros provided w and υ are sufficiently small (depending only on Σ).

4.6 Second-Order Estimate for $\psi^{\mu}_{\mathcal{T},t\nu}$, Case 2a

We now understand all cases except for (2a) and (3b) of Corollary 4.11. Let $\{h_1,h_2\}=\{\hat{1},\hat{2}\}$ in Case (2a) and $\{\hat{2},\hat{3}\}$ in (3b). By dimension count as in the proof of Lemma 4.9, $\mathcal{D}_{\mathcal{T},h_1}$ and $\mathcal{D}_{\mathcal{T},h_2}$ do not vanish on $\mathcal{M}_{\mathcal{T}}(\mu)$ in these two cases. By Corollary 6.3, $\pi^{\perp}_b\circ\mathcal{D}_{\mathcal{T},h_2}$ is transversal to zero, where $\pi^{\perp}_b$ denotes the projection onto the orthogonal complement E_1 of the image of $\mathcal{D}_{\mathcal{T},h_1}$ in $\mathrm{ev}^*T\mathbb{P}^n$. Since

$$\alpha_{\mathcal{T}}(\upsilon) = \big(\mathcal{D}_{\mathcal{T},h_1}b_\upsilon\big)s_{\Sigma,x_{\tilde{h}_1(\mathcal{T})}}(\tilde{\upsilon}_{h_1}) + \big(\mathcal{D}_{\mathcal{T},h_2}b_\upsilon\big)s_{\Sigma,x_{\tilde{h}_2(\mathcal{T})}}(\tilde{\upsilon}_{h_2}),$$

$\alpha_{\mathcal{T}}$ can fail to have the full rank only on the zero set of $\pi^{\perp}_b\circ\mathcal{D}_{\mathcal{T},h_2}$. Furthermore, $s_{\Sigma,x_{\tilde{h}_1}}$ and $s_{\Sigma,x_{\tilde{h}_2}}$ must have the same image in $\mathcal{H}^{0,1}_{\Sigma}$. This is automatic in Case (3b), since $\tilde{h}_1(\mathcal{T})=\tilde{h}_2(\mathcal{T})=\hat{1}$, but in Case (2a), this means that $x_{\hat{1}}$ and $x_{\hat{2}}$ differ by the nontrivial holomorphic automorphism of Σ; see [GH, p254].

We first treat Case (2a); so we can assume $h_1=\hat{1}$, $h_2=\hat{2}$. Let $\mathcal{S}\equiv\mathcal{S}_{\mathcal{T},2}$ denote the subset of $\mathcal{M}_{\mathcal{T}}$ on which the section $\alpha_{\mathcal{T}}$ has rank one. By Corollary 6.3, this is a complex submanifold of $\mathcal{M}_{\mathcal{T}}$. Furthermore, $\mathcal{S}=\mathcal{S}_0\times\mathcal{S}_1$, where $\mathcal{S}_1$ is the subspace of $\mathcal{U}_{\bar{\mathcal{T}}}$ on which the operator $\bar{\mathcal{D}}_{\mathcal{T},2}$, defined as in the proof of Lemma 4.9, has rank one,

$$\mathcal{S}_0 = \big\{(x_{\hat{1}},-x_{\hat{1}})\colon x_{\hat{1}}\in\Sigma^*\big\},$$

$-x_{\hat{1}}\in\Sigma$ denotes the image of $x_{\hat{1}}$ under the nontrivial automorphism of Σ, and Σ^* is the subset of Σ which is not fixed by this automorphism, i.e. the complement of the points $z_1,\ldots,z_6$ described in Subsection 4.5. By Corollary 6.3, $\mathcal{S}_1$ is a complex submanifold of $\mathcal{U}_{\bar{\mathcal{T}}}$. The normal bundle of $\mathcal{S}$ in $\mathcal{M}_{\mathcal{T}}$ is

$$\mathcal{N}\mathcal{S} = \mathcal{N}\mathcal{S}_0\oplus\mathcal{N}\mathcal{S}_1, \quad \text{where} \quad \mathcal{N}\mathcal{S}_0 = \pi^*_{\Sigma,\hat{2}}T\Sigma, \;\; \mathcal{N}\mathcal{S}_1 = L^*_{\hat{2}}\mathcal{T}\otimes E_1,$$

and $\pi_{\Sigma,h}\colon\mathcal{S}_0\subset\Sigma\times\Sigma\longrightarrow\Sigma$ is the projection on the hth component. Let $(\Phi_{\mathcal{S}},\Phi^{\mu}_{\mathcal{S}})$ be a regularization of $\mathcal{S}_{\mathcal{T},2}(\mu)\equiv\mathcal{S}\cap\mathcal{M}_{\mathcal{T}}(\mu)$. This regularization can be chosen so that

$$\pi^{\perp}_{\phi_{\mathcal{S}}(b,X)}\mathcal{D}_{\mathcal{T},\hat{2}}\tilde{\phi}_{\mathcal{S}}(b,X) = \Pi_{b,\tilde{\phi}_{\mathcal{S}}(b,X)}X \quad \forall(b,X)\in\mathcal{N}\tilde{\mathcal{S}}_1 = E_1, \tag{4.22}$$

where $\tilde{\phi}_{\mathcal{S}}$ is the lift of $\phi_{\mathcal{S}}$ to $\mathcal{M}^{(0)}_{\mathcal{T}}$. We also assume that $\Phi^{\mu}_{\mathcal{S}}$ is given by the $g_{\mathbb{P}^n,b}$-parallel transport on $\mathcal{N}_b\mathcal{S}_1$. Since the section s is invariant under the automorphism group of Σ, we identify $\pi^*_{\Sigma,\hat{2}}T\Sigma|_{\mathcal{S}_0}$ with $\pi^*_{\Sigma,\hat{1}}T\Sigma|_{\mathcal{S}_0}$. If $(b;w)\in\mathcal{N}\mathcal{S}_0$ is sufficiently small, let

$$x_{\hat{2}}(w) = \exp_{b,x_{\hat{2}}} w.$$

The bundle $\mathcal{NS}$ carries a natural norm induced by the $g_{\mathbb{P}^n,\mathrm{ev}}$-metric on $\mathbb{P}^n$ and $g_{\cdot,\hat{0}}$-metric on Σ. Denote by $F\mathcal{S}$ and $F^{\emptyset}\mathcal{S}$ the bundles described in Subsection 2.4 corresponding to the submanifold $\mathcal{S}_{\mathcal{T},2}$. If $(w,X,\upsilon)\in F\mathcal{S}=\mathcal{NS}\oplus F\mathcal{T}$, put

$$\tilde{\alpha}(w,X,\upsilon)=\Pi^{-1}_{b,\phi_{\mathcal{S}}(b,X)}\Big(\big(\mathcal{D}_{\mathcal{T},\hat{1}}\phi_{\mathcal{S}}(b,X)\big)s_{\Sigma,x_{\hat{1}}}(\upsilon_{\hat{1}})+\big(\mathcal{D}_{\mathcal{T},\hat{2}}\phi_{\mathcal{S}}(b,X)\big)s_{\Sigma,x_{\hat{2}}(w)}(\upsilon_{\hat{2}})\Big)$$
$$+\Big(\big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}b\big)s^{(2)}_{b,x_{\hat{1}}}(\upsilon_{\hat{1}})+\big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{2}}b\big)s^{(2)}_{b,x_{\hat{1}}}(\upsilon_{\hat{2}})\Big).$$

If $(w,X,\upsilon)\in F^{\emptyset}\mathcal{S}|_{\mathcal{S}_{\mathcal{T},2}(\mu)}$ is sufficiently small, let

$$\tilde{\alpha}^{\mu}(w,X,\upsilon)=\Big(\big(\mathcal{D}^{\mu}_{\mathcal{S},t\nu,\hat{1}}(w,X,\upsilon)\big)s_{\Sigma,x_{\hat{1}}}(\upsilon_{\hat{1}})+\big(\mathcal{D}^{\mu}_{\mathcal{S},t\nu,\hat{2}}(w,X,\upsilon)\big)s_{\Sigma,x_{\hat{2}}(w)}(\upsilon_{\hat{2}})\Big)$$
$$+\Big(\big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}b\big)s^{(2)}_{b,x_{\hat{1}}}(\upsilon_{\hat{1}})+\big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{2}}b\big)s^{(2)}_{b,x_{\hat{1}}}(\upsilon_{\hat{2}})\Big),$$

where, with $\varphi^{\mu}_{\mathcal{S},t\nu}$ as in Theorem 2.7,

$$\mathcal{D}^{\mu}_{\mathcal{S},t\nu,h}(w,X,\upsilon)=\Pi^{-1}_{\phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,\upsilon),\phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,\upsilon)}\Pi^{-1}_{b,\phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,\upsilon)}\big(\mathcal{D}_{\mathcal{T},h}\phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,\upsilon)\big).$$

Lemma 4.19 *There exist $\delta,C\in C^{\infty}(\mathcal{S};\mathbb{R}^+)$ such that for all $\varpi=[(b,w,X,\upsilon)]\in F^{\emptyset}\mathcal{S}_{\delta}$,*

$$\big\|\pi^{0,1}_{\Phi_{\mathcal{S}}(\varpi),-}\bar{\partial}u_{\Phi_{\mathcal{S}}(\varpi)}+\tilde{R}_{\Phi_{\mathcal{S}}(\varpi)}Pi_{b,\phi_{\mathcal{S}}(X)}\tilde{\alpha}(w,X,\upsilon)\big\|_2\le C(b)|\varpi||\upsilon|^2.$$

Proof: The proof is analogous to the proof of Lemma 4.15; here we use Proposition 4.4 with two terms for $h=\hat{1}$ and two terms for $h=\hat{2}$.

Lemma 4.20 *There exist $\delta,C\in C^{\infty}(\mathcal{S}_{\mathcal{T},2}(\mu);\mathbb{R}^+)$ such that for all $\varpi=[(b,w,X,\upsilon)]\in F^{\emptyset}\mathcal{S}_{\delta}$,*

$$\big\|\psi^{\mu}_{\mathcal{S},t\nu}(\varpi)-(t\bar{\nu}_{ev(b)}+\tilde{\alpha}^{\mu}(w,X,\upsilon))\big\|_2\le C(b)(t+|\varpi|^{\frac{1}{p}})\big(t+|\upsilon|^2+|w||\upsilon_{\hat{2}}|\big).$$

Proof: As in the proof of Lemmas 4.13 and 4.16, we need to obtain an appropriate estimate on

$$\big\|D^*_{\Phi_{\mathcal{S}}(\varpi)}R_{\Phi_{\mathcal{S}}(\varpi)}X_i\psi_2\big\|_{L^1},$$

where ψ_2 is a $(0,1)$-form vanishing at $x_{\hat{1}}$ and with norm 1. From equation (2.11), we see that the L^1-norm over the small annulus centered at $x_{\hat{1}}$ is bounded by $C(b)|\upsilon_{\hat{1}}|^2$; see also the proof of Lemma 4.13. Furthermore, since $x_{\hat{2}}$ is "dual" to $x_{\hat{1}}$, ψ_2 also vanishes at $x_{\hat{2}}$. Thus, the L^1-norm over the small annulus centered at $x_{\hat{2}}(w)$ is bounded by $C(b)(|w|+|\upsilon_{\hat{2}}|)|\upsilon_{\hat{2}}|$ as can be seen from equation (2.11).

Let $\tilde{s}^{(2,+)}_{b,x}\in T^*_x\Sigma$ be given by $s^{(2,+)}_{b,x}(\upsilon,\upsilon)=\tilde{s}^{(2,+)}_{b,x}(\upsilon)s_{\Sigma,x}(\upsilon)$. For any $b\in\mathcal{S}_{\mathcal{T},2}(\mu)$, define

$$\kappa(b)\in L^*_{\hat{2}}\mathcal{T}\otimes L_{\hat{1}}\mathcal{T}-\{0\}\quad\text{and}\quad\mu(b)\in L^*_{\hat{2}}\mathcal{T}\otimes L_{\hat{1}}\mathcal{T}\quad\text{by}$$
$$\big(\mathcal{D}_{\mathcal{T},\hat{2}}b\big)=\kappa(b)\big(\mathcal{D}_{\mathcal{T},\hat{1}}b\big),\quad\pi_b\big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}b\big)=\mu(b)\big(\mathcal{D}^{(1)}_{\mathcal{T},\hat{1}}b\big),$$

where $\pi_b:\mathrm{ev}^*T\mathbb{P}^n\longrightarrow\mathrm{Im}(\mathcal{D}_{\mathcal{T},\hat{1}})$ is the orthogonal projection map. If $(w,X,\upsilon)\in F^{\emptyset}\mathcal{S}|_{\mathcal{S}_{\mathcal{T},2}(\mu)}$ is sufficiently small, let $\tilde{\kappa}(w,X,\upsilon)\in\mathbb{C}^*$ be given by

$$\pi_{\phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,\upsilon)}\big(\mathcal{D}_{\mathcal{T},\hat{2}}\phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,\upsilon)\big)=\tilde{\kappa}(w,X,\upsilon)\big(\mathcal{D}_{\mathcal{T},\hat{1}}\phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,\upsilon)\big).$$

Note that by Theorem 2.7, $|\tilde{\kappa}(w,X,\upsilon)-\kappa(b)| \le C(b)(t+|\varpi|^{\frac{1}{p}})$. Let

$$Y^t(w,X,\upsilon)=\big(\mathcal{D}^{\mu}_{\mathcal{S},t\nu,\hat{1}}(w,X,\upsilon)\big)s_{\Sigma,x_{\hat{1}}}\big(\upsilon_{\hat{1}}+\tilde{\kappa}(w,X,\upsilon)\upsilon_{\hat{2}}+\mu(b)\tilde{s}^{(2,+)}_{\Sigma,x_{\hat{1}}}(\upsilon_{\hat{1}})\upsilon_{\hat{1}}\big),\ Y^{\perp}\big(X,\upsilon_{\hat{2}}\big)=Xs_{\Sigma,x_{\hat{1}}}\big(\upsilon_{\hat{2}}\big);$$
$$^{(2)}\alpha^-_{\mathcal{T};2}\big(w,\upsilon_{\hat{2}}\big) = \big(\mathcal{D}_{\mathcal{T},\hat{1}}b\big)s^{(2,-)}_{b,x_{\hat{1}}}\big(w,\upsilon_{\hat{2}}\big) + \big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}b\big)s^{(2,-)}_{b,x_{\hat{1}}}\big(\kappa(b)\upsilon_{\hat{2}}\big) + \big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{2}}b\big)s^{(2,-)}_{b,x_{\hat{1}}}\big(\upsilon_{\hat{2}}\big);$$
$$r^+_{\mathcal{T};2}(w,\upsilon) = \big(\mathcal{D}^{(1)}_{\mathcal{T},\hat{1}}(b)\big)s^{(2,+)}_{b,x_{\hat{1}}}\big(w,\upsilon_{\hat{2}}\big) + \pi^{\perp}_b\big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}(b)\big)s^{(2,+)}_{b,x_{\hat{1}}}\big(\kappa(b)\upsilon_{\hat{2}}\big) + \big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{2}}(b)\big)s^{(2,+)}_{b,x_{\hat{1}}}\big(\upsilon_{\hat{2}}\big).$$

Let $Y=Y^t+Y^{\perp}$ and $\bar{\nu}^{\pm}_b=\pi^{\pm}_{x_{\hat{1}}}\bar{\nu}_b$.

Corollary 4.21 *There exist $\delta, C\in C^{\infty}(\mathcal{S}_{\mathcal{T},2}(\mu);\mathbb{R}^+)$ such that for all $\varpi=[(b,w,X,\upsilon)]\in F^{\emptyset}\mathcal{S}_{\delta}$,*

$$\big\|\pi^+_{x_{\hat{1}}}\psi^{\mu}_{\mathcal{S},t\nu}(\varpi)-(t\bar{\nu}^+_b+Y(w,X,\upsilon)+r^+_{\mathcal{T};2}(w,\upsilon))\big\|_2 \le C(b)(t+|\varpi|^{\frac{1}{p}})\big(|\upsilon|^2+|w||\upsilon_{\hat{2}}|+|Y|\big);$$
$$\big\|\pi^-_{x_{\hat{1}}}\psi^{\mu}_{\mathcal{S},t\nu}(\varpi)-(t\bar{\nu}^-_b+{}^{(2)}\alpha^-_{\mathcal{T};2}(w,\upsilon_{\hat{2}}))\big\|_2 \le C(b)(t+|\varpi|^{\frac{1}{p}})\big(|\upsilon|^2+|w||\upsilon_{\hat{2}}|+|Y|\big).$$

Proof: The proof is similar to that of Corollary 4.17, but we use

$$\big|s_{\Sigma,x_{\hat{2}}(w)}(\upsilon_{\hat{2}})-(s_{\Sigma,x_{\hat{1}}}(\upsilon_{\hat{2}})+s^{(2)}_{b,x_{\hat{1}}}(w,\upsilon_{\hat{2}}))\big| \le C(b)|w|^2|\upsilon_{\hat{2}}|.$$

We also use $|\mathcal{D}^{\mu}_{\mathcal{S},t\nu,\hat{1}}(w,X,\upsilon)| \ge C(b)^{-1}$.

The next step is to apply Corollary 3.6. Let

$$F^+=\mathcal{H}^+_{\Sigma}\otimes\mathrm{ev}^*T\mathbb{P}^n,\quad F^-=\pi^*_{\Sigma,\hat{1}}T\Sigma\oplus F_{\hat{2}}\mathcal{T},\quad \mathcal{O}^{\pm}=\mathcal{H}^{\pm}_{\Sigma}\otimes\mathrm{ev}^*T\mathbb{P}^n,\quad \tilde{F}^-=\pi^*_{\Sigma,\hat{1}}T\Sigma\otimes F_{\hat{2}}\mathcal{T}\oplus F_{\hat{2}}\mathcal{T}^{\otimes 2};$$
$$\phi\big([b;w,\upsilon_{\hat{2}}]\big)=\big[b,w\otimes\upsilon_{\hat{2}},\upsilon_{\hat{2}}\otimes\upsilon_{\hat{2}}\big],\quad \alpha^-\big(\phi(w,\upsilon_{\hat{2}})\big)\equiv{}^{(2)}\alpha^-_{\mathcal{T},2}(w,\upsilon_{\hat{2}}),\quad \pi^+r(w,\upsilon)=r^+_{\mathcal{T};1}(w,\upsilon).$$

Note that $\alpha^-\in\Gamma(\mathcal{S};\tilde{F}^{-*}\otimes\mathcal{O}^-)$ is well-defined. Since the map

$$(w,X,\upsilon)\longrightarrow\big(Y(w,X,\upsilon),w,\upsilon_{\hat{2}}\big)$$

is injective on $F^{\emptyset}\mathcal{S}$ as long as $\delta\in C^{\infty}(\mathcal{S}_{\mathcal{T},2}(\mu);\mathbb{R}^+)$ is sufficiently small, we can view $\psi^{\mu}_{\mathcal{S},t\nu}$ as a map on an open subset of $F^-\oplus F^+$.

Corollary 4.22 *Suppose d is a positive integer, $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ is a simple bubble type, with $\hat{I}=\{\hat{1},\hat{2}\}$, $M_{\hat{0}}\mathcal{T}=\emptyset$, $d_{\hat{0}}=0$, and $\sum_{i\in I}d_i=d$, and μ is an N-tuple of constraints in general position such that*

$$codim_{\mathbb{C}}\mu = d(n+1)-n(g-1)+N.$$

Let $\nu\in\Gamma(\Sigma\times\mathbb{P}^n;\Lambda^{0,1}\pi^*_{\Sigma}T^*\Sigma\otimes\pi^*_{\mathbb{P}^n}T\mathbb{P}^n)$ be a generic section. Then there exists a compact subset $\tilde{K}_{\mathcal{T},2}$ of $\mathcal{S}_{\mathcal{T},2}(\mu)$ with the following property. If K is a compact subset of $\mathcal{S}_{\mathcal{T},2}(\mu)$ containing $\tilde{K}_{\mathcal{T},1}$, there exist a neighborhood U_K of K in $\bar{C}^{\infty}_{(d;[N])}(\Sigma;\mu)$ and $\epsilon_K>0$ such that for any $t\in(0,\epsilon_K)$, the signed cardinality of $U_K\cap\mathcal{M}_{\Sigma,d,t\nu}(\mu)$ equals to twice the signed number of zeros of the map

$$\pi^*_{\Sigma}T\Sigma^{\otimes 2}\otimes\big(L_{\hat{2}}\bar{\mathcal{T}}\oplus L_{\hat{2}}\bar{\mathcal{T}}^{\otimes 2}\big)\big|_{\Sigma^*\times\mathcal{S}_{\bar{\mathcal{T}};2}(\mu)}\longrightarrow\mathcal{H}^-_{\Sigma}\otimes ev^*T\mathbb{P}^n,$$
$$\big[(x,b);(w,v)\big]\longrightarrow\bar{\nu}^-_b+\big(\mathcal{D}_{\bar{\mathcal{T}},\hat{2}}b\big)s^{(2,-)}_x(w,v)+\big(\mathcal{D}^{(2)}_{\bar{\mathcal{T}},\hat{1}}b\big)s^{(2,-)}_x\big(\kappa(b)v\big)+\big(\mathcal{D}^{(2)}_{\bar{\mathcal{T}},\hat{2}}b\big)s^{(2,-)}_x(v). \tag{4.23}$$

Proof: The proof is similar to that of Corollary 4.14. We only need to see that the section α^- defined above has rank two. If $d_{\hat{1}}=d_{\hat{2}}=1$, the space $\mathcal{S}_{\mathcal{T},2}(\mu)=\emptyset$, since any two tangent lines in $\mathbb{P}^n$ agree, and no line passes through all of the constraints $\mu_1,\ldots,\mu_N$ if $n=3$. Thus, it can be assumed that $d_{\hat{1}}\ge 2$. Note that $S_{\mathcal{T},2}(\mu)$ is one-dimensional, with the only dimension coming from the singular point $x_{\hat{1}}\in\Sigma$. Thus, by Corollary 6.3, if the constraints $\mu_1,\ldots,\mu_N$ are in general position, the image of $\mathcal{D}^{(2)}_{\mathcal{T};\hat{1}}$ does not lie in the linear span of $\mathcal{D}_{\mathcal{T},\hat{2}}b$ and $\mathcal{D}^{(2)}_{\mathcal{T},\hat{2}}b$. Furthermore, $\mathcal{D}_{\mathcal{T},\hat{2}}b\neq 0$.

4.7 Second-Order Estimate for $\psi^{\mu}_{\mathcal{T},t\nu}$, Case 2b

We now treat Case (3b) of Corollary 4.11; we can assume $h_1=\hat{2}$, $h_2=\hat{3}$. Let $\mathcal{S}\equiv\mathcal{S}_{\mathcal{T},2}$ denote the subset of $\mathcal{M}_{\mathcal{T}}$ on which the operator $\bar{\mathcal{D}}_{\mathcal{T},2}$ of Lemma 4.9 has rank one. Similarly to the case of Subsection 4.6, $\mathcal{S}$ is a regular submanifold of $\mathcal{M}_{\mathcal{T}}$ with normal bundle $\mathcal{N}\mathcal{S}=L^*_{\hat{3}}\mathcal{T}\otimes E_1$. As before, we can choose a regularization $\big(\Phi_{\mathcal{S}},\Phi^{\mu}_{\mathcal{S}}\big)$ of $\mathcal{S}_{\mathcal{T},2}(\mu)\equiv\mathcal{S}\cap\mathcal{M}_{\mathcal{T}}(\mu)$ such that

$$\pi^{\perp}_{\tilde{\phi}_{\mathcal{S}}(b,X)}\mathcal{D}_{\mathcal{T},\hat{3}}\tilde{\phi}_{\mathcal{S}}(b,X)=\Pi_{b,\tilde{\phi}_{\mathcal{S}}(b,X)}X\quad\forall(b,X)\in\mathcal{N}\tilde{\mathcal{S}}_1=E_1,\tag{4.24}$$

where $\tilde{\phi}_{\mathcal{S}}$ is the lift of $\phi_{\mathcal{S}}$ to $\mathcal{M}^{(0)}_{\mathcal{T}}$, and $\Phi^{\mu}_{\mathcal{S}}$ is given by the $g_{\mathbb{P}^n,b}$-parallel transport on $\mathcal{N}_b\mathcal{S}$. Denote by $F\mathcal{S}$ and $F^{\emptyset}\mathcal{S}$ the bundles described in Subsection 2.4 corresponding to the submanifold $\mathcal{S}_{\mathcal{T},2}$. If (X,υ) is a sufficiently small element of $F\mathcal{S}=\mathcal{N}\mathcal{S}\oplus F\mathcal{T}$, let

$$\begin{aligned}\tilde{\alpha}(X,\upsilon)&=\big(\mathcal{D}_{\mathcal{T},\hat{2}}\phi_{\mathcal{S}}(b,X)\big)s_{\Sigma,\tilde{x}_{\hat{2}}(\upsilon)}(\tilde{\upsilon}_{\hat{2}})+\big(\mathcal{D}_{\mathcal{T},\hat{3}}\phi_{\mathcal{S}}(b,X)\big)s_{\Sigma,\tilde{x}_{\hat{3}}(\upsilon)}(\tilde{\upsilon}_{\hat{3}});\\ \tilde{\alpha}^{\mu}(X,\upsilon)&=\big(\mathcal{D}^{\mu}_{\mathcal{S},t\nu,\hat{2}}(X,\upsilon)\big)s_{\Sigma,\tilde{x}_{\hat{2}}(\upsilon)}(\tilde{\upsilon}_{\hat{2}})+\big(\mathcal{D}^{\mu}_{\mathcal{S},t\nu,\hat{3}}(X,\upsilon)\big)s_{\Sigma,x_{\hat{3}}(\upsilon)}(\upsilon_{\hat{3}}),\end{aligned}$$

where, with $\varphi^{\mu}_{\mathcal{S},t\nu}$ as in Theorem 2.7,

$$\mathcal{D}^{\mu}_{\mathcal{S},t\nu,h}(X,\upsilon)=\Pi^{-1}_{\phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(X,\upsilon),\phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(X,\upsilon)}\Pi^{-1}_{b,\phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(X,\upsilon)}\big(\mathcal{D}_{\mathcal{T},h}\phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(X,\upsilon)\big).$$

Lemma 4.23 *There exist $\delta,C\in C^{\infty}(\mathcal{S}_{\mathcal{T},2};\mathbb{R}^+)$ such that for all $\varpi=[(b,X,\upsilon)]\in F^{\emptyset}\mathcal{S}_{\delta}$,*

$$\big\|\pi^{0,1}_{\Phi_{\mathcal{S}}(\varpi),-}\bar{\partial}u_{\Phi_{\mathcal{S}}(\varpi)}+\tilde{R}_{\Phi_{\mathcal{S}}(\varpi)}\tilde{\alpha}(X,\upsilon)\big\|_2\le C(b)\big(|\tilde{\upsilon}_{\hat{2}}|^2+|\tilde{\upsilon}_{\hat{3}}|^2\big).$$

Proof: This lemma is immediate from Proposition 4.4 applied with one term for each $h=\hat{2},\hat{3}$.

Lemma 4.24 *There exist $\delta,C\in C^{\infty}(\mathcal{S}_{\mathcal{T},2}(\mu);\mathbb{R}^+)$ such that for all $\varpi=[(b,X,\upsilon)]\in F^{\emptyset}\mathcal{S}_{\delta}$,*

$$\big\|\psi^{\mu}_{\mathcal{S},t\nu}(\varpi)-(t\bar{\nu}_b+\tilde{\alpha}^{\mu}(X,\upsilon))\big\|_2\le C(b)(t+|\varpi|^{\frac{1}{p}})\big(t+|\upsilon_{\hat{1}}|(|\tilde{\upsilon}_{\hat{2}}|+|\tilde{\upsilon}_{\hat{3}}|)\big).$$

Proof: As usually, we only need to obtain a good bound on

$$\big\|D^*_{\Phi_{\mathcal{S}}(\varpi)}R_{\Phi_{\mathcal{S}}(\varpi)}X_i\psi_2\|_{L^1},$$

where the notation is as in the proof of Lemma 4.20. By equation (2.11), the L^1-norm on the small annulus centered at $\tilde{x}_{\hat{2}}(\upsilon)$ is bounded by $|\tilde{\upsilon}_{\hat{2}}|^2$. Since $g_{b,\hat{0}}$-distance between $\tilde{x}_{\hat{2}}(\upsilon)$ and $\tilde{x}_{\hat{3}}(\upsilon)$ is bounded by $C(b)|\upsilon_{\hat{1}}|$, the L_1-norm over the annulus centered at $\tilde{x}_{\hat{3}}(\upsilon)$ is bounded by $|\upsilon_{\hat{1}}||\tilde{\upsilon}_{\hat{3}}|$.

For any $b\in\mathcal{S}_{\mathcal{T},2}(\mu)$, let $\kappa(b)\in L_{\hat{3}}\mathcal{T}^*\otimes L_{\hat{2}}\mathcal{T}$ be given by $\mathcal{D}_{\mathcal{T},\hat{3}}b=\kappa(b)(\mathcal{D}_{\mathcal{T},\hat{2}}b)$. For $(X,\upsilon)\in F^{\emptyset}\mathcal{S}|_{\mathcal{S}_{\mathcal{T},2}(\mu)}$ sufficiently small, we define the nonzero element $\tilde{\kappa}(X,\upsilon)$ of $L_{\hat{3}}\mathcal{T}^*\otimes L_{\hat{2}}\mathcal{T}$ by

$$\pi_{\phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(X,\upsilon)}\big(\mathcal{D}_{\mathcal{T},\hat{3}}\phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(X,\upsilon)\big)=\tilde{\kappa}(X,\upsilon)\big(\mathcal{D}_{\mathcal{T},\hat{2}}\phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(X,\upsilon)\big).$$

Note that by Theorem 2.7, $|\tilde{\kappa}(X,\upsilon)-\kappa(b)|\le C(b)(t+|\varpi|^{\frac{1}{p}})$. Let

$$\begin{aligned}Y^t(X,\upsilon)&=\big(\mathcal{D}^{\mu}_{\mathcal{S},t\nu,\hat{2}}(X,\upsilon)\big)\big(s_{\Sigma,x_{\hat{1}}}(\tilde{\upsilon}_{\hat{2}}+\tilde{\kappa}(X,\upsilon)\tilde{\upsilon}_{\hat{3}})+s^{(2,+)}_{b,x_{\hat{1}}}(\upsilon_{\hat{1}},x_{\hat{2}}\tilde{\upsilon}_{\hat{2}}+x_{\hat{3}}\tilde{\kappa}(b)\tilde{\upsilon}_{\hat{3}}));\\ Y^{\perp}(X,\upsilon)&=Xs_{\Sigma,x_{\hat{1}}}(\tilde{\upsilon}_{\hat{3}}),\quad {}^{(2)}\alpha^{-}_{\mathcal{T};2}(\upsilon)=\big(\mathcal{D}_{\mathcal{T},\hat{2}}b\big)s^{(2,-)}_{b,x_{\hat{1}}}\big(\upsilon_{\hat{1}},x_{\hat{2}}\tilde{\upsilon}_{\hat{2}}+x_{\hat{3}}\kappa(b)\tilde{\upsilon}_{\hat{3}}\big).\end{aligned}$$

Let $Y=Y^t+Y^{\perp}$ and $\bar{\nu}^{\pm}_b=\pi^{\pm}_{x_{\hat{1}}}\bar{\nu}_b$.

Corollary 4.25 *There exist $\delta, C\in C^{\infty}(\mathcal{S}_{\mathcal{T},2}(\mu);\mathbb{R}^+)$ such that for all $\varpi=[(b,X,\upsilon)]\in F^{\emptyset}\mathcal{S}_{\delta}$,*

$$\begin{aligned}\big\|\pi_{x_{\hat{1}}}\psi^{\mu}_{\mathcal{S},t\nu}(X,\varpi)-(t\bar{\nu}_b^{+}+Y(X,\upsilon))\big\|_2 &\le C(b)(t+|\varpi|^{\frac{1}{p}})\big(t+|\upsilon_{\hat{1}}|(|\tilde{v}_{\hat{2}}|+|\tilde{v}_{\hat{3}}|)+|Y^{\perp}(X,\upsilon)|\big);\\ \big\|\pi^{-}_{x_{\hat{1}}}\psi^{\mu}_{\mathcal{S},t\nu}(\varpi)-(t\bar{\nu}_b^{-}+{}^{(2)}\alpha^{-}_{\mathcal{T};2}(\upsilon))\big\|_2 &\le C(b)(t+|\varpi|^{\frac{1}{p}})\big(t+|\upsilon_{\hat{1}}|(|\tilde{v}_{\hat{2}}|+|\tilde{v}_{\hat{3}}|)\big).\end{aligned}$$

Proof: This claim is proved similarly to Corollary 4.21.

The next step is to apply Lemma 3.2. Let

$$\begin{gathered}F^{+}=\mathcal{H}^{+}_{\Sigma}\otimes E_1,\quad F^{-}=F\mathcal{T},\quad \mathcal{O}^{\pm}=\mathcal{H}^{\pm}_{\Sigma}\otimes \mathrm{ev}^*T\mathbb{P}^n,\quad \tilde{F}^{-}=\pi_{\Sigma}^*T\Sigma\otimes F_{\hat{2}}\mathcal{T};\\ \phi\big([b;\upsilon]\big)=\big[b,\upsilon_{\hat{1}}\otimes(x_{\hat{2}}\tilde{v}_{\hat{2}}+x_{\hat{3}}\kappa(b)\tilde{v}_{\hat{3}})\big],\quad \alpha^{-}(\phi(\upsilon))\equiv {}^{(2)}\alpha^{-}_{\mathcal{T},2}(\upsilon),\quad \alpha(X,\upsilon)=Y(X,\upsilon)+{}^{(2)}\alpha^{-}_{\mathcal{T},2}(\upsilon).\end{gathered}$$

Note that $\alpha^{-}\in\Gamma(\mathcal{S};\tilde{F}^{-*}\otimes\mathcal{O}^{-})$ is well-defined. Since the map

$$(X,\upsilon)\longrightarrow \big(Y^{\perp}(X,\upsilon),\upsilon\big)$$

is injective on $F^{\emptyset}\mathcal{S}$, we can view $\psi^{\mu}_{\mathcal{S},t\nu}$ as a map on an open subset of $F^{+}\oplus F^{-}$.

Corollary 4.26 *Suppose d is a positive integer, $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ is a simple bubble type, with $\hat{I}=\{\hat{1},\hat{2},\hat{3}\}$, $H_{\hat{1}}\mathcal{T}=\{\hat{2},\hat{3}\}$, $d_{\hat{0}}=0$, and $\sum_{i\in I}d_i=d$, and μ is an N-tuple of constraints in general position such that*

$$\mathit{codim}_{\mathbb{C}}\mu=d(n+1)-n(g-1)+N.$$

Let $\nu\in\Gamma(\Sigma\times\mathbb{P}^n;\Lambda^{0,1}\pi_{\Sigma}^*T^*\Sigma\otimes\pi_{\mathbb{P}^n}^*T\mathbb{P}^n)$ be a generic section. For every compact subset K of $\mathcal{S}_{\mathcal{T},2}(\mu)$, such that $x_{\hat{1}}(b)\in\Sigma^*$ for all $b\in K$, there exist a neighborhood U_K of K in $\bar{C}^{\infty}_{(d;[N])}(\Sigma;\mu)$, where and $\epsilon_K>0$ such that for any $t\in(0,\epsilon_K)$, $U_K\cap\mathcal{M}_{\Sigma,d,t\nu}(\mu)=\emptyset$.

Proof: The set $\mathcal{S}^*_{\mathcal{T},2}(\mu)\equiv\{b\in\mathcal{S}_{\mathcal{T},2}(\mu)\colon x_{\hat{1}}\in\Sigma^*\}$ is an open subset of $\mathcal{S}_{\mathcal{T},2}(\mu)$ on which the section α^{-} has full rank, since $\mathcal{D}_{\mathcal{T},\hat{2}}$ does not vanish on $\mathcal{S}_{\mathcal{T},2}(\mu)$. Note that the dimension of $\mathcal{S}_{\mathcal{T},2}(\mu)$ is 1, the rank of $\tilde{F}^{-}$ is also 1, while the rank $\mathcal{O}^{-}$ is 3. Thus, the claim follows from Theorem 2.7, Lemma 3.2, and Corollary 4.25, provided

$$|\upsilon_{\hat{1}}|\big(|\tilde{v}_{\hat{2}}|+|\tilde{v}_{\hat{3}}|\big)\le C(b)\big(|\upsilon_{\hat{1}}||x_{\hat{2}}\tilde{v}_{\hat{2}}+x_{\hat{3}}\kappa(b)\tilde{v}_{\hat{3}}|+|Y^{t}(X,\upsilon)|\big)$$

for some $C\in C^{\infty}(\mathcal{S}^*_{\mathcal{T},2}(\mu);\mathbb{R}^+)$. By definition of $Y^t(X,\upsilon)$,

$$|\tilde{v}_{\hat{2}}+\kappa(b)\tilde{v}_{\hat{3}}|\le|Y^t(X,\upsilon)|+C(b)|x_{\hat{2}}\tilde{v}_{\hat{2}}+x_{\hat{3}}\kappa(b)\tilde{v}_{\hat{3}}|.$$

Since $x_{\hat{2}}\neq x_{\hat{3}}$,

$$\begin{aligned}|\upsilon_{\hat{1}}|\big(|\tilde{v}_{\hat{2}}|+|\tilde{v}_{\hat{3}}|\big)&\le C(b)|\upsilon_{\hat{1}}|\big(|\tilde{v}_{\hat{2}}+\kappa(b)\tilde{v}_{\hat{3}}|+|x_{\hat{2}}\tilde{v}_{\hat{2}}+x_{\hat{3}}\kappa(b)\tilde{v}_{\hat{3}}|\big)\\ &\le C'(b)|\upsilon_{\hat{1}}|\big(|x_{\hat{2}}\tilde{v}_{\hat{2}}+x_{\hat{3}}\kappa(b)\tilde{v}_{\hat{3}}|+|Y^t(X,\upsilon)|\big).\end{aligned}$$

4.8 Third-Order Estimate for $\psi^{\mu}_{\mathcal{T},t\nu}$, Case 2

It remains to consider gluing along the subset $\mathcal{S}^{(m)}_{\mathcal{T},2}(\mu)$ of $\mathcal{S}_{\mathcal{T},2}(\mu)$ consisting of bubble maps b such that $x_{\hat{1}}(b)=z_m$, one of the six distinguished points of Σ. Let

$$\mathcal{S}=\mathcal{S}^{(m)}_{\mathcal{T},2}=\{b\in\mathcal{S}_{\mathcal{T},2}\colon x_{\hat{1}}(b)=z_m\}.$$

The normal bundle of $\mathcal{S}^{(m)}_{\mathcal{T},2}$ in $\mathcal{M}_{\mathcal{T}}$ is $\mathcal{N}\mathcal{S}=T_{z_m}\Sigma\oplus\mathcal{N}\mathcal{S}_1$, where $\mathcal{N}\mathcal{S}_1$ is the normal bundle of $\mathcal{S}_{\mathcal{T},2}$ in $\mathcal{M}_{\mathcal{T}}$ described in the previous subsection. Let $\big(\Phi_{\mathcal{S}},\Phi^{\mu}_{\mathcal{S}}\big)$ be a regularization of $\mathcal{S}^{(m)}_{\mathcal{T},2}(\mu)$ induced by the regularization of $\mathcal{S}_{\mathcal{T},2}(\mu)$ described in Subsection 4.7. In particular,

$$\pi^{\perp}_{\phi_{\mathcal{S}}(b,X)}\mathcal{D}_{\mathcal{T},\hat{3}}\tilde{\phi}_{\mathcal{S}}(b,w,X)=\Pi_{b,\tilde{\phi}_{\mathcal{S}}(b,w,X)}X\quad\forall(b,w,X)\in T_{z_m}\Sigma\oplus\mathcal{N}\tilde{\mathcal{S}}_1=T_{z_m}\Sigma\oplus E_1,$$

where $\tilde{\phi}_{\mathcal{S}}$ is the lift of $\phi_{\mathcal{S}}$ to $\mathcal{M}^{(0)}_{\mathcal{T}}$. We also assume that $\Phi^{\mu}_{\mathcal{S}}$ is given by the $g_{\mathbb{P}^n,b}$-parallel transport on $\mathcal{N}_b\mathcal{S}_1$. The bundle $\mathcal{N}\mathcal{S}$ carries a natural norm induced by the $g_{\mathbb{P}^n,\mathrm{ev}}$-metric on $\mathbb{P}^n$ and $g_{\cdot,\hat{0}}$-metric on Σ. Denote by $F\mathcal{S}$ and $F^{\emptyset}\mathcal{S}$ the bundles described in Subsection 2.4 corresponding to the submanifold $\mathcal{S}^{(m)}_{\mathcal{T},2}$. If $(b,w,X,\upsilon)\in F^{\emptyset}\mathcal{S}$ is sufficiently small, let

$$\tilde{x}_h(w,\upsilon)=\tilde{x}_h(\phi_{\mathcal{S}}(w,X,\upsilon))=\tilde{x}_h(\phi_{\mathcal{S}}(w,0,\upsilon))\in\Sigma,\quad h=\hat{2},\hat{3}.$$

We identify a small neighborhood of z_m in Σ with a neighborhood of 0 in $T_{z_m}\Sigma$ via the $g_{b,\hat{0}}$-exponential map. Put

$$\begin{aligned}\tilde{\alpha}(w,X,\upsilon)=\Pi^{-1}_{b,\phi_{\mathcal{S}}(b,X)}\big((\mathcal{D}_{\mathcal{T},\hat{2}}\phi_{\mathcal{S}}(b,X))s_{\Sigma,\tilde{x}_{\hat{2}}(w,\upsilon)}(\tilde{v}_{\hat{2}})+(\mathcal{D}_{\mathcal{T},\hat{3}}\phi_{\mathcal{S}}(b,X))s_{\Sigma,\tilde{x}_{\hat{3}}(w,\upsilon)}(\tilde{v}_{\hat{3}})\\+(\mathcal{D}^{(2)}_{\mathcal{T},\hat{2}}\phi_{\mathcal{S}}(b,X))s^{(2)}_{b,z_m}(\tilde{v}_{\hat{2}})+(\mathcal{D}^{(2)}_{\mathcal{T},\hat{3}}\phi_{\mathcal{S}}(b,X))s^{(2)}_{b,z_m}(\tilde{v}_{\hat{3}})\big);\\ \tilde{\alpha}^{\mu}(w,X,\upsilon)=\big((\mathcal{D}^{\mu}_{\mathcal{S},t\nu,\hat{2}}(w,X,\upsilon))s_{\Sigma,\tilde{x}_{\hat{2}}(w,\upsilon)}(\tilde{v}_{\hat{2}})+(\mathcal{D}^{\mu}_{\mathcal{S},t\nu,\hat{3}}(w,X,\upsilon))s_{\Sigma,x_{\hat{3}}(w,\upsilon)}(v_{\hat{3}})\big)\\+\big((\mathcal{D}^{\mu,(2)}_{\mathcal{S},t\nu,\hat{2}}b)s^{(2)}_{b,z_m}(\tilde{v}_{\hat{2}})+(\mathcal{D}^{\mu,(2)}_{\mathcal{S},t\nu,\hat{3}}b)s^{(2)}_{b,z_m}(\tilde{v}_{\hat{3}})\big),\end{aligned}$$

where, with $\varphi^{\mu}_{\mathcal{S},t\nu}$ as in Theorem 2.7,

$$\mathcal{D}^{\mu,(k)}_{\mathcal{S},t\nu,h}(w,X,\upsilon)=\Pi^{-1}_{\phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,\upsilon),\phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,\upsilon)}\Pi^{-1}_{b,\phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,\upsilon)}\big(\mathcal{D}^{(k)}_{\mathcal{T},h}\phi_{\mathcal{S}}\Phi^{\mu}_{\mathcal{S}}\varphi^{\mu}_{\mathcal{S},t\nu}(w,X,\upsilon)\big).$$

With $\kappa(b)$ as in the previous subsection, let

$$\begin{gathered}\alpha^{+}(\upsilon)=\big(\mathcal{D}_{\mathcal{T},\hat{2}}b\big)s_{\Sigma,z_m}\big(\tilde{v}_{\hat{2}}+\kappa(b)\tilde{v}_{\hat{3}}\big),\quad\alpha^{-}_{\hat{2}}(w,\upsilon)=\big(\mathcal{D}_{\mathcal{T},\hat{2}}b\big)s^{(3,-)}_{b,z_m}\big(\tilde{x}_{\hat{2}}(w,\upsilon),(x_{\hat{2}}-x_{\hat{3}})v_{\hat{1}},\tilde{v}_{\hat{2}}\big);\\ \alpha^{-}_{\hat{3}}(w,\upsilon)=\big(\mathcal{D}_{\mathcal{T},\hat{3}}b\big)s^{(3,-)}_{b,z_m}\big(\tilde{x}_{\hat{3}}(w,\upsilon),(x_{\hat{3}}-x_{\hat{2}})v_{\hat{1}},\tilde{v}_{\hat{3}}\big).\end{gathered}$$

Lemma 4.27 *There exist $\delta,C\in C^{\infty}(\mathcal{S}^{(m)}_{\mathcal{T},2};\mathbb{R}^+)$ such that for all $\varpi=(b,w,X,\upsilon)\in F^{\emptyset}\mathcal{S}_{\delta}$,*

$$\big\|\pi^{0,1}_{\Phi_{\mathcal{S}}(\varpi),-}\bar{\partial}u_{\Phi_{\mathcal{S}}(\varpi)}-\tilde{R}_{\Phi_{\mathcal{S}}(\varpi)}\tilde{\alpha}(w,X,\upsilon)\big\|_2\le C(b)|\varpi|\big(|\tilde{v}_{\hat{2}}|^2+|\tilde{v}_{\hat{3}}|^2\big).$$

Proof: This lemma follows from Proposition 4.4 applied with first- and second-order terms.

Lemma 4.28 *There exist $\delta,C>0$ such that for all $\varpi=(b,w,X,\upsilon)\in F^{\emptyset}\mathcal{S}_{\delta}\big|_{\mathcal{S}^{(m)}_{\mathcal{T};2}(\mu)}$,*

$$\big\|\psi^{\mu}_{\mathcal{S},t\nu}(\varpi)-(t\bar{\nu}_b+\tilde{\alpha}^{\mu}(w,X,\upsilon))\big\|_2\le C(t+|\varpi|^{\frac{1}{p}})\big(t+(|v_{\hat{1}}|^2+|v_{\hat{1}}||w|)(|\tilde{v}_{\hat{2}}|+|\tilde{v}_{\hat{3}}|)\big).$$

Proof: Note that the space $\mathcal{S}^{(m)}_{\mathcal{T},2}(\mu)$ is zero-dimensional and compact if $n=3$. As before, we need to bound

$$\big\|D^*_{\Phi_{\mathcal{S}}(\varpi)}R_{\Phi_{\mathcal{S}}(\varpi)}X_i\psi_2\big\|_{L^1},$$

where the notation is as in the proof of Lemma 4.20. By equation (2.11), the L^1-norm on the annulus centered at $\tilde{x}_{\hat{2}}=\tilde{x}_{\hat{2}}(w,\upsilon)$ is bounded by $(|\tilde{x}_{\hat{2}}||\tilde{v}_{\hat{2}}|+|\tilde{v}_{\hat{2}}|^2)|\tilde{v}_{\hat{2}}|$, while the norm over the other annulus is bounded by $(|\tilde{x}_{\hat{2}}||v_{\hat{1}}|+|v_{\hat{1}}|^2)|\tilde{v}_{\hat{3}}|$, since the $g_{b,\hat{0}}$-distance between $\tilde{x}_{\hat{2}}$ and $\tilde{x}_{\hat{3}}$ is bounded by $C|v_{\hat{1}}|$. See the proof of Lemma 4.16 for more detail. The claim follows from $\tilde{x}_{\hat{2}}=w+x_{\hat{2}}v_{\hat{1}}$.

Lemma 4.29 *There exist $\delta, C>0$ such that for all $\varpi=(b,w,X,\upsilon)\in F^{\emptyset}\mathcal{S}_\delta\big|_{\mathcal{S}^{(m)}_{\mathcal{T};2}(\mu)}$,*

$$\begin{aligned}\big\|\tilde{\alpha}^\mu(w,X,\upsilon)-\alpha^+(w,\upsilon)\big\|_2&\le C(t+|\varpi|^{\frac{1}{p}})\big(|\tilde{v}_{\hat{2}}|+|\tilde{v}_{\hat{3}}|\big);\\ \big\|\pi^-_{\tilde{x}_{\hat{2}}(w,\upsilon)}\tilde{\alpha}^\mu(w,X,\upsilon)-\alpha^-_{\hat{3}}(w,\upsilon)\big\|&\le C(t+|\varpi|^{\frac{1}{p}})\big(|v_{\hat{1}}|+|w|\big)|v_{\hat{1}}|\big(|\tilde{v}_{\hat{2}}|+|\tilde{v}_{\hat{3}}|\big);\\ \big\|\pi^-_{\tilde{x}_{\hat{3}}(w,\upsilon)}\tilde{\alpha}^\mu(w,X,\upsilon)-\alpha^-_{\hat{2}}(w,\upsilon)\big\|&\le C(t+|\varpi|^{\frac{1}{p}})\big(|v_{\hat{1}}|+|w|\big)|v_{\hat{1}}|\big(|\tilde{v}_{\hat{2}}|+|\tilde{v}_{\hat{3}}|\big).\end{aligned}$$

Proof: The first bound is clear from the definition of $\tilde{\alpha}^\mu$, since

$$\big(\mathcal{D}_{\mathcal{T},\hat{3}}b\big)=\kappa(b)\big(\mathcal{D}_{\mathcal{T},\hat{2}}b\big),\qquad |\varphi(w,X,\upsilon)|_b\le C(t+|\varpi|^{\frac{1}{p}}).$$

Since $s^{(2,-)}_{b,z_m}=0$,

$$\big|\pi^-_{\tilde{x}_{\hat{2}}}s^{(2)}_{\tilde{x}_h}(\tilde{v}_h)\big|\le C\big(|\tilde{x}_{\hat{2}}|+|v_{\hat{1}}|\big)|\tilde{v}_h|^2. \tag{4.25}$$

where $\tilde{x}_h=\tilde{x}_h(w,\upsilon)$. Since $\tilde{x}_{\hat{3}}-\tilde{x}_{\hat{2}}=(x_{\hat{3}}-x_{\hat{2}})v_{\hat{1}}$,

$$\Big|s_{b,\tilde{x}_{\hat{3}}}(\tilde{v}_{\hat{3}})-\big(s_{b,\tilde{x}_{\hat{2}}}(\tilde{v}_{\hat{3}})+s^{(2)}_{b,\tilde{x}_{\hat{2}}}((x_{\hat{3}}-x_{\hat{2}})v_{\hat{1}},\tilde{v}_{\hat{3}})+s^{(3)}_{b,\tilde{x}_{\hat{2}}}((x_{\hat{3}}-x_{\hat{2}})v_{\hat{1}},(v_{\hat{3}}-v_{\hat{2}})v_{\hat{1}},\tilde{v}_{\hat{3}})\big)\Big|\le C|v_{\hat{1}}|^3|\tilde{v}_{\hat{3}}|.$$

Since $\pi^-_{\tilde{x}_{\hat{2}}}s_{\Sigma,\tilde{x}_{\hat{2}}}=0$ and $s^{(2,-)}_{b,z_m}=0$,

$$\begin{aligned}\big|\pi^-_{\tilde{x}_{\hat{2}}}s^{(2)}_{b,\tilde{x}_{\hat{2}}}((x_{\hat{3}}-x_{\hat{2}})v_{\hat{1}},\tilde{v}_{\hat{3}})-s^{(3,-)}_{b,z_m}(\tilde{x}_{\hat{2}},(x_{\hat{3}}-x_{\hat{2}})v_{\hat{1}},\tilde{v}_{\hat{3}})\big|&\le C|\tilde{x}_{\hat{2}}|^2|v_{\hat{1}}||\tilde{v}_{\hat{3}}|;\\ \big|\pi^-_{\tilde{x}_{\hat{2}}}s^{(3)}_{b,\tilde{x}_{\hat{2}}}((x_{\hat{3}}-x_{\hat{2}})v_{\hat{1}},(x_{\hat{3}}-x_{\hat{2}})v_{\hat{1}},\tilde{v}_{\hat{3}})-s^{(3,-)}_{b,z_m}((x_{\hat{3}}-x_{\hat{2}})v_{\hat{1}},(x_{\hat{3}}-x_{\hat{2}})v_{\hat{1}},\tilde{v}_{\hat{3}})\big|&\le C|\tilde{x}_{\hat{2}}||v_{\hat{1}}|^2|\tilde{v}_{\hat{3}}|.\end{aligned}$$

Putting the last three equations together, we see that

$$\Big|\pi^-_{\tilde{x}_{\hat{2}}}s_{b,\tilde{x}_{\hat{3}}}(\tilde{v}_{\hat{3}})-s^{(3,-)}_{b,z_m}\big(\tilde{x}_{\hat{3}},(x_{\hat{3}}-x_{\hat{2}})v_{\hat{1}},\tilde{v}_{\hat{3}}\big)\Big|\le C\big(|\tilde{x}_{\hat{2}}|+|v_{\hat{1}}|\big)\big(|\tilde{x}_{\hat{2}}||v_1|+|v_{\hat{1}}|^2\big)|\tilde{v}_{\hat{3}}|. \tag{4.26}$$

The second bound follows from equations (4.25) and (4.26). The last estimate is proved similarly.

Corollary 4.30 *There exist $\delta, C>0$ such that for all $\varpi=(b,w,X,\upsilon)\in F^{\emptyset}\mathcal{S}_\delta\big|_{\mathcal{S}^{(m)}_{\mathcal{T};2}(\mu)}$,*

$$\big\|\psi^\mu_{\mathcal{S},t\nu}(\varpi)-(t\bar{\nu}_b+\tilde{\alpha}^\mu(\varpi))\big\|_2\le C(t+|\varpi|^{\frac{1}{p}})\big(t+|\tilde{\alpha}^\mu(\varpi)|\big).$$

Proof: In light of Lemma 4.28, it is sufficient to show that

$$\big(|\upsilon_{\hat{1}}|+|w|\big)|\upsilon_{\hat{1}}|\big(|\tilde{\upsilon}_{\hat{2}}|+|\tilde{\upsilon}_{\hat{3}}|\big) \le C\big|\tilde{\alpha}^{\mu}(w,X,\upsilon)\big| \tag{4.27}$$

for some $C>0$. Since $\big(\mathcal{D}_{\mathcal{T},\hat{2}}b\big)s_{\Sigma,z_m}$, $\big(\mathcal{D}_{\mathcal{T},\hat{2}}b\big)s^{(3,-)}_{b,z_m}$ and $\big(\mathcal{D}_{\mathcal{T},\hat{3}}b\big)s^{(3,-)}_{b,z_m}$ are nonzero, by Lemma 4.29

$$\begin{aligned}\big|\tilde{\upsilon}_{\hat{2}}+\kappa(b)\tilde{\upsilon}_{\hat{3}}\big| &\le C\big(|\tilde{\alpha}^{\mu}(w,X,\upsilon)|+(t+|\varpi|^{\frac{1}{p}})(|\tilde{\upsilon}_{\hat{2}}|+|\tilde{\upsilon}_{\hat{3}}|)\big);\\ |\tilde{x}_h||\upsilon_{\hat{1}}||\tilde{\upsilon}_h| &\le C\big(|\tilde{\alpha}^{\mu}(w,X,\upsilon)|+(t+|\varpi|^{\frac{1}{p}})(|\upsilon_{\hat{1}}|+|w|)|\upsilon_{\hat{1}}|(|\tilde{\upsilon}_{\hat{2}}|+|\tilde{\upsilon}_{\hat{3}}|)\big).\end{aligned}$$

Since $\kappa(b)\neq 0$, $x_{\hat{2}}\neq x_{\hat{3}}$, and $\tilde{x}_h=w+x_h\upsilon_{\hat{1}}$, we obtain

$$\begin{aligned}\big(|\upsilon_{\hat{1}}|+|w|\big)|\upsilon_{\hat{1}}|\big(|\tilde{\upsilon}_{\hat{2}}|+|\tilde{\upsilon}_{\hat{3}}|\big) &\le C\big(|\tilde{x}_{\hat{2}}|+|\tilde{x}_{\hat{3}}|\big)|\upsilon_{\hat{1}}|\big(|\tilde{\upsilon}_{\hat{2}}|+|\tilde{\upsilon}_{\hat{3}}|\big)\\ &\le C'\big(|\tilde{x}_{\hat{2}}||\upsilon_{\hat{1}}|(|\tilde{\upsilon}_{\hat{2}}|+|\tilde{\upsilon}_{\hat{2}}+\kappa(b)\tilde{\upsilon}_{\hat{3}}|)+|\tilde{x}_{\hat{3}}||\upsilon_{\hat{1}}|(|\tilde{\upsilon}_{\hat{3}}|+|\tilde{\upsilon}_{\hat{3}}+\kappa(b)\tilde{\upsilon}_{\hat{3}}|)\big)\\ &\le C''\big(|\tilde{\alpha}^{\mu}(w,X,\upsilon)|+(t+|\varpi|^{\frac{1}{p}})(|\upsilon_{\hat{1}}|+|w|)|\upsilon_{\hat{1}}|(|\tilde{\upsilon}_{\hat{2}}|+|\tilde{\upsilon}_{\hat{3}}|)\big).\end{aligned} \tag{4.28}$$

If δ is sufficiently small, estimate (4.27) follows from (4.28).

The next step is to apply Lemma 3.2. Let

$$\begin{gathered}F^+=L^*_{\hat{3}}\mathcal{T}\otimes E_1,\quad F^-=T_{z_m}\Sigma\oplus F\mathcal{T},\quad \mathcal{O}^{\pm}=\mathcal{H}^{\pm}_{\Sigma}\otimes\mathrm{ev}^*T\mathbb{P}^n,\quad \tilde{F}^-=\pi^*_{\Sigma}T\Sigma^{\otimes 3}\otimes L_{\hat{2}}\mathcal{T}^{\otimes 3};\\ \phi\big([b,w,\upsilon]\big)=\big[b,(w+x_{\hat{2}}\upsilon_{\hat{1}})\otimes((x_{\hat{2}}-x_{\hat{2}})\upsilon_{\hat{1}})\otimes\tilde{\upsilon}_{\hat{2}}\big];\\ \alpha^-\big(\phi(w,\upsilon)\big)\equiv\alpha^-_{\hat{2}}(w,\upsilon),\quad \alpha(X,w,\upsilon)=\alpha^{\mu}(X,w,\upsilon).\end{gathered}$$

Note that $\alpha^-\in\Gamma(\mathcal{S};\tilde{F}^{-*}\otimes\mathcal{O}^-)$ is well-defined.

Corollary 4.31 *Suppose d is a positive integer, $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ is a simple bubble type, with $\hat{I}=\{\hat{1},\hat{2},\hat{3}\}$, $H_{\hat{1}}\mathcal{T}=\{\hat{2},\hat{3}\}$, $d_{\hat{0}}=0$, and $\sum_{i\in I}d_i=d$, and μ is an N-tuple of constraints in general position such that*

$$\mathit{codim}_{\mathbb{C}}\mu=d(n+1)-n(g-1)+N.$$

Let $\nu\in\Gamma(\Sigma\times\mathbb{P}^n;\Lambda^{0,1}\pi^*_{\Sigma}T^*\Sigma\otimes\pi^*_{\mathbb{P}^n}T\mathbb{P}^n)$ be a generic section. There exist a neighborhood U of $\mathcal{S}^{(m)}_{\mathcal{T},2}(\mu)$ in $\bar{C}^{\infty}_{(d;[N])}(\Sigma;\mu)$, and $\epsilon>0$ such that for any $t\in(0,\epsilon)$, $U\cap\mathcal{M}_{\Sigma,d,t\nu}(\mu)=\emptyset$.

Proof: Analogously to the proof of Corollary 4.18, we apply Lemma 3.2 to the map

$$(w,\upsilon,X)\longrightarrow\pi^+_{z_m}\pi^+_{x_{\hat{3}}(w,\upsilon)}\psi^{\mu}_{\mathcal{S},t\nu}(w,\upsilon,X)+\pi^-_{z_m}\pi^-_{x_{\hat{3}}(w,\upsilon)}\psi^{\mu}_{\mathcal{S},t\nu}(w,\upsilon,X)$$

instead of $\psi^{\mu}_{\mathcal{S},t\nu}$. The claim then follows from Theorem 2.7, Lemma 3.2, and Corollary 4.30.

4.9 Summary of Section 4

We conclude Section 4 by reviewing the main results so far. Throughout this subsection,

$$\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$$

is a simple bubble type, with $d=\sum d_h$ and $d_{\hat{0}}=0$, and μ is an N-tuple of constraints in general position such that $\mathrm{codim}_{\mathbb{C}}\mu=d(n+1)-n(g-1)+N$.

If $|\hat{I}|>n$, by Corollaries 4.10 and 4.11, there exist a neighborhood $U_{\mathcal{T}}$ of $\bar{\mathcal{M}}_{\mathcal{T}}(\mu)$ in $\bar{C}^{\infty}_{(d;[N])}(\Sigma;\mu)$ and $\epsilon_{\mathcal{T}}>0$ such that for all $t\in(0,\epsilon_{\mathcal{T}})$, $U_{\mathcal{T}}\cap\mathcal{M}_{\Sigma,d,t\nu}(\mu)=\emptyset$. This is also true if $H_{\hat{0}}\mathcal{T}\neq\hat{I}$ or $M_{\hat{0}}\mathcal{T}\neq\emptyset$. If $n=2$, this statement is just Corollary 4.10. If $n=3$, we only need to consider Cases (1), (2b), and (3b) of Corollary 4.11. Case (3b) follows from Corollaries 4.7, 4.26, and 4.31. The claim for Case (2b) is obtained from Corollaries 4.7, 4.14, 4.18 and the same claim for Case (3b). Finally, in Case (1), we use Corollaries 4.7, 4.26, and 4.31, the statement of Corollary 4.11 for $|\hat{I}|\geq 2$, and the just stated result for Case (2b).

If $|\hat{I}|\leq n$, $H_{\hat{0}}\mathcal{T}=\hat{I}$, and $M_{\hat{0}}\mathcal{T}=\emptyset$, i.e. $\mathcal{T}$ is a primitive bubble type, by the previous paragraph and Corollaries 4.7, 4.14, 4.18, and 4.22, there exist a neighborhood $U_{\mathcal{T}}$ of $\bar{\mathcal{M}}_{\mathcal{T}}(\mu)$ in $\bar{C}^{\infty}_{(d;[N])}(\Sigma;\mu)$ and $\epsilon_{\mathcal{T}}>0$ such that for all $t\in(0,\epsilon_{\mathcal{T}})$, the signed cardinality $n_{\mathcal{T}}(\mu)$ of $U_{\mathcal{T}}\cap\mathcal{M}_{\Sigma,d,t\nu}(\mu)$ is the sum of the numbers given by these four corollaries applied to $\mathcal{T}$. If $|\hat{I}|=1$,

$$n_1(\mu)\equiv n_{\mathcal{T}}(\mu)=n_1^{(1)}(\mu)+2n_1^{(2)}(\mu)+18n_1^{(3)}(\mu), \tag{4.29}$$

where the numbers $n_1^{(k)}(\mu)$ are described as follows. The number $n_1^{(1)}(\mu)$ is the signed number of zeros of the affine map

$$\psi_1^{(1)}\colon T\Sigma\otimes L_{\hat{1}}\bar{\mathcal{T}}\longrightarrow\mathcal{H}_{\Sigma}^{0,1}\otimes\mathrm{ev}^*T\mathbb{P}^n,\qquad \psi_1^{(1)}(x,[b,v_{\hat{1}}])=\bar{\nu}_b+(\mathcal{D}_{\mathcal{T},\hat{1}}b)s_{\Sigma,x}(v_{\hat{1}}), \tag{4.30}$$

where the bundles are considered over $\Sigma\times\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)=\bar{\mathcal{M}}_{\mathcal{T}}(\mu)$ and $\hat{1}$ is the unique element of $\hat{I}$. Note that this number is the same as the number of zeros of the map in (4.11), since $\Sigma\times\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)-\mathcal{M}_{\mathcal{T}}(\mu)$ is a finite union of smooth manifolds of dimension less than the dimension of $\mathcal{M}_{\mathcal{T}}(\mu)$. Thus, if ν is generic, $\psi_1^{(1)}$ has no zeros over $\Sigma\times\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)-\mathcal{M}_{\mathcal{T}}(\mu)$. The number $n_1^{(2)}(\mu)$ is the signed number of zeros of the affine map

$$\psi_1^{(2)}\colon T\Sigma^{\otimes 2}\otimes L_{\hat{1}}\bar{\mathcal{T}}^{\otimes 2}\longrightarrow\mathcal{H}_{\Sigma}^{-}\otimes\mathrm{ev}^*T\mathbb{P}^n,\qquad \psi_2^{(2)}(x,[b,v_{\hat{1}}])=\bar{\nu}_b^-+(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}b)s^{(2,-)}_{\Sigma,x}(v_{\hat{1}}), \tag{4.31}$$

where the bundles are considered over $\Sigma\times\bar{\mathcal{S}}_1(\mu)$ and $\bar{\mathcal{S}}_1(\mu)$ is the closure in $\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)$ of the space

$$\mathcal{S}_1(\mu)=\big\{b\in\mathcal{U}_{\bar{\mathcal{T}}}(\mu)\colon\mathcal{D}_{\mathcal{T},\hat{1}}|_b=0\big\}. \tag{4.32}$$

If $n=2$, $\mathcal{S}_1(\mu)$ is a finite set and thus $\bar{\mathcal{S}}_1(\mu)=\mathcal{S}_1(\mu)$. If $n=3$, $\mathcal{S}_1(\mu)$ is one-dimensional over $\mathbb{C}$. The boundary $\bar{\mathcal{S}}_1(\mu)-\mathcal{S}_1(\mu)$ is a finite set, as can be seen from the estimate on $\mathcal{D}_{\bar{\mathcal{T}},\hat{1}}$ of Theorem 2.8. Thus, in either case, the maps in (4.31) and (4.16) have the same zeros. Finally, the number $n_1^{(3)}(\mu)$ is the signed number of zeros of the affine map

$$\begin{gathered}\psi_1^{(3)}\colon T\Sigma^{\otimes 3}\otimes\big(L_{\hat{1}}\bar{\mathcal{T}}^{\otimes 2}\oplus L_{\hat{1}}\bar{\mathcal{T}}^{\otimes 3}\big)\longrightarrow\mathcal{H}_{\Sigma}^{-}\otimes\mathrm{ev}^*T\mathbb{P}^n,\\ \psi_1^{(3)}(x,[b,v_{\hat{1}},w_{\hat{1}}])=\bar{\nu}_b^-+(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}b)s^{(3,-)}_{b,z_m}(v_{\hat{1}})+(\mathcal{D}^{(3)}_{\mathcal{T},\hat{1}})s^{(3,-)}_{b,z_m}(w_{\hat{1}}),\end{gathered} \tag{4.33}$$

where the bundles are considered over $\bar{\mathcal{S}}_1(\mu)$ and z_m is one of the six distinguished points of Σ. By the same argument as above, this number is precisely the number of zeros of the map in (4.21).

If $|\hat{I}|=2$ and $n=2$, $n_{\mathcal{T}}(\mu)=n_{\mathcal{T}}^{(1)}(\mu)$ is the signed number of zeros of the affine map

$$\begin{gathered}\psi_{\mathcal{T}}^{(1)}\colon T\Sigma_{\hat{1}}\otimes L_{\hat{1}}\bar{\mathcal{T}}\oplus T\Sigma_{\hat{2}}\otimes L_{\hat{2}}\bar{\mathcal{T}}\longrightarrow\mathcal{H}_{\Sigma}^{0,1}\otimes\mathrm{ev}^*T\mathbb{P}^n,\\ \psi_{\mathcal{T}}^{(1)}\big(x_{\hat{1}},x_{\hat{2}},[b,v_{\hat{1}},v_{\hat{2}}]\big)=\bar{\nu}_b+(\mathcal{D}_{\mathcal{T},\hat{1}}b)s_{\Sigma,x_{\hat{1}}}(v_{\hat{1}})+(\mathcal{D}_{\mathcal{T},\hat{2}}b)s_{\Sigma,x_{\hat{2}}}(v_{\hat{2}}),\end{gathered} \tag{4.34}$$

where the bundles are considered over $\Sigma^2\times\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)=\Sigma_{\hat{1}}\times\Sigma_{\hat{2}}\times\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)$ and $\hat{1},\hat{2}$ are the two elements of $\hat{I}$. By the same argument as before, the number $n_{\mathcal{T}}^{(1)}(\mu)$ is the same as the number of zeros of the map (4.7). If $|\hat{I}|=2$ and $n=3$,

$$n_{\mathcal{T}}(\mu)=n_{\mathcal{T}}^{(1)}(\mu)+2n_{\mathcal{T}}^{(2)}(\mu), \tag{4.35}$$

where $n_{\mathcal{T}}^{(1)}(\mu)$ is defined the same way as in the $n=2$ case, while $n_{\mathcal{T}}^{(2)}(\mu)$ is the signed number of zeros of the affine map

$$\psi_{\mathcal{T}}^{(2)}: T\Sigma^{\otimes 2}\otimes\big(L_{\hat{2}}\bar{\mathcal{T}}\oplus L_{\hat{2}}\bar{\mathcal{T}}^{\otimes 2}\big)\longrightarrow\mathcal{H}_{\Sigma}^{-}\otimes\mathrm{ev}^*T\mathbb{P}^n, \tag{4.36}$$
$$\psi_{\mathcal{T}}^{(2)}\big(x,[b,v_{\hat{2}},w_{\hat{2}}]\big)=\bar{\nu}_b^{-}+(\mathcal{D}_{\mathcal{T},\hat{2}}b)s_{\Sigma,x}^{(2,-)}(w_{\hat{2}})+(\mathcal{D}_{\mathcal{T},\hat{1}}^{(2)}b)s_{\Sigma,x}^{(2,-)}(\kappa(b)v_{\hat{2}})+(\mathcal{D}_{\mathcal{T},\hat{2}}^{(2)}b)s_{\Sigma,x}^{(2,-)}(v_{\hat{2}}),$$

where the bundles are viewed over $\Sigma\times\mathcal{S}_{\bar{\mathcal{T}}}(\mu)$,

$$\mathcal{S}_{\bar{\mathcal{T}}}(\mu)=\big\{b\in\mathcal{U}_{\bar{\mathcal{T}}}(\mu):\pi_{[b]}^{\perp}\circ\mathcal{D}_{\mathcal{T},\hat{2}}|_b=0\big\}, \tag{4.37}$$

E_1 is the quotient of $\mathrm{ev}^*T\mathbb{P}^n$ by $\mathrm{Im}(\mathcal{D}_{\mathcal{T},\hat{1}})$, $\pi^{\perp}:\mathrm{ev}^*T\mathbb{P}^n\longrightarrow E_1$ is the projection map, and $\kappa(b)\in L_{\hat{2}}^*\bar{\mathcal{T}}\otimes L_{\hat{1}}\bar{\mathcal{T}}$ is a nonzero homomorphism. Note that $\mathcal{S}_{\bar{\mathcal{T}}}(\mu)$ is a finite set with our choice of constraints. Finally, if $|\hat{I}|=3$ and $n=3$, $n_{\mathcal{T}}(\mu)=n_{\mathcal{T}}^{(1)}(\mu)$ is the signed number of zeros of the affine map

$$\psi_{\mathcal{T}}^{(1)}: T\Sigma_{\hat{1}}\otimes L_{\hat{1}}\bar{\mathcal{T}}\oplus T\Sigma_{\hat{2}}\otimes L_{\hat{2}}\bar{\mathcal{T}}\oplus T\Sigma_{\hat{3}}\otimes L_{\hat{3}}\bar{\mathcal{T}}\longrightarrow\mathcal{H}_{\Sigma}^{0,1}\otimes\mathrm{ev}^*T\mathbb{P}^n, \tag{4.38}$$
$$\psi_{\mathcal{T}}^{(1)}\big(x_{\hat{1}},x_{\hat{2}},x_{\hat{3}},[b,v_{\hat{1}},v_{\hat{2}},v_{\hat{3}}]\big)=\bar{\nu}_b+(\mathcal{D}_{\mathcal{T},\hat{1}}b)s_{\Sigma,x_{\hat{1}}}(v_{\hat{1}})+(\mathcal{D}_{\mathcal{T},\hat{2}}b)s_{\Sigma,x_{\hat{2}}}(v_{\hat{2}})+(\mathcal{D}_{\mathcal{T},\hat{3}}b)s_{\Sigma,x_{\hat{3}}}(v_{\hat{3}}),$$

where the bundles are considered over $\Sigma^3\times\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)=\Sigma_{\hat{1}}\times\Sigma_{\hat{2}}\times\Sigma_{\hat{3}}\times\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)$ and $\hat{1},\hat{2},\hat{3}$ are the three elements of $\hat{I}$. As before, the number $n_{\mathcal{T}}^{(1)}(\mu)$ is precisely the number of zeros of the map (4.7). If $m\ge 2$ and $k\ge 1$, we denote by $n_m^{(k)}(\mu)$ the sum of the numbers $n_{\mathcal{T}}^{(k)}(\mu)$ over all equivalence classes of primitive bubble types $\mathcal{T}$ with $|\hat{I}|=m$.

5 Computations

5.1 The Numbers $n_m^{(1)}(\mu)$ with $m=n$

Our goal now is to compute the numbers $n_{\mathcal{T}}^{(k)}(\mu)$ for any primitive bubble type $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$, and thus the genus-two enumerative invariants for $\mathbb{P}^2$ and $\mathbb{P}^3$. Most of this section is devoted to expressing the numbers $n_{\mathcal{T}}^{(k)}(\mu)$ in terms of intersection numbers of tautological classes of various spaces of stable rational maps that pass through the constraints μ. These are shown to be computable in [P2]. The procedure for counting the zeros of affine maps between vector bundles is described in Section 3. We start with the easiest cases.

Lemma 5.1 *If $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ is a primitive bubble type with $|\hat{I}|=n$ and μ is an N-tuple of constraints in general position such that*

$$\mathit{codim}_{\mathbb{C}}\mu=(n+1)\sum_{i\in I}d_i-n+N,$$

the set $\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)$ is finite and $n_{\mathcal{T}}^{(1)}(\mu)=2^n|\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)|$.

Proof: The first statement is clear by dimension counting. By equations (4.34) and (4.38), we need to apply Lemma 3.14 with

$$\bar{\mathcal{M}} = \begin{cases} \Sigma_{\hat{1}} \times \Sigma_{\hat{2}} \times \bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu), & \text{if } n=2; \\ \Sigma_{\hat{1}} \times \Sigma_{\hat{2}} \times \Sigma_{\hat{3}} \times \bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu), & \text{if } n=3; \end{cases} \qquad E = \begin{cases} T\Sigma_{\hat{1}} \otimes L_{\hat{1}}\bar{\mathcal{T}} \oplus T\Sigma_{\hat{2}} \otimes L_{\hat{2}}\bar{\mathcal{T}}, & \text{if } n=2; \\ T\Sigma_{\hat{1}} \otimes L_{\hat{1}}\bar{\mathcal{T}} \oplus T\Sigma_{\hat{2}} \otimes L_{\hat{2}}\bar{\mathcal{T}} \oplus T\Sigma_{\hat{3}} \otimes L_{\hat{3}}\bar{\mathcal{T}}, & \text{if } n=3, \end{cases}$$

$\mathcal{O} = \mathcal{H}^{0,1}_{\Sigma} \otimes \mathrm{ev}^* T\mathbb{P}^n$, and α given by (4.34) and (4.38). By Lemma 4.9, $\alpha \in \Gamma(\bar{\mathcal{M}}; E^* \otimes \mathcal{O})$ has full rank on every fiber of E. Thus by Lemma 3.14,

$$n^{(1)}_{\mathcal{T}}(\mu) = \big\langle e(\mathcal{O}/\alpha(E)), [\bar{\mathcal{M}}] \big\rangle = \big\langle c(\mathcal{O}) c(E)^{-1}, [\bar{\mathcal{M}}] \big\rangle. \tag{5.1}$$

Since $\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)$ is a finite set,

$$E \approx \begin{cases} T\Sigma_{\hat{1}} \oplus T\Sigma_{\hat{2}}, & \text{if } n=2; \\ T\Sigma_{\hat{1}} \oplus T\Sigma_{\hat{2}} \oplus T\Sigma_{\hat{3}}, & \text{if } n=3; \end{cases} \qquad \mathcal{O} \approx \bar{\mathcal{M}} \times \mathbb{C}^{2n}.$$

Let $y_h = c_1(T\Sigma_h)$. Thus, if $n=2$, by (5.1)

$$n^{(1)}_{\mathcal{T}}(\mu) = \big\langle (1+(y_{\hat{1}}+y_{\hat{2}})+y_{\hat{1}}y_{\hat{2}})^{-1}, [\bar{\mathcal{M}}] \big\rangle = \big\langle y_{\hat{1}}y_{\hat{2}}, [\Sigma_{\hat{1}} \times \Sigma_{\hat{2}}] \big\rangle |\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)| = 4|\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)|,$$

since $\langle y_h, [\Sigma_h] \rangle = -2$. If $n=3$, we similarly obtain

$$n^{(1)}_{\mathcal{T}}(\mu) = \big\langle -y_{\hat{1}}y_{\hat{2}}y_{\hat{3}}, [\Sigma_{\hat{1}} \times \Sigma_{\hat{2}} \times \Sigma_{\hat{3}}] \big\rangle |\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)| = 8|\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)|,$$

as claimed.

Let $\tau_n(\mu)$ denote the sum of the numbers $|\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)|$ taken over all equivalence classes of primitive bubble types $\mathcal{T}$ with $|\hat{I}|=n$. This is the number of n-component connected curves of total degree d passing through the constraints $\mu_1, \ldots, \mu_N$ in $\mathbb{P}^n$ with a choice of a node which belongs to all n components. From Lemma 5.1, we immediately conclude:

Corollary 5.2 *If $n=2$, $n^{(1)}_2(\mu) = 4\tau_2(\mu)$. If $n=3$, $n^{(1)}_3(\mu) = 8\tau_3(\mu)$.*

5.2 The Numbers $n^{(2)}_m(\mu)$ and $n^{(3)}_m(\mu)$ with $m = n-1$

In this subsection, we describe the numbers $n^{(2)}_{\mathcal{T}}(\mu)$ and $n^{(3)}_{\mathcal{T}}(\mu)$ with $|\hat{I}|=n-1$ topologically. The similarity between these cases is that $\mathcal{U}_{\bar{\mathcal{T}}}(\mu)$ is two-dimensional (over $\mathbb{C}$), while $\mathcal{S}_{\bar{\mathcal{T}}}(\mu)$ is a finite set; see Subsection 4.9 for notation.

The numbers $n^{(2)}_{\mathcal{T}}(\mu)$ with $|\hat{I}| = n-1=1$ and $|\hat{I}|=n-1=2$ are the signed cardinalities of the zero sets of the affine maps in (4.31) and (4.36), respectively. By Subsections 4.4 and 4.6, the linear part α of the affine map $\psi^{(2)}_{\mathcal{T}}$ has full rank in these cases, except over the zero set of $s^{(2,-)}_{\Sigma}$. In order to simplify our computations, we replace $s^{(2,-)}_{\Sigma}$ by another section that has no zeros on Σ, but so that the corresponding affine maps have the same number of zeros as the maps in (4.31) and (4.36). The section

$$s^{(2,-)}_{\Sigma} \in \Gamma(\Sigma; T^*\Sigma^{\otimes 2} \otimes \mathcal{H}^-_{\Sigma})$$

has transverse zeros at the points $z_1,\ldots,z_6\in\Sigma$; see Subsection 4.5. Thus, it induces a nonvanishing section

$$\tilde{s}_\Sigma^{(2,-)}\in\Gamma(\Sigma;\tilde{T}\Sigma^*\otimes\mathcal{H}_\Sigma^-),\quad \text{where}\quad \tilde{T}\Sigma=T\Sigma^{\otimes2}\otimes\mathcal{O}(z_1)\otimes\ldots\otimes\mathcal{O}(z_6)$$

and $\mathcal{O}(z_m)$ denotes the holomorphic line bundle corresponding to the divisor z_m on Σ. The bundles $\tilde{T}\Sigma$ and $T\Sigma^{\otimes2}$ can be identified on Σ^*, the complement of the six points, in such a way that $\tilde{s}_\Sigma^{(2,-)}=\eta s_\Sigma^{(2,-)}$ on Σ^* for some $\eta\in C^\infty(\Sigma^*;\mathbb{R}^+)$. Let $\tilde{\psi}_{\mathcal{T}}^{(2)}$ denote the affine maps obtained by replacing $T\Sigma^{\otimes2}$ and $s_\Sigma^{(2,-)}$ by $\tilde{T}\Sigma$ and $\tilde{s}_\Sigma^{(2,-)}$, respectively, in (4.31) and (4.36) (depending on $\mathcal{T}$). Since $\psi_{\mathcal{T}}^{(2)}$ and $\tilde{\psi}_{\mathcal{T}}^{(2)}$ have no zeros over $\{z_m\}$ if ν is generic and $s_\Sigma^{(2,-)}$ and $\tilde{s}_\Sigma^{(2,-)}$ differ by a nonzero multiple on Σ^*, there is a sign-preserving bijection between the zeros of $\psi_{\mathcal{T}}^{(2)}$ and of $\tilde{\psi}_{\mathcal{T}}^{(2)}$. Furthermore, the linear part of $\tilde{\psi}_{\mathcal{T}}^{(2)}$ has full rank on every fiber.

Denote by $\mathcal{S}_2(\mu)$ the union of the spaces $\mathcal{S}_{\bar{\mathcal{T}}}(\mu)$ defined by equation (4.37) taken over all equivalence classes of appropriate bubble types $\mathcal{T}$. This set can be identified with the degree-d two-component rational curves in $\mathbb{P}^3$ that are connected at a tacnode and pass through the constraints μ. Similarly, in the $n=2$ case, the set $\mathcal{S}_1(\mu)$ corresponds to the degree-d cuspidal rational curves passing through the constraints.

Lemma 5.3 *If* $n=2$, $n_1^{(2)}(\mu)=2|\mathcal{S}_1(\mu)|$ *and* $n_1^{(3)}(\mu)=|\mathcal{S}_1(\mu)|$. *If* $n=3$, $n_2^{(2)}(\mu)=2|\mathcal{S}_2(\mu)|$.

Proof: Let $\mathcal{T}=(\Sigma,[N],I;j,\underline{d})$ be a bubble type that contributes to one of these numbers. By dimension counting and Corollary 6.3, $\mathcal{S}_{\bar{\mathcal{T}}}(\mu)$ is zero-dimensional and compact. Thus, in all cases the bundles $L_h\bar{\mathcal{T}}$ and $\mathrm{ev}^*T\mathbb{P}^n$ of equations (4.31), (4.33) and (4.36) are trivial. If $n=2$ and $k=2$, we are in the case of (4.31). By the above, we can apply Lemma 3.14 with

$$E=\tilde{T}\Sigma,\quad \mathcal{O}=\mathcal{H}_\Sigma^-\oplus\mathcal{H}_\Sigma^-,$$

and $\alpha\in\Gamma(\Sigma\times\mathcal{S}_{\bar{\mathcal{T}}}(\mu);E^*\otimes\mathcal{O})$ that has full rank. We obtain

$$n_{\mathcal{T}}^{(2)}(\mu)=\big\langle c_1(\mathcal{O})-c_1(E),\big[\Sigma\times\mathcal{S}_{\bar{\mathcal{T}}}(\mu)\big]\big\rangle=\big(4+(4-6)\big)|\mathcal{S}_{\bar{\mathcal{T}}}(\mu)|=2|\mathcal{S}_{\bar{\mathcal{T}}}(\mu)|.$$

If $n=3$ and $k=2$, we are in the case of (4.36) and apply Lemma 3.14 with

$$E=\tilde{T}\Sigma\oplus\tilde{T}\Sigma,\quad \mathcal{O}=\mathcal{H}_\Sigma^-\oplus\mathcal{H}_\Sigma^-\oplus\mathcal{H}_\Sigma^-,$$

and $\alpha\in\Gamma(\Sigma\times\mathcal{S}_{\bar{\mathcal{T}}}(\mu);E^*\otimes\mathcal{O})$ that again has full rank. Thus,

$$n_{\mathcal{T}}^{(2)}(\mu)=\big\langle c_1(\mathcal{O})-c_1(E),\big[\Sigma\times\mathcal{S}_{\bar{\mathcal{T}}}(\mu)\big]\big\rangle=(6-4)|\mathcal{S}_{\bar{\mathcal{T}}}(\mu)|=2|\mathcal{S}_{\bar{\mathcal{T}}}(\mu)|.$$

Finally, if $n=2$ and $k=3$, we are in the case of (4.33). Note that all the bundles involved are trivial and the linear part of $\psi_{\mathcal{T}}^{(2)}$ is an isomorphism on every fiber. Thus, $n_{\mathcal{T}}^{(3)}(\mu)=|\mathcal{U}_{\bar{\mathcal{T}}}(\mu)|$.

The next step is to compute the cardinalities of the sets $\mathcal{S}_{n-1}(\mu)$. In order to simplify our answers, it is convenient to introduce cohomology classes $c_1(\mathcal{L}_k^*\mathcal{T})$ closely related to $c_1(L_k^*\mathcal{T})$. Suppose $\mathcal{T}=(S^2,M,I;j,\underline{d})$ is a bubble type. and $\{\mathcal{T}_k=(S^2,M_k,I_k;j_k,\underline{d}_k)\}$ are the corresponding simple types; see [Z1]. For any $k\in I-\hat{I}$ and nonempty subset M_0 of $M_k\mathcal{T}$, we define bubble types $\mathcal{T}(M_0)$ and $\mathcal{T}/M_0$ as follows. Let

$$\mathcal{T}/M_0=\big(S^2,I,M-M_0;j|_{M-M_0},\underline{d}\big).$$

Let $\mathcal{T}(M_0) \equiv (S^2, M, \hat{I} \sqcup_k \hat{1}; j', \underline{d}')$ be given by

$$j'_l = \begin{cases} k, & \text{if } l \in M_0; \\ \hat{1}, & \text{if } l \in M_k\mathcal{T} - M_0; \\ j_l, & \text{otherwise}; \end{cases} \qquad d'_i = \begin{cases} 0, & \text{if } i = k; \\ d_k, & \text{if } i = \hat{1}; \\ d_i, & \text{otherwise.} \end{cases}$$

The tuples $\mathcal{T}/M_0$ and $\mathcal{T}(M_0)$ are bubble types as long as $d_k \neq 0$ or $M_0 \neq M_{\hat{0}}\mathcal{T}$. Then,

$$\bar{\mathcal{U}}_{\mathcal{T}(M_0)}(\mu) = \bar{\mathcal{M}}_{0,\{\hat{1}\}\sqcup M_0} \times \bar{\mathcal{U}}_{\mathcal{T}/M_0}\Big(\bigcap_{l \in M_0} \mu_l; \mu\Big), \tag{5.2}$$

where $\bar{\mathcal{M}}_{0,\{\hat{1}\}\sqcup M_0}$ denotes the Deligne-Mumford moduli space of rational curves with $(\{\hat{0},\hat{1}\}\sqcup M_0)$-marked points. If $l \in M_k\mathcal{T}$ for some $k \in I - \hat{I}$, we denote $\mathcal{T}(\{l\})$ by $\mathcal{T}(l)$. If $\mathcal{T}$ is a basic bubble type, by Theorem 2.8 and decomposition (5.2), $\bar{\mathcal{U}}_{\mathcal{T}(M_0)}(\mu)$ is an oriented topological suborbifold of $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ of (real) codimension two. Thus,

$$c_1(\mathcal{L}_k^*\mathcal{T}) \equiv c_1(L_k^*\mathcal{T}) - \sum_{M_0 \subset M_k, M_0 \neq \emptyset} PD_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)}\big[\bar{\mathcal{U}}_{\mathcal{T}(M_0)}(\mu)\big] \in H^2\big(\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\big), \tag{5.3}$$

where $PD_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)}\big[\bar{\mathcal{U}}_{\mathcal{T}(M_0)}(\mu)\big]$ denotes the Poincare Dual of $\big[\bar{\mathcal{U}}_{\mathcal{T}(M_0)}(\mu)\big]$ in $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$, is a well-defined cohomology class. Since our constraints μ are disjoint, $\bar{\mathcal{U}}_{\mathcal{T}(M_0)}(\mu) = \emptyset$ if $|M_0| \geq 2$. Furthermore, it is well-known in algebraic geometry that for any $l \in M_k$ the normal bundle of $\bar{\mathcal{U}}_{\mathcal{T}(l)}(\mu)$ in $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ is $L_{\hat{1}}\mathcal{T}(l)$; see [P2]. Thus, if μ is an M-tuple of disjoint constraints,

$$\big[\bar{\mathcal{U}}_{\mathcal{T}(l)}(\mu)\big] \cap c_1(\mathcal{L}_k^*\mathcal{T}) = \big[\bar{\mathcal{U}}_{\mathcal{T}(l)}(\mu)\big] \cap c_1(L_{\hat{1}}^*\mathcal{T}(l)) = \big[\bar{\mathcal{U}}_{\mathcal{T}(l)}(\mu)\big] \cap c_1(\mathcal{L}_{\hat{1}}^*\mathcal{T}(l)), \tag{5.4}$$

since $L_k\mathcal{T}|_{\bar{\mathcal{U}}_{\mathcal{T}(l)}}$ is the trivial line bundle. The above fact from algebraic geometry is only used to simplify notation and is not really needed for our computations. In addition, (5.4) can deduced from Subsection 5.7.

In the $n=3$ case, we denote by $\bar{\mathcal{V}}_2(\mu)$ the disjoint union of the spaces $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ taken over equivalence classes of basic bubble types $\mathcal{T} = (S^2, M, I; j, \underline{d})$ with $|I| = 2$. While the components of $\bar{\mathcal{V}}_2(\mu)$ are unordered, we can still define the chern classes

$$c_1(\mathcal{L}_1^*) + c_1(\mathcal{L}_2^*),\ c_1^2(\mathcal{L}_1^*) + c_1^2(\mathcal{L}_2^*),\ c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*) \in H^*\big(\bar{\mathcal{V}}_2(\mu)\big).$$

In the notation of the previous paragraph, $c_1(\mathcal{L}_i^*)$ denotes the cohomology class $c_1(\mathcal{L}_{k_i}^*\mathcal{T}_{k_i})$, where we write $I = \{k_1, k_2\}$. If $\mathcal{T}^* = (S^2, M, \{\hat{0}\}; \hat{0}, d)$, we denote by $\bar{\mathcal{V}}_1(\mu)$ the space $\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$ and by $c_1(\mathcal{L}^*) \in H^2(\bar{\mathcal{V}}_1(\mu))$ the cohomology class $c_1(\mathcal{L}_{\hat{0}}^*\mathcal{T}^*)$.

Lemma 5.4 *If $d \geq 1$, the number of rational degree-d cuspidal curves passing through a tuple μ of $3d-2$ points in general position in $\mathbb{P}^2$ is given by*

$$\big|\mathcal{S}_1(\mu)\big| = \big\langle 3a^2 + 3ac_1(\mathcal{L}^*) + c_1^2(\mathcal{L}^*), \big[\bar{\mathcal{V}}_1(\mu)\big]\big\rangle - \tau_2(\mu),$$

where $a = ev^*(\mathcal{O}(1))$.

Proof: (1) This result is well-known in algebraic geometry; see [V]. Nevertheless, for the sake of completeness, we include a proof. Let $\mathcal{T}^*$ be as above. By definition, $\mathcal{S}_1(\mu)$ is the intersection of the zero set of the section

$$\mathcal{D} \equiv \mathcal{D}_{\mathcal{T}^*} \in \Gamma\big(\bar{\mathcal{V}}_1(\mu); L^* \otimes \mathrm{ev}^* T\mathbb{P}^2\big), \qquad \text{where} \quad L = L_{\hat{0}}\mathcal{T}^*,$$

with $\mathcal{V}_1(\mu) = \mathcal{U}_{\mathcal{T}^*}(\mu)$. Thus, by Corollary 3.13, with $\partial\bar{\mathcal{V}}_1(\mu) = \bar{\mathcal{V}}_1(\mu) - \mathcal{V}_1(\mu)$,

$$\begin{aligned} |\mathcal{S}_1(\mu)| &= \big\langle c_2\big(L^* \otimes \mathrm{ev}^* T\mathbb{P}^2\big), \big[\bar{\mathcal{V}}_1(\mu)\big]\big\rangle - \mathcal{C}_{\partial\bar{\mathcal{V}}_1(\mu)}(\mathcal{D}) \\ &= \big\langle 3a^2 + 3ac_1(L^*) + c_1^2(L^*), \big[\bar{\mathcal{V}}_1(\mu)\big]\big\rangle - \mathcal{C}_{\partial\bar{\mathcal{V}}_1(\mu)}(\mathcal{D}). \end{aligned} \tag{5.5}$$

(2) Suppose $\mathcal{T} = (S^2, [N], I; j, \underline{d}) < \mathcal{T}^*$, where $N = 3d-2$, is a bubble type such that $\mathcal{D}$ vanishes somewhere on $\mathcal{U}_{\mathcal{T}}(\mu)$. Since the complex dimension of $\mathcal{U}_{\mathcal{T}}(\mu)$ is at most one, by Corollary 6.3 $d_{\hat{0}} = 0$. Let

$$\rho_{\mathcal{T}} \in \Gamma\big(\mathcal{U}_{\mathcal{T}}(\mu); \mathrm{Polyn}(\mathcal{FT}; \tilde{\mathcal{F}}\mathcal{T})\big) \quad \text{and} \quad \alpha_{\mathcal{T}} \in \Gamma\big(\mathcal{U}_{\mathcal{T}}(\mu); \mathrm{Hom}(\tilde{\mathcal{F}}\mathcal{T}; L^* \otimes \mathrm{ev}^* T\mathbb{P}^n)\big)$$

be the sections defined in equation (2.21). Recall that with appropriate identifications

$$\big|\mathcal{D}(\gamma_{\mathcal{T}}^{\mu}(\upsilon)) - \alpha_{\mathcal{T}}(\rho_{\mathcal{T}}(\upsilon))\big| \le C(b_{\upsilon})|\upsilon|^{\frac{1}{p}}\big|\rho_{\mathcal{T}}(\upsilon)\big| \quad \forall \upsilon \in \mathcal{FT}_{\delta}, \tag{5.6}$$

where $\delta, C \in C^{\infty}(\mathcal{U}_{\mathcal{T}}(\mu); \mathbb{R}^+)$ and $\gamma_{\mathcal{T}}^{\mu}\colon \mathcal{FT}_{\delta} \longrightarrow \bar{\mathcal{V}}_1(\mu)$ is an identification of neighborhoods of $\mathcal{U}_{\mathcal{T}}(\mu)$, which is smooth on the preimage of $\mathcal{V}_1(\mu)$. Note that $|\hat{I}| \in \{1, 2\}$ if $\mathcal{U}_{\mathcal{T}}(\mu)$ is nonempty. By the proof of Lemma 4.9, $\alpha_{\mathcal{T}}$ has full rank on every fiber $\tilde{\mathcal{F}}\mathcal{T} \longrightarrow \mathcal{U}_{\mathcal{T}}(\mu)$. Thus, by equation (5.6) and Corollary 3.13,

$$\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)}(\mathcal{D}) = 0 \qquad \text{if} \quad H_{\hat{0}}\mathcal{T} \neq \hat{I}.$$

(3) Suppose $|H_{\hat{0}}\mathcal{T}| = |\hat{I}| = 1$. Then $\mathcal{T} = \mathcal{T}^*(l)$ for some $l \in [N]$ and $\tilde{\mathcal{F}}\mathcal{T} = \mathcal{FT} \approx L_{\hat{1}}\mathcal{T}$. Since $\alpha_{\mathcal{T}} \circ \rho_{\mathcal{T}}$ has constant rank over $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$, by Corollary 3.13 and Lemma 3.14,

$$\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)}(\mathcal{D}) = \big\langle c_1(L^* \otimes \mathrm{ev}^* T\mathbb{P}^2) - c_1\big(L_{\hat{1}}\mathcal{T}\big), \big[\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\big]\big\rangle = \big\langle 3a + c_1(L_{\hat{1}}^*\mathcal{T}), \big[\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\big]\big\rangle.$$

If $|H_{\hat{0}}\mathcal{T}| = |\hat{I}| = 2$, $\alpha_{\mathcal{T}} \circ \rho_{\mathcal{T}}$ is an isomorphism on every fiber. Thus, $\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)}(\mathcal{D}) = |\mathcal{U}_{\mathcal{T}}(\mu)|$ by Corollary 3.13. Combining these contributions to the euler class of $L^* \otimes \mathrm{ev}^* T\mathbb{P}^2$ gives

$$\begin{aligned} \mathcal{C}_{\partial\bar{\mathcal{U}}}(\mathcal{D}_{\mathcal{T}^*}) &= \sum_{l \in [N]} \big\langle 3a + c_1(L_{\hat{1}}^*\mathcal{T}^*(l)), \big[\mathcal{U}_{\mathcal{T}^*(l)}(\mu)\big]\big\rangle + \sum_{[\mathcal{T}], |H_{\hat{0}}\mathcal{T}| = |\hat{I}| = 2} |\bar{\mathcal{U}}_{\mathcal{T}}(\mu)| \\ &= \sum_{l \in [3d-2]} \big\langle 3a + c_1(L_{\hat{1}}^*\mathcal{T}^*(l)), \big[\mathcal{U}_{\mathcal{T}^*(l)}(\mu)\big]\big\rangle + \tau_2(\mu). \end{aligned} \tag{5.7}$$

The claim follows by plugging equation (5.7) into (5.5) and using equations (5.3) and (5.4).

Lemma 5.5 *If $d \ge 1$, the number of two-component rational degree-d curves connected at a tacnode and passing through a tuple μ of p points and q lines in general position in $\mathbb{P}^3$, where $2p+q = 4d-3$, is given by*

$$|\mathcal{S}_2(\mu)| = \big\langle 6a^2 + 4a(c_1(\mathcal{L}_1^*) + c_1(\mathcal{L}_2^*)) + (c_1^2(\mathcal{L}_1^*) + c_1^2(\mathcal{L}_2^*)) + c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*), \big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle - 3\tau_3(\mu).$$

Proof: (1) Let $\mathcal{T}^*=(S^2,[N],I^*;j^*,\underline{d}^*)$ be a basic bubble type such that $I^*=\{k_1,k_2\}$ is a two-element set, $d^*_{k_1},d^*_{k_2}>0$, $d^*_{k_1}+d^*_{k_2}=d$, and $N=p+q$. Denote by $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$ the corresponding simple types. The proof is similar to that of Lemma 5.4, but we pass to the projectivization $\mathbb{P}E$ (over $\mathbb{C}$) of the bundle

$$E=L_1\oplus L_2\longrightarrow\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu),\quad\text{where}\quad L_i=L_{k_i}\mathcal{T}_i^*.$$

The section $\bar{\mathcal{D}}_{\mathcal{T}^*,2}$ of Lemma 4.9 induces a section $\mathcal{D}\in\Gamma(\mathbb{P}E;\gamma_E^*\otimes\mathrm{ev}^*T\mathbb{P}^3)$ such that $\mathcal{S}_{\mathcal{T}^*}(\mu)$ corresponds to the intersection of the zero set of $\mathcal{D}$ with $\mathbb{P}E|_{\mathcal{U}_{\mathcal{T}^*}(\mu)}$. If $\mathbb{P}E'$ denotes the restriction of $\mathbb{P}E$ to $\partial\bar{\mathcal{U}}\equiv\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)-\mathcal{U}_{\mathcal{T}^*}(\mu)$, by Corollary 3.13,

$$\begin{aligned}|\bar{\mathcal{S}}_{\mathcal{T}^*}(\mu)|&=\big\langle c_3(\gamma_E^*\otimes\mathrm{ev}^*T\mathbb{P}^3),\big[\mathbb{P}E\big]\big\rangle-\mathcal{C}_{\mathbb{P}E'}(\mathcal{D})\\&=\big\langle 6a^2+4a(c_1(L_1^*)+c_1(L_2^*))+(c_1^2(L_1^*)+c_1^2(L_2^*))+c_1(L_1^*)c_1(L_2^*),\big[\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)\big]\big\rangle-\mathcal{C}_{\mathbb{P}E'}(\mathcal{D}).\end{aligned}\tag{5.8}$$

The second equality above is obtained by applying (3.16).
(2) Suppose $\mathcal{T}=(S^2,[N],I;j,\underline{d})<\mathcal{T}^*$ is a bubble type such that $\mathcal{D}$ vanishes somewhere on $\mathbb{P}E|_{\mathcal{U}_{\mathcal{T}}(\mu)}$. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the corresponding simple types. Since the constraints are disjoint, up to interchanging the indices, we must have

$$\mathcal{T}_1=\mathcal{T}_1^*,\qquad\mathcal{T}_2=\big(S^2,M_2,I_2;j|_{M_2},\underline{d}|_{I_2}\big)<\mathcal{T}_2^*\quad\text{with}\quad d_{k_2}=0.$$

Furthermore, $\mathcal{D}_{\mathcal{T}_1^*,k_1}$ does not vanish on $\mathcal{U}_{\mathcal{T}}(\mu)$; see the proof of Lemma 4.9. Thus, $\mathcal{D}$ vanishes only the subspace

$$\mathcal{Z}_{\mathcal{T}}\equiv\mathbb{P}L_2\big|_{\mathcal{U}_{\mathcal{T}}(\mu)}=\big\{\big(b,L_2|_b\big):b\in\mathcal{U}_{\mathcal{T}}(\mu)\big\}.$$

The map $\gamma_{\mathcal{T}}^{\mu}$ of Theorem 2.8 induces an identification of a neighborhood of 0 in

$$\mathcal{FS}\equiv\pi_E^*\mathcal{FT}\oplus\pi_E^*L_2^*\otimes\pi_E^*L_1\longrightarrow\mathcal{Z}_{\mathcal{T}}$$

with a neighborhood of $\mathcal{Z}_{\mathcal{T}}$ in $\mathbb{P}E$. Similarly to the $n=2$ case, with appropriate identifications,

$$\begin{gathered}\big|\mathcal{D}(\gamma_{\mathcal{T}}^{\mu}(\upsilon,u))-\tilde{\alpha}_{\mathcal{T}}(\tilde{\rho}_{\mathcal{T}}(\upsilon,u))\big|\le C(b_\upsilon)|\upsilon|^{\frac{1}{p}}\big|\rho_{\mathcal{T}}(\upsilon)\big|\quad\forall(\upsilon,u)\in\mathcal{FS}_\delta,\\ \text{where}\quad\tilde{\rho}_{\mathcal{T}}(\upsilon,u)=\big(\rho_{\mathcal{T}}(\upsilon),u\big)\in\tilde{F}\mathcal{S}\equiv\pi^*\tilde{\mathcal{F}}\mathcal{T}\oplus\pi_E^*L_2^*\otimes\pi_E^*L_1\longrightarrow\mathcal{Z}_{\mathcal{T}},\end{gathered}\tag{5.9}$$

and $\tilde{\alpha}_{\mathcal{T}}$ has full rank on every fiber by (2.21) and Lemma 4.9. Thus, similarly to the proof of Lemma 5.4, and $\mathcal{C}_{\mathbb{P}E'|_{\mathcal{Z}_{\mathcal{T}}}}(\mathcal{D})=0$ if $H_{k_2}\mathcal{T}\neq\hat{I}_2$, and only two cases remain to be considered.
(3) If $|H_{k_2}\mathcal{T}|=|\hat{I}_2|=1$, $\tilde{\alpha}_{\mathcal{T}}\circ\tilde{\rho}_{\mathcal{T}}$ has full rank over all of $\bar{\mathcal{Z}}_{\mathcal{T}}$. Thus, by Corollary 3.13 and Lemma 3.14,

$$\mathcal{C}_{\mathcal{Z}_{\mathcal{T}}}(\mathcal{D})=\big\langle c_1(\gamma_E^*\otimes\mathrm{ev}^*T\mathbb{P}^3)-c_1(\mathcal{FS}),\big[\bar{\mathcal{Z}}_{\mathcal{T}}\big]\big\rangle=\big\langle 4a+c_1(L_{\hat{1}}^*\mathcal{T}_2)+c_1(L_1^*),\big[\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\big]\big\rangle;\tag{5.10}$$

note that $c_1(\gamma_E^*)=c_1(L_2^*)=0$ over $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$. If $|H_{k_2}\mathcal{T}|=|\hat{I}_2|=2$, $\tilde{\alpha}_{\mathcal{T}}\circ\tilde{\rho}_{\mathcal{T}}$ is an isomorphism on every fiber, and thus

$$\mathcal{C}_{\mathbb{P}E'|_{\mathcal{Z}_{\mathcal{T}}}}(\mathcal{D})=|\mathcal{Z}_{\mathcal{T}}|=|\mathcal{U}_{\mathcal{T}}(\mu)|.\tag{5.11}$$

Note that the sum of $|\mathcal{U}_{\mathcal{T}}(\mu)|$ over all equivalence classes of bubble types $\mathcal{T}^*$ and $\mathcal{T}<\mathcal{T}^*$ is $3\tau_3(\mu)$, since one of the three components of the image of each bubble map in $\mathcal{U}_{\mathcal{T}}(\mu)$ is distinguished by the bubble type $\mathcal{T}$. As before, we now sum up equations (5.10) and (5.11) over all equivalence classes of bubble types $\mathcal{T}<\mathcal{T}^*$ of the appropriate form, plug the result back into (5.8) and use equations (5.3) and (5.4). The claim follows by summing the result over all equivalence classes of basic simple bubble types $\mathcal{T}^*$.

5.3 The Numbers $n_m^{(1)}$ with $m = n-1$

In this subsection, we give topological formulas for the numbers $n_{\mathcal{T}}^{(1)}$ with $|\hat{I}| = n-1$. As before, the reason these two cases are similar is that the complex dimension of $\mathcal{U}_{\mathcal{T}}(\mu)$ is two.

Lemma 5.6 *If $n=2$, $n_1^{(1)}(\mu) = 2\langle 6a^2 + 3ac_1(\mathcal{L}^*), [\bar{\mathcal{V}}_1(\mu)]\rangle$.*

Proof: (1) Let N, $\mathcal{T}^*$, L, $\mathcal{D}$ be as in the proof of Lemma 5.4. Since s_Σ does not vanish on Σ, by equation (4.30) and Lemma 3.14,

$$\begin{aligned} n_1^{(1)}(\mu) &= \sum_{k=0}^{k=3} \langle c_k(\mathcal{O}) c_1^{3-k}(T^*\Sigma \otimes L^*), [\Sigma \times \bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)]\rangle - \mathcal{C}_{\Sigma\times\mathcal{D}^{-1}(0)}(\alpha^\perp), \\ &= 2\langle 15a^2 + 12ac_1(L^*) + 3c_1^2(L^*), [\bar{\mathcal{V}}_1(\mu)]\rangle - \mathcal{C}_{\Sigma\times\mathcal{D}^{-1}(0)}(\alpha^\perp), \end{aligned} \tag{5.12}$$

where $\mathcal{O} = \mathcal{H}_\Sigma^{0,1} \otimes \mathrm{ev}^* T\mathbb{P}^2$ and $\alpha \in \Gamma(\Sigma \times \bar{\mathcal{V}}_1(\mu); T^*\Sigma \otimes L^* \otimes \mathcal{O})$ is the linear part of the affine map $\psi_1^{(1)}$ of (4.30).

(2) We first compute $\mathcal{C}_{\Sigma\times\mathcal{S}_1(\mu)}(\alpha^\perp)$. Since $\mathcal{V}_1(\mu)$ is a complex manifold and $\mathcal{D}$ is transverse to the zero set in $L^* \otimes \mathrm{ev}^* T\mathbb{P}^2$ by Corollary 6.3, we can identify a neighborhood of 0 in

$$F \equiv L^* \otimes \mathrm{ev}^* T\mathbb{P}^2 \longrightarrow \mathcal{S}_1(\mu)$$

with a neighborhood of $\mathcal{S}_1(\mu)$ in $\mathcal{V}_1(\mu)$ via a map γ in such a way that

$$\Pi_{b,\gamma(b,X)}^{-1}\big(\mathcal{D}\gamma(b,X)\big) = X \quad \forall (b,X) \in F_\delta. \tag{5.13}$$

Then with appropriate identifications,

$$\alpha^\perp(\gamma(X)) = \pi^\perp \circ X s_\Sigma \equiv \alpha_{\mathcal{S}}(X),$$

where $\pi^\perp : \mathcal{O} \longrightarrow \mathcal{O}^\perp$ is the quotient projection map. In particular, $\alpha_{\mathcal{S}}$ has full rank if $\bar{\nu}_b \notin \mathcal{H}_\Sigma^+ \otimes \mathrm{ev}^* T\mathbb{P}^2$ for all $b \in \mathcal{S}_1(\mu)$, i.e. ν is generic. Furthermore,

$$\big(T^*\Sigma \otimes L^* \otimes \mathcal{O}^\perp\big)/\big(\mathrm{Im}\ \alpha_{\mathcal{S}}\big) \approx T^*\Sigma \otimes \big((\mathcal{H}_\Sigma^- \otimes \mathbb{C}^2)/\mathbb{C}\big).$$

Thus, by Corollary 3.13,

$$\mathcal{C}_{\Sigma\times\mathcal{S}_1(\mu)}(\alpha^\perp) = \langle e(T^*\Sigma^{\otimes 3}), [\Sigma \times \mathcal{S}_1(\mu)]\rangle = 6|\mathcal{S}_1(\mu)|. \tag{5.14}$$

(3) It remains to compute the contribution to $\mathcal{C}_{\Sigma\times\mathcal{D}^{-1}(0)}(\alpha^\perp)$ from $\Sigma \times (\bar{\mathcal{V}}_1(\mu) - \mathcal{V}_1(\mu))$. Suppose

$$\mathcal{T} = (S^2, [N], I; j, \underline{d}) < \mathcal{T}^*$$

is a bubble type such that $\mathcal{D}$ vanishes somewhere on $\mathcal{U}_{\mathcal{T}}(\mu)$. As in the proof of Lemma 5.4, $|\hat{I}| \in \{1,2\}$ and $d_{\hat{0}} = 0$. Furthermore,

$$\big|\alpha^\perp(x, \gamma_{\mathcal{T}}^\mu(\upsilon)) - \tilde{\alpha}_{\mathcal{T}}(x, b; \rho_{\mathcal{T}}(\upsilon))\big| \le C(b_\upsilon)|\upsilon|^{\frac{1}{p}} |\rho_{\mathcal{T}}(\upsilon)| \quad (x, b; \upsilon) \in \mathcal{F}\mathcal{T}_\delta,$$

where $\tilde{\alpha}_{\mathcal{T}} = \pi^\perp \circ (s_\Sigma \otimes \alpha_{\mathcal{T}})$. If ν is generic, $\tilde{\alpha}_{\mathcal{T}}$ has full rank on every fiber, since s_Σ has no zeros. Thus, by Corollary 3.13,

$$\mathcal{C}_{\Sigma\times\mathcal{U}_{\mathcal{T}}(\mu)}(\alpha^\perp) = 0 \qquad \text{if} \;\; H_{\hat{0}}\mathcal{T} \neq \hat{I}.$$

If $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=1$, $\tilde{\alpha}_{\mathcal{T}}\circ\tilde{\rho}_{\mathcal{T}}$ has full rank over all $\Sigma\times\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$, and thus by Corollary 3.13 and Lemma 3.14

$$\mathcal{C}_{\Sigma\times\mathcal{U}_{\mathcal{T}}(\mu)}(\alpha^{\perp})=\big\langle c(T^*\Sigma\otimes\mathcal{O}^{\perp})c(L_{\hat{1}}\mathcal{T})^{-1}, \big[\Sigma\times\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\big]\big\rangle=2\big\langle 12a+3c_1(L_{\hat{1}}^*\mathcal{T}), \big[\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\big]\big\rangle. \tag{5.15}$$

If $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2$,

$$\big(T^*\Sigma\otimes L^*\otimes\mathcal{O}^{\perp}\big)/\big(\text{Im }\tilde{\alpha}_{\mathcal{T}}\circ\tilde{\rho}_{\mathcal{T}}\big)\approx T^*\Sigma\otimes\big((\mathcal{H}_{\Sigma}^{-}\otimes\mathbb{C}^2)/\mathbb{C}\big).$$

Thus, similarly to the computation in (2) above,

$$\mathcal{C}_{\Sigma\times\mathcal{U}_{\mathcal{T}}(\mu)}(\alpha^{\perp})=6|\mathcal{U}_{\mathcal{T}}(\mu)|. \tag{5.16}$$

Summing equations (5.15) and (5.16) over all equivalence classes of $\mathcal{T}<\mathcal{T}^*$, we obtain

$$\mathcal{C}_{\Sigma\times(\bar{\mathcal{V}}_1(\mu)-\mathcal{V}_1(\mu))}(\alpha^{\perp})=2\sum_{l\in[N]}\big\langle 12a+3c_1(L_{\hat{1}}^*\mathcal{T}), \big[\mathcal{U}_{\mathcal{T}^*(l)}(\mu)\big]\big\rangle+6\tau_2(\mu). \tag{5.17}$$

The claim follows by plugging (5.14) and (5.17) into (5.12) and using (5.3), (5.4), and Lemma 5.4.

Lemma 5.7 *If* $n=3$, $n_2^{(1)}(\mu)=4\big\langle 10a^2+4a(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*))+c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*), \big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle$.

Proof: (1) We use the same notation as in the proof of Lemma 5.5. By Section 4.9 and equation (4.34), $n_2^{(1)}(\mu)=N(\alpha_2)$, where

$$\alpha_2\in\Gamma\big(\Sigma^2\times\bar{\mathcal{V}}_2(\mu);\text{Hom}(\tilde{E};\mathcal{O})\big),\quad \tilde{E}=T\Sigma_1\otimes L_1\oplus T\Sigma_2\otimes L_2,\quad \mathcal{O}=\mathcal{H}_{\Sigma}^{0,1}\otimes\text{ev}^*T\mathbb{P}^3,$$
$$\alpha_2\big(x_1,x_2,b;v_1\otimes\upsilon_1,v_2\otimes\upsilon_2\big)=(\mathcal{D}\upsilon_1)(s_{\Sigma,x_1}v_1)+(\mathcal{D}\upsilon_2)(s_{\Sigma,x_2}v_2).$$

Here the bundles $L_i\longrightarrow\bar{\mathcal{V}}_2(\mu)$ and the sections $\mathcal{D}_i\in\Gamma\big(\bar{\mathcal{V}}_2(\mu);L_i^*\otimes\text{ev}^*T\mathbb{P}^2\big)$ are defined as follows. If $b\in\mathcal{U}_{\mathcal{T}^*}(\mu)\subset\mathcal{V}_2(\mu)$, $\mathcal{T}^*=(S^2,[N],I^*;j^*,\underline{d}^*)$, and $I^*=\{k_1,k_2\}$, we let $L_i\big|_b=L_{k_i}\mathcal{T}$ and $\mathcal{D}_i=\mathcal{D}_{\mathcal{T},k_i}$. These bundles and sections are well-defined once we fix a representative for each equivalence class of such bubble types $\mathcal{T}^*$ and order the elements of the corresponding set I^*.
(2) By Lemma 3.14,

$$\begin{aligned} n_1^{(1)}(\mu)&=\sum_{k=0}^{k=5}\big\langle c_k(\mathcal{O})\lambda_{\tilde{E}}^{5-k}, \big[\mathbb{P}\tilde{E}\big]\big\rangle-\mathcal{C}_{\tilde{\alpha}^{-1}(0)}(\tilde{\alpha}^{\perp}), \\ &=4\big\langle 28a^2+16a(c_1(L_1^*)+c_1(L_2^*))+3(c_1^2(L_1^*)+c_1^2(L_2^*))+4c_1(L_1^*)c_1(L_2^*),\big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle-\mathcal{C}_{\tilde{\alpha}^{-1}(0)}(\tilde{\alpha}^{\perp}),\end{aligned} \tag{5.18}$$

where $\tilde{\alpha}\in\Gamma(\mathbb{P}\tilde{E};\gamma_{\tilde{E}}^*\otimes\mathcal{O})$ is the section induced by α_2. Let

$$\Sigma^{(\pm)}=\big\{(x_1,x_2)\in\Sigma_1^*\times\Sigma_2^*\colon x_1=\pm x_2\big\},\quad \Sigma^{(0)}=\big\{(z_m,z_m)\colon m\in[6]\big\};$$
$$\mathcal{S}_2^{(\pm)}=\Sigma^{(\pm)}\times\mathcal{S}_2(\mu),\quad \mathcal{S}_2^{(0)}=\Sigma^{(0)}\times\mathcal{S}_2(\mu),$$

where $+x_2\equiv x_2$ and $-x_2$ is the image of x_2 under the nontrivial automorphism of Σ. The zero set of $\tilde{\alpha}$ is the union of a section of $\mathbb{P}\tilde{E}$ over $\mathcal{S}_2^{(\pm)}$, $\mathcal{S}_2^{(0)}$, and $\Sigma^2\times\mathcal{U}_{\mathcal{T}}(\mu)$, where $\mathcal{T}$ is as in the proof of Lemma 5.5.
(3) The above section over $\Sigma^2\times\mathcal{U}_{\mathcal{T}}(\mu)$ is given by

$$\mathcal{Z}_{\mathcal{T}}\equiv\tilde{\alpha}^{-1}(0)\cap\mathbb{P}\tilde{E}|_{\Sigma^2\times\mathcal{U}_{\mathcal{T}}(\mu)}=\big\{(x_1,x_2,b,T_{x_1}\Sigma_1\otimes L_1|_b)\colon(x_1,x_2,b)\in\Sigma^2\times\mathcal{U}_{\mathcal{T}}(\mu)\big\}.$$

The map $\gamma_{\mathcal{T}}^{\mu}$ of Theorem 2.8 induces identifications of neighborhoods of $\mathcal{Z}_{\mathcal{T}}$ in

$$\mathcal{FS} = \pi_{\tilde{E}}^*\big(\mathcal{FT} \oplus T^*\Sigma_1\otimes L_1^*\otimes T\Sigma_2\otimes L_2\big)$$

and in $\mathbb{P}\tilde{E}$ as well as of appropriate bundles such that

$$\begin{gathered}\big|\tilde{\alpha}(\gamma_{\mathcal{T}}^{\mu}(\upsilon,u)) - \tilde{\alpha}_{\mathcal{T}}(\rho_{\mathcal{T}}(\upsilon),u)\big| \le C(b_\upsilon)|\upsilon|^{\frac{1}{p}}|\rho_{\mathcal{T}}(\upsilon)| \quad \forall(\upsilon,u)\in\mathcal{FS}_\delta, \quad \text{where}\\ \tilde{\alpha}_{\mathcal{T}} \in \Gamma\big(\mathcal{Z}_{\mathcal{T}};\mathrm{Hom}(\tilde{\mathcal{F}}\mathcal{S};\gamma_{\tilde{E}}^*\otimes\mathcal{O})\big), \quad \tilde{\mathcal{F}}\mathcal{S} = \pi_{\tilde{E}}^*\big(\tilde{\mathcal{F}}\mathcal{T} \oplus T^*\Sigma_1\otimes L_1^*\otimes T\Sigma_2\otimes L_2\big),\\ \tilde{\alpha}_{\mathcal{T}}(x_1,x_2,b;\tilde{\upsilon},u) = \{\alpha_{\mathcal{T}}(\tilde{\upsilon})\}\otimes s_{x_1} + (\mathcal{D}_2\otimes s_{x_2})\circ u.\end{gathered}$$

By the proof of Lemma 4.9, $\tilde{\alpha}_{\mathcal{T}}$ is nondegenerate. The same is true of $\tilde{\alpha}^{\perp}$ as long as $\bar{\nu}\in\Gamma(\mathbb{P}\tilde{E};\mathcal{O})$ is generic. Thus, if $\hat{I}\neq H_{\hat{0}}\mathcal{T}$, $\mathcal{Z}_{\mathcal{T}}$ is $\tilde{\alpha}^{\perp}$-hollow and $\mathcal{C}_{\mathcal{Z}_{\mathcal{T}}}(\tilde{\alpha}^{\perp})=0$ by Corollary 3.13. If $|H_{k_1}\mathcal{T}|=|\hat{I}|=1$, i.e. $\mathcal{T}=\mathcal{T}^*(l)$ for some $l\in[N]$, $\tilde{\alpha}_{\mathcal{T}}$ has full rank on $\bar{\mathcal{Z}}_{\mathcal{T}}\approx\Sigma^2\times\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$. Thus, by Corollary 3.13,

$$\mathcal{C}_{\mathcal{Z}_{\mathcal{T}}}(\tilde{\alpha}^{\perp}) = \big\langle c(\gamma_{\tilde{E}}^*\otimes\mathcal{O}^{\perp})c\big(\mathcal{FS}\big)^{-1}, [\bar{\mathcal{Z}}_{\mathcal{T}}]\big\rangle = 4\big\langle 16a+4c_1(L_2^*)+3c_1(L_{\hat{1}}^*\mathcal{T}), \big[\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\big]\big\rangle, \tag{5.19}$$

since $F\mathcal{S}\approx L_{\hat{1}}\mathcal{T}\oplus T^*\Sigma_1\otimes T\Sigma_2\otimes L_2$. If $|H_{k_1}\mathcal{T}|=|\hat{I}|=2$, we similarly obtain

$$\mathcal{C}_{\mathcal{Z}_{\mathcal{T}}}(\tilde{\alpha}^{\perp}) = \big\langle c(\gamma_{\tilde{E}}^*\otimes\mathcal{O}^{\perp})c\big(\mathcal{FS}\big)^{-1}, [\bar{\mathcal{Z}}_{\mathcal{T}}]\big\rangle = 12|\mathcal{U}_{\mathcal{T}}(\mu)| \tag{5.20}$$

Note that $F\mathcal{S}\approx\mathbb{C}^2\oplus T^*\Sigma_1\otimes T\Sigma_2$ in this case. Summing up equations (5.19) and (5.20) over all equivalence classes of bubble types $\mathcal{T}$ of the appropriate form and using (5.3) and (5.4), we obtain

$$\mathcal{C}_{\mathbb{P}\tilde{E}|\partial\mathcal{V}_2(\mu)}(\tilde{\alpha}^{\perp}) = 4\sum_{[\mathcal{T}^*]}\sum_{l\in M_i^*, i\neq j}\big\langle 16a+3c_1(\mathcal{L}_i^*)+4c_1(\mathcal{L}_j^*), \big[\mathcal{U}_{\mathcal{T}^*(l)}(\mu)\big]\big\rangle - 36\big|\mathcal{V}_3(\mu)\big|, \tag{5.21}$$

where the outer sum is taken over equivalence classes of bubbles $\mathcal{T}^*$ as in (1) above.
(4) It remains to compute $\mathcal{C}_{\mathbb{P}\tilde{E}|\mathcal{S}_2^{\pm}}(\tilde{\alpha}^{\perp})$ and $\mathcal{C}_{\mathbb{P}\tilde{E}|\mathcal{S}_2^{(0)}}(\tilde{\alpha}^{\perp})$. Note that

$$\tilde{\alpha}^{-1}(0)\cap\mathbb{P}\tilde{E}|_{\mathcal{S}_2^{(\pm)}} = \mathcal{Z}_2^{(\pm)} \equiv \big\{(x,\pm x,b;[v\otimes\upsilon_1, v\otimes\upsilon_2])\in\mathbb{P}\tilde{E}|_{\mathcal{S}_2^{(\pm)}} : \mathcal{D}|_{(b;[\upsilon_1,\upsilon_2])}=0\big\},$$

where $\mathcal{D}$ is the section of $\gamma_E^*\otimes\mathrm{ev}^*T\mathbb{P}^3$ defined in the proof of Lemma 5.5. Identify neighborhoods of $\mathcal{Z}_2^{(\pm)}$ in

$$\mathcal{FS} \equiv T\Sigma\oplus\gamma_E^*\otimes\mathrm{ev}^*T\mathbb{P}^3 \approx T\Sigma\oplus\mathbb{C}^3$$

and in $\mathbb{P}\tilde{E}$ via a map γ in such a way that

$$\begin{gathered}\big|\tilde{\alpha}(\gamma(w,X)) - \alpha_{\mathcal{S}}(w,X)\big| \le C(x,b)\big(|w|+|X|\big)|w \quad \forall(w,X)\in\mathcal{FS}_\delta,\\ \text{where} \qquad \alpha_{\mathcal{S}}\in\Gamma\big(\mathcal{Z}_{2;2};\mathrm{Hom}(\mathcal{FS};\gamma_{\tilde{E}}^*\otimes\mathcal{O})\big),\\ \big\{\alpha_{\mathcal{S}}(w,X)\big\}(v\otimes\upsilon) = (X\upsilon)(s_x v)+\big(\mathcal{D}_2\upsilon_2\big)\big(s_{b,x}^{(2)}(w,v)\big)\in\mathcal{O}, \quad \text{if} \ \ v\in T_x\Sigma,\ \upsilon=(\upsilon_1,\upsilon_2)\in\gamma_E.\end{gathered}$$

Since $s_x^{(2)}=\pi_x^-\circ s_{b,x}^{(2)}$ does not vanish on Σ^*, $\alpha_{\mathcal{S}}$ has full rank on $\mathcal{Z}_{2;2}^{(\pm)}$ and extends over $\bar{\mathcal{Z}}_2^{(\pm)}\approx\Sigma\times\mathcal{S}_2(\mu)$. This extension is a regular polynomial in the sense of Definition 3.9. Furthermore,

$$\pi_{\bar{\nu}}^{\perp}\alpha_{\mathcal{S}} : \gamma_E^*\otimes\mathrm{ev}^*T\mathbb{P}^3 \longrightarrow \gamma_{\tilde{E}}^*\otimes\pi_{\bar{\nu}}^{\perp}\big(\mathcal{H}_{\Sigma}^+\otimes\mathrm{ev}^*T\mathbb{P}^3\big)$$

is an isomorphism. Thus, by Corollary 3.13, $\mathcal{C}_{\mathcal{Z}_2^{(\pm)}}(\tilde{\alpha}^{\perp})=N(\alpha_{\mathcal{S}}^-)$, where

$$\alpha_{\mathcal{S}}^- \in \Gamma\big(\bar{\mathcal{Z}}_2; \mathrm{Hom}(T\Sigma; \mathcal{O}_2)\big), \quad \mathcal{O}_2 = \gamma_{\tilde{E}}^* \otimes \big(\mathcal{H}_\Sigma^- \otimes \mathrm{ev}^* T\mathbb{P}^3\big)^\perp \approx T^*\Sigma \otimes \big(\mathcal{H}_\Sigma^- \otimes \mathbb{C}^3\big)^\perp,$$
$$\big\{\alpha_{\mathcal{S}}^-(w)\big\}(v \otimes v_2) = \pi_{\pi_x^- \bar{\nu}}^\perp\big((\mathcal{D}_2 v_2) s_x^{(2,-)}(w,v)\big).$$

As in the previous section, we can replace $T\Sigma$ with

$$\tilde{T}'\Sigma \equiv T\Sigma \otimes \mathcal{O}(z_1) \otimes \ldots \otimes \mathcal{O}(z_6)$$

and $s^{(2,-)}$ with $\tilde{s}^{(2,-)} \in \Gamma(\Sigma; \tilde{T}'\Sigma^* \otimes \mathcal{H}_\Sigma^-)$ above to obtain a non-vanishing linear map $\tilde{\alpha}_{\mathcal{S}}^-$ such that $N(\alpha_{\mathcal{S}}^-)=N(\tilde{\alpha}_{\mathcal{S}}^-)$. Thus, by Lemma 3.14,

$$\mathcal{C}_{\mathcal{Z}_2^{(+)} \cup \mathcal{Z}_2^{(-)}}(\tilde{\alpha}^\perp) = 2\big\langle c_1(\mathcal{O}_2) - c_1(\tilde{T}'\Sigma), [\mathcal{Z}_2^{(+)}]\big\rangle = 2(10-4)|\mathcal{S}_2(\mu)| = 12|\mathcal{S}_2(\mu)|. \tag{5.22}$$

(5) We next show that $\mathcal{C}_{\mathbb{P}\tilde{E}|\mathcal{S}_2^{(0)}}(\tilde{\alpha}^\perp)=0$. Similarly to (4),

$$\alpha^{-1}(0) \cap \mathbb{P}\tilde{E}|_{\mathcal{S}_2^{(0)}} = \mathcal{Z}_2^{(0)} \equiv \big\{(z_m, z_m, b; [v \otimes v_1, v \otimes v_2]) \in \mathbb{P}\tilde{E}|_{\mathcal{S}_2^{(0)}} : \mathcal{D}|_{(b;[v_1,v_2])} = 0\big\}.$$

We can identify neighborhoods of $\mathcal{Z}_2^{(0)}$ in

$$\mathcal{FS} \equiv T\Sigma_1 \oplus T\Sigma_2 \oplus \gamma_E^* \otimes \mathrm{ev}^* T\mathbb{P}^3 \approx \mathbb{C}^2 \oplus \mathbb{C}^3$$

and in $\mathbb{P}\tilde{E}$ via a map γ in such a way that

$$\big|\pi_{w_1}^- \circ \tilde{\alpha}(\gamma(w_1,w_2,X)) - \alpha_{\mathcal{S}}^-(w_1,w_2,X)\big| \le C\big|X, w_1, w_2\big||w_1||w_1 - w_2| \quad \forall (w_1,w_2,X) \in \mathcal{FS}_\delta,$$
$$\text{where} \quad \big\{\alpha_{\mathcal{S}}^-(w_1,w_2)\big\}(v \otimes v_1, v \otimes v_2) = (\mathcal{D}_2 v_2) s_{z_m}^{(3)}(w_1, w_2 - w_1, v) \in \mathcal{H}_\Sigma^-(z_m) \otimes \mathrm{ev}^* T\mathbb{P}^3.$$

Since the rank of $(\mathcal{H}_\Sigma^-(z_m) \otimes \mathrm{ev}^* T\mathbb{P}^3)/\mathbb{C}\pi_{z_m}^- \bar{\nu}$ is two, while the rank of $T_{z_m}\Sigma_1 \otimes T_{z_m}\Sigma_2$ is one, it follows that $\mathcal{Z}_2^{(0)}$ is $\tilde{\alpha}^\perp$-hollow, and $\mathcal{C}_{\mathbb{P}\tilde{E}|\mathcal{S}_2^{(0)}}(\tilde{\alpha}^\perp)=0$ by Corollary 3.13. The lemma is obtained by plugging (5.21) and (5.22) into (5.18), using (5.3) and (5.4), and Lemma 5.5.

5.4 Behavior of $\mathcal{D}^{(2)}$ and $\mathcal{D}^{(3)}$ near $\bar{\mathcal{S}}_1(\mu) - \mathcal{S}_1(\mu)$

If $n=3$, the space $\mathcal{S}_1(\mu)$ is not compact. In order to be able to compute the numbers $n_1^{(k)}(\mu)$, we thus must understand the structure of $\bar{\mathcal{S}}_1(\mu)$ as well as the behavior of $\mathcal{D}_{\mathcal{T}^*,\hat{0}}^{(2)}$ and $\mathcal{D}_{\mathcal{T}^*,\hat{0}}^{(3)}$, where $\mathcal{T}^*=(S^2,[N],\{\hat{0}\};\hat{0},d)$, near $\bar{\mathcal{S}}_1(\mu)-\mathcal{S}_1(\mu)$.

If $\mathcal{T}=(S^2,[N],I;j,\underline{d})<\mathcal{T}^*$, from Theorem 2.8 one should expect that the normal bundle, or cone, $\mathcal{FS}$ of $\mathcal{S}_\mathcal{T}(\mu) \equiv \mathcal{U}_\mathcal{T}(\mu) \cap \bar{\mathcal{S}}_1(\mu)$ in $\bar{\mathcal{S}}_1(\mu)$ is the closure of the set

$$\big\{[\upsilon=(b,(v_h)_{h\in\hat{I}})] \in \mathcal{F}^{(\emptyset)}\mathcal{T}\big|_{\mathcal{S}_\mathcal{T}(\mu)} : \sum_{h\in\chi(\mathcal{T})} \prod_{i\in\hat{I}, i\le h} v_i\,(\mathcal{D}_{\mathcal{T},h} b) = 0\big\} \tag{5.23}$$

in $\mathcal{FT}$. The next lemma shows that this is indeed the case. By a dimension-counting argument, if the set in (5.23) is not empty, either $|\chi(\mathcal{T})|=1$ or $\chi(\mathcal{T})=\{h_1,h_2\}$ is a two-element set, $\iota_{h_1}=\iota_{h_2}$, $\mathcal{D}_{\mathcal{T},h_1}b\neq 0$, and $\mathcal{D}_{\mathcal{T},h_2}b\neq 0$. In the first case $\mathcal{FS}=\mathcal{FT}|_{\mathcal{S}_\mathcal{T}(\mu)}$, while in the second $\mathcal{FS}$ is a codimension-one subbundle of $\mathcal{FT}|_{\mathcal{S}_\mathcal{T}(\mu)}$.

Let $\mathcal{NS}\longrightarrow\mathcal{FS}$ denote the normal bundle of $\mathcal{FS}$ in $\mathcal{FT}\longrightarrow\mathcal{U}_\mathcal{T}(\mu)$. While for the purposes of Lemma 5.8, we can use any identification of neighborhoods of $\mathcal{FS}$ in $\mathcal{NS}$ and in $\mathcal{FT}\longrightarrow\mathcal{U}_\mathcal{T}(\mu)$, in order to simplify the statement of Lemma 5.10, we choose a fairly natural one. More precisely, denote by $\mathcal{FS}^\perp$ a subspace of $\mathcal{FT}|_{\mathcal{S}_\mathcal{T}(\mu)}$ complementary to $\mathcal{FS}$ and by $\pi_\mathcal{S}\colon\mathcal{NS}^{(1)}\longrightarrow\mathcal{S}_\mathcal{T}(\mu)$ the normal bundle of $\mathcal{S}_\mathcal{T}(\mu)$ in $\mathcal{U}_\mathcal{T}(\mu)$. Choose a norm on $\mathcal{NS}^{(1)}$ and an identification $\phi_\mathcal{S}\colon\mathcal{NS}^{(1)}_\delta\longrightarrow\mathcal{S}_\mathcal{T}(\mu)$ of neighborhoods of $\mathcal{S}_\mathcal{T}(\mu)$ in $\mathcal{NS}^{(1)}$ and in $\mathcal{U}_\mathcal{T}(\mu)$. Let $\Phi_\mathcal{S}\colon\pi_\mathcal{S}^*\mathcal{FT}\longrightarrow\mathcal{FT}$ be a lift of $\phi_\mathcal{S}$ such that $\Phi_\mathcal{S}$ restricts to the identity over $\mathcal{S}_\mathcal{T}(\mu)\subset\mathcal{NS}^{(1)}_\delta$. Let $\pi\colon\mathcal{FT}\longrightarrow\mathcal{S}_\mathcal{T}(\mu)$ be the bundle projection. Then

$$\mathcal{NS}=\pi^*\mathcal{NS}^{(1)}\oplus\mathcal{FS}^\perp,\quad\text{and}\quad\tilde\phi_\mathcal{S}\colon\mathcal{NS}_\delta\longrightarrow\mathcal{FT},\quad\tilde\phi_\mathcal{S}\big((b,v),(X,v^\perp)\big)=\Phi_\mathcal{S}\big((b,X),v+v^\perp\big),$$

is an identification of neighborhoods of $\mathcal{FS}$ in $\mathcal{NS}$ and $\mathcal{FT}\longrightarrow\mathcal{U}_\mathcal{T}(\mu)$.

Lemma 5.8 *For every bubble type $\mathcal{T}=(S^2,[N],I;j,\underline{d})<\mathcal{T}^*$, there exist $\delta,C\in C^\infty(\mathcal{S}_\mathcal{T}(\mu);\mathbb{R}^+)$ and a section $\varphi_\mathcal{S}\in\Gamma(\mathcal{FS}_\delta;\mathcal{NS})$ such that*

$$\|\varphi_\mathcal{S}(\upsilon)\|\le C(b_\upsilon)|\upsilon|^{\frac{1}{p}},\qquad\|\varphi_{\mathcal{FS}^\perp}(\upsilon)\|\le C(b_\upsilon)|\upsilon|^{1+\frac{1}{p}},$$

where $\varphi_{\mathcal{FS}^\perp}$ denotes the $\mathcal{FS}^\perp$-component of $\varphi_\mathcal{S}$, and the map

$$\gamma_\mathcal{S}\colon\mathcal{FS}_\delta\longrightarrow\bar{\mathcal{S}}_1(\mu),\qquad\gamma_\mathcal{S}(\upsilon)=\gamma_\mathcal{T}^\mu\big(\tilde\phi_\mathcal{S}\varphi_\mathcal{S}(\upsilon)\big),$$

is a homeomorphism onto an open neighborhood of $\mathcal{S}_\mathcal{T}(\mu)$ in $\bar{\mathcal{S}}_1(\mu)$, which is smooth and orientation-preserving on the preimage of $\mathcal{S}_1(\mu)$.

Proof: (1) The proof is similar to that of Lemma 3.32 in [Z1], and so we only describe the differences. If $\mathcal{S}_\mathcal{T}(\mu)\neq\emptyset$, $\mathcal{T}$ must have one of the three forms described by Lemma 5.10. In Case (1), we apply Subsection 3.7 in [Z1], which contains an application of the Implicit Function Theorem, to $\mathcal{D}_{\mathcal{T}^*,\hat 0}$ instead of the evaluation maps. By Theorem 2.8,

$$\big|\Pi^{-1}_{b,\tilde\gamma^\mu_\mathcal{T}(\upsilon)}(\mathcal{D}_{\mathcal{T}^*,\hat 0}\tilde\gamma^\mu_\mathcal{T}(\upsilon))-(\mathcal{D}_{\mathcal{T}^*,\hat 0}b_\upsilon)\big|\le C'(b)|\upsilon|^{\frac{1}{p}}\qquad\forall\upsilon\in\mathcal{FS}_\delta.$$

This estimate suffices for applying an argument similar to the proof of Lemma 3.32 in [Z1].
(2) In Case (2) of Lemma 5.10, instead of the section $\mathcal{D}_{\mathcal{T}^*,\hat 0}$ of $L^*_{\hat 0}\mathcal{T}^*\otimes\mathrm{ev}^*T\mathbb{P}^3$, we consider the section $\tilde{\mathcal{D}}$ of $\big(L_{\hat 0}\mathcal{T}^*\otimes\mathcal{FT}\big)^*\otimes\mathrm{ev}^*T\mathbb{P}^3$ on a neighborhood of $\mathcal{U}_\mathcal{T}(\mu)$ in $\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$ defined by

$$\tilde{\mathcal{D}}\big|_{\tilde\gamma^\mu_\mathcal{T}(b,v_{\hat 1})}(v_{\hat 0},v_{\hat 1})=\mathcal{D}_{\mathcal{T}^*,\hat 0}\big|_{\tilde\gamma^\mu_\mathcal{T}(b,v_{\hat 1})}(v_{\hat 0})\in\mathrm{ev}^*T\mathbb{P}^3.$$

This section is well-defined outside of $\mathcal{U}_\mathcal{T}(\mu)$ and by Theorem 2.8 extends over $\mathcal{U}_\mathcal{T}(\mu)$ by

$$\tilde{\mathcal{D}}\big|_b(v_{\hat 0}\otimes v_{\hat 1})=v_{\hat 0}v_{\hat 1}\big(\mathcal{D}_{\mathcal{T},\hat 1}b\big).$$

The restriction of this section to $\mathcal{U}_\mathcal{T}(\mu)$ vanishes transversally at $\mathcal{S}_\mathcal{T}(\mu)$ by Corollary 6.3, while its zero set on $\mathcal{U}_{\mathcal{T}^*}(\mu)$ is the same as the zero set of $\mathcal{D}_{\mathcal{T}^*,\hat 0}$. By Theorem 2.8, with appropriate identifications,

$$\big|\tilde{\mathcal{D}}|_{\gamma^\mu_\mathcal{T}(b,\upsilon)}-\tilde{\mathcal{D}}|_b\big|\le C'(b)|\upsilon|^{\frac{1}{p}}\qquad\forall\upsilon\in\mathcal{FS}_\delta.$$

(3) In the final case of Lemma 5.10, we replace $\mathcal{D}_{\mathcal{T}^*,\hat{0}}$ by a bundle section over the blowup of $\mathcal{FT}$ along $\mathcal{U}_{\mathcal{T}}(\mu)$. Let

$$\Omega_{\mathcal{T}} = \big\{(b,v,\ell)\colon (b,v)\in\mathcal{FT},\ v\in\ell\in\mathbb{P}\mathcal{FT}|_b\big\}, \quad \Omega^*_{\mathcal{T}} = \big\{(b,v,\ell)\in\Omega_{\mathcal{T}}\colon v\neq 0\big\}, \quad \mathcal{E}_{\mathcal{T}} = \Omega_{\mathcal{T}} - \Omega^*_{\mathcal{T}}.$$

Denote by $\gamma\longrightarrow\Omega_{\mathcal{T}}$ the tautological line bundle. The normal bundle $\tilde{\mathcal{N}}\mathcal{S}$ of $\gamma\longrightarrow\mathbb{P}\mathcal{FS}$ in $\gamma\longrightarrow\mathcal{E}_{\mathcal{T}}$ is given by

$$\begin{gathered}\tilde{\mathcal{N}}\mathcal{S} = \pi^*_\gamma\pi^*_{\mathcal{FT}}\mathcal{NS}^{(1)} \oplus \pi^*_\gamma\big(\gamma^*\otimes\pi^*_{\mathcal{FT}}\mathcal{FS}^{\perp}\big),\\ \tilde{\phi}_{\tilde{\mathcal{N}}\mathcal{S}}\big((b,\ell,v),X,\sigma\big) = \big(\phi_{\mathcal{S}}(b,X),\big[\Phi_{\mathcal{S}}(v+\sigma(v))\big],v+\sigma(v)\big),\end{gathered}$$

where $\pi_\gamma\colon\gamma\longrightarrow\Omega_{\mathcal{T}}$ is the bundle projection map. The bundle $L_{\hat{0}}\mathcal{T}^*$ pulls back to a bundle $\tilde{L}$ over a neighborhood Ω_δ of $\mathcal{E}_{\mathcal{T}}$ in $\Omega_{\mathcal{T}}$. We define a section $\tilde{\mathcal{D}}$ of $\big(\tilde{L}\otimes\gamma\big)^*\otimes\mathrm{ev}^*T\mathbb{P}^3$ over Ω_δ by

$$\tilde{\mathcal{D}}\big|_{(b,v_{\hat{1}},v_{\hat{2}},\ell)}\big(v_{\hat{0}},v_{\hat{1}},v_{\hat{2}}\big) = \mathcal{D}_{\mathcal{T}^*,\hat{0}}\big|_{\tilde{\gamma}^{\mu}_{\mathcal{T}}(b,v_{\hat{1}},v_{\hat{2}})}(v_{\hat{0}}) \in \mathrm{ev}^*T\mathbb{P}^3.$$

This section is well-defined outside of $\mathcal{E}_{\mathcal{T}}(\mu)$ and by Theorem 2.8 extends over $\mathcal{E}_{\mathcal{T}}(\mu)$ by

$$\tilde{\mathcal{D}}\big|_b\big(v_{\hat{0}},v_{\hat{1}},v_{\hat{2}}\big) = v_{\hat{0}}\big(v_{\hat{1}}(\mathcal{D}_{\mathcal{T},\hat{1}}b) + v_{\hat{2}}(\mathcal{D}_{\mathcal{T},\hat{2}}b)\big).$$

The restriction of this section to $\mathcal{E}_{\mathcal{T}}(\mu)$ vanishes transversally at $\mathbb{P}\mathcal{FS}\longrightarrow\mathcal{S}_{\mathcal{T}}(\mu)$ by Corollary 6.3, while its zero set on Ω^* corresponds to the zero set of $\mathcal{D}_{\mathcal{T}^*,\hat{0}}$ on $\gamma^{\mu}_{\mathcal{T}}\big(\mathcal{FT}_\delta - \mathcal{U}_{\mathcal{T}}(\mu)\big)$. By Theorem 2.8, with appropriate identifications,

$$\big|\tilde{\mathcal{D}}|_{(b,v_{\hat{1}},v_{\hat{2}},\ell)} - \tilde{\mathcal{D}}|_{(b,\ell)}\big| \le C'(b)|v|^{\frac{1}{p}}.$$

Thus, we can apply the arguments of Lemma 3.32 in [Z1] to $\tilde{\mathcal{D}}$ to describe its zero set near $\mathcal{E}_{\mathcal{T}}$. We obtain a section $\tilde{\varphi}_{\mathcal{S}}\in\Gamma(\gamma_\delta|_{\mathbb{P}\mathcal{FS}};\tilde{\mathcal{N}}\mathcal{S})$ such that $\|\tilde{\varphi}_{\mathcal{S}}(v)\|\le C(b_v)|v|^{\frac{1}{p}}$, and the map

$$\tilde{\gamma}_{\mathcal{S}}\colon\gamma_\delta|_{\mathbb{P}\mathcal{FS}}\longrightarrow\Omega_{\mathcal{T}}, \quad \tilde{\gamma}_{\mathcal{S}}(v) = \tilde{\phi}_{\tilde{\mathcal{N}}\mathcal{S}}(\varphi_{\mathcal{S}}(v)),$$

is a homeomorphism onto an open neighborhood of $\mathbb{P}\mathcal{FS}$ in $\tilde{\mathcal{D}}^{-1}(0)$. This section $\tilde{\varphi}_{\mathcal{S}}$ induces the required section $\varphi_{\mathcal{S}}$ with the claimed properties.

Corollary 5.9 *For every bubble type $\mathcal{T}=(S^2,[N],I;j,\underline{d})<\mathcal{T}^*$, there exist $\delta\in C^\infty(\mathcal{S}_{\mathcal{T}}(\mu);\mathbb{R}^+)$ and a map*

$$\gamma_{\mathcal{S}}:\big(\mathcal{NS}^{(1)}\oplus\mathcal{FT}\big)_\delta\big|_{\mathcal{S}_{\mathcal{T}}(\mu)}\longrightarrow\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$$

such that $\gamma_{\mathcal{S}}$ is a homeomorphism onto an open neighborhood of $\mathcal{S}_{\mathcal{T}}(\mu)$ in $\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$, which is smooth and orientation-preserving on the preimage of $\mathcal{U}_{\mathcal{T}^*}(\mu)$, and with appropriate identifications,

$$\mathcal{D}\gamma_{\mathcal{S}}(X,v) = \begin{cases} X, & \textit{in Case (1) with } X\in L^*\otimes ev^*T\mathbb{P}^3;\\ Xv_{\hat{1}}, & \textit{in Case (2) with } X\in L^*_{\hat{1}}\mathcal{T}\otimes ev^*T\mathbb{P}^3,\end{cases}$$

where the cases are the ones described by Lemma 5.10.

Proof: The proof is just a modification of the proof of Lemma 5.8. We work with the sections $\bar{\mathcal{D}}\equiv\mathcal{D}$ and $\bar{\mathcal{D}}\equiv\tilde{\mathcal{D}}$ in Cases (1) and (2), respectively. Choose an identification $\gamma\colon\mathcal{NS}_{\delta}^{(1)}\longrightarrow\mathcal{U}_{\mathcal{T}}(\mu)$ of neighborhoods of $\mathcal{S}_{\mathcal{T}}(\mu)$ in $\mathcal{NS}^{(1)}$ and in $\mathcal{U}_{\mathcal{T}}(\mu)$ as well as of the appropriate line bundle over these neighborhoods such that

$$\bar{\mathcal{D}}\big|_{(b,X)}=X\quad\forall X\in\mathcal{NS}_{\delta}^{(1)}.$$

By the same argument as in the proof of Lemma 5.8, for any $(Y,\upsilon)\in N\mathcal{S}^{(1)}\oplus\mathcal{FT}$, there exists a unique $Z\in N\mathcal{S}^{(1)}$, such that

$$\bar{\mathcal{D}}\big|_{\gamma_{\mathcal{T}}^{\mu}(\Phi_{\mathcal{S}}(X+Y;\upsilon))}=\bar{\mathcal{D}}\big|_{X}=X.$$

Furthermore, $|Z|\leq C(b)\big(|Y|+|\upsilon|^{\frac{1}{p}}\big)$.

Lemma 5.10 *If $d\geq1$, μ is a tuple of p points and q lines in general position in $\mathbb{P}^3$ with $2p+q=4d-3$, and $N=p+q$, the set $\bar{\mathcal{S}}_1(\mu)-\mathcal{S}_1(\mu)$, is finite. Furthermore, if*

$$\mathcal{T}=(S^2,[N],I;j,\underline{d})<\mathcal{T}^*\quad\textit{and}\quad\mathcal{S}_{\mathcal{T}}(\mu)\neq\emptyset,$$

(1) $\hat{I}=\{\hat{1}\}$, $d_{\hat{0}}>0$, and the images of $\mathcal{D}^{(2)}_{\mathcal{T}^*,\hat{0}}$ and $\mathcal{D}^{(3)}_{\mathcal{T}^*,\hat{0}}$ are linearly independent in every fiber of $ev^*T\mathbb{P}^3$ over $\mathcal{S}_{\mathcal{T}}(\mu)$;
(2) OR $\hat{I}=\{\hat{1}\}$, $d_{\hat{0}}=0$, $d_{\hat{1}}=d$, and for all $\upsilon=[b,v_{\hat{1}}]\in\mathcal{FS}_{\delta}$,

$$\big|\Pi^{-1}_{b,\gamma_{\mathcal{S}}(\upsilon)}(\mathcal{D}^{(2)}\gamma_{\mathcal{S}}(\upsilon))-v_{\hat{1}}^2(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}b)\big|\leq C|v_{\hat{1}}|^{2+\frac{1}{p}};$$
$$\big|\Pi^{-1}_{b,\gamma_{\mathcal{S}}(\upsilon)}\big((\mathcal{D}^{(3)}\gamma_{\mathcal{S}}(\upsilon))-3x_{\hat{1}}(\mathcal{D}^{(2)}\gamma_{\mathcal{S}}(\upsilon))\big)-v_{\hat{1}}^3(\mathcal{D}^{(3)}_{\mathcal{T},\hat{1}}b)\big|\leq C|v_{\hat{1}}|^{3+\frac{1}{p}};$$

(3) OR $\hat{I}=\{\hat{1},\hat{2}\}$, $d_{\hat{0}}=0$, and for all $\upsilon=[b,v_{\hat{1}},v_{\hat{2}}]\in\mathcal{FS}$

$$\big|\Pi^{-1}_{b,\gamma_{\mathcal{S}}(\upsilon)}(\mathcal{D}^{(2)}\gamma_{\mathcal{S}}(\upsilon))-2\big(x_{\hat{1}}v_{\hat{1}}(\mathcal{D}_{\mathcal{T},\hat{1}}b)+x_{\hat{2}}v_{\hat{2}}(\mathcal{D}_{\mathcal{T},\hat{2}}b)\big)\big|\leq C|\upsilon|^{1+\frac{1}{p}};$$
$$\big|\Pi^{-1}_{b,\gamma_{\mathcal{S}}(\upsilon)}\big(2(\mathcal{D}^{(3)}\gamma_{\mathcal{S}}(\upsilon))-3(x_{\hat{1}}+x_{\hat{2}})(\mathcal{D}^{(2)}\gamma_{\mathcal{S}}(\upsilon))\big)-3\big(x_{\hat{1}}-x_{\hat{2}}\big)\big(v_{\hat{1}}^2(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}b)-v_{\hat{2}}^2(\mathcal{D}^{(2)}_{\mathcal{T},\hat{2}}b)\big)\big|\leq C|\upsilon|^{2+\frac{1}{p}}.$$

Proof: (1) The statement about the possible structures of $\mathcal{T}$ is easily seen from Theorem 2.8 and dimension count. The finiteness claim then also follows by dimension count. In Case (1), if $d_{\hat{0}}\geq3$, by Corollary 6.3, the images of $\mathcal{D}^{(2)}_{\mathcal{T}^*,\hat{0}}$ and $\mathcal{D}^{(3)}_{\mathcal{T}^*,\hat{0}}$ are transversal and thus linearly independent over the finite set $\mathcal{S}_{\mathcal{T}}(\mu)$. On the other hand, if $d_{\hat{0}}<3$, $\mathcal{S}_{\mathcal{T}}(\mu)=\emptyset$; see Subsection 4.5.
(2) The four inequalities in the lemma will be obtained by refining the proof of the analytic estimate of Theorem 2.8. We use the same notation. Combining equations (2.22), (2.23), (2.24), and (2.26), we obtain

$$\big(\mathcal{D}^{(m)}\tilde{\gamma}_{\mathcal{T}}(\upsilon)\big)=m\sum_{h\in\chi(\mathcal{T})}\sum_{k=1}^{k=m}\frac{a_{k,h}(\upsilon)}{k}\tilde{\upsilon}^k\big(\mathcal{D}^{(k)}_{\mathcal{T},h}b\big)-\frac{m}{2\pi i}\sum_{h\in\chi(\mathcal{T})}\int_{A_h^-(\upsilon)}\xi_{\upsilon}w^{m-1}dw,\tag{5.24}$$

where the integral is computed by using the same trivializations as before. This equality holds for any bubble type. If $\gamma_{\mathcal{T}}(\upsilon)\in\mathcal{S}_1(\mu)$ and $\mathcal{T}$ is as in (2) of the lemma, (5.24) with $m=1,2,3$ gives

$$0=v_{\hat{1}}\big(\mathcal{D}^{(1)}_{\mathcal{T},\hat{1}}b\big)-\frac{1}{2\pi i}\int_{|x_{\hat{1}}-w|=\epsilon}\xi_\upsilon dw; \tag{5.25}$$

$$\big(\mathcal{D}^{(2)}\tilde{\gamma}_{\mathcal{T}}(\upsilon)\big)=2x_{\hat{1}}v_{\hat{1}}\big(\mathcal{D}^{(1)}_{\mathcal{T},\hat{1}}b\big)+v_{\hat{1}}^2\big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}b\big)-\frac{1}{\pi i}\int_{|x_{\hat{1}}-w|=\epsilon}\xi_\upsilon w dw; \tag{5.26}$$

$$\big(\mathcal{D}^{(3)}\tilde{\gamma}_{\mathcal{T}}(\upsilon)\big)=3x_{\hat{1}}^2v_{\hat{1}}\big(\mathcal{D}^{(1)}_{\mathcal{T},\hat{1}}b\big)+3x_{\hat{1}}v_{\hat{1}}^2\big(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}b\big)+2v_{\hat{1}}^3\big(\mathcal{D}^{(3)}_{\mathcal{T},\hat{1}}b\big)-\frac{3}{2\pi i}\int_{|x_{\hat{1}}-w|=\epsilon}\xi_\upsilon w^2 dw. \tag{5.27}$$

where $\epsilon=4\delta(b_\upsilon)^{-1}|v_{\hat{1}}|$. Subtracting $2x_{\hat{1}}$ times the first equation from the second, we obtain

$$\big|(\mathcal{D}^{(2)}\tilde{\gamma}_{\mathcal{T}}(\upsilon))-v_{\hat{1}}^2(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}b)\big|\le C(b)|v_{\hat{1}}|^{2+\frac{1}{p}}. \tag{5.28}$$

Similarly, subtracting $3x_{\hat{1}}$ times (5.26) from and adding $3x_{\hat{1}}^2$ times (5.25) to (5.27), we obtain

$$\big|\big((\mathcal{D}^{(3)}\tilde{\gamma}_{\mathcal{T}}(\upsilon))-3x_{\hat{1}}(\mathcal{D}^{(2)}\tilde{\gamma}_{\mathcal{T}}(\upsilon))\big)-2v_{\hat{1}}^3(\mathcal{D}^{(3)}_{\mathcal{T},\hat{1}}b)\big|\le C(b)|v_{\hat{1}}|^{3+\frac{1}{p}}. \tag{5.29}$$

If $\upsilon\in\mathcal{FS}$ is sufficiently small, the claim in Case (2) follows from equations (5.28) and (5.29) along with Lemma 5.8 and our choice of $\tilde{\phi}_{\mathcal{S}}$. Note that if $\upsilon\in\mathcal{FS}$, we have to apply (5.28) and (5.29) with υ replaced by $\Phi^\mu_{\mathcal{T}}\varphi^\mu_{\mathcal{T}}\tilde{\phi}_{\mathcal{S}}\varphi_{\mathcal{S}}(\upsilon)$, where $\Phi^\mu_{\mathcal{T}}$ and $\varphi^\mu_{\mathcal{T}}$ are as in Subsection 3.9 of [Z1]. However, applying the bounds on $\varphi^\mu_{\mathcal{T}}$ and $\tilde{\phi}_{\mathcal{S}}$, we obtain the claimed estimates.
(3) In Case (3), we proceed similarly. The analog of equation (5.26) gives

$$\big|(\mathcal{D}^{(2)}\tilde{\gamma}_{\mathcal{T}}(\upsilon))-2\big(x_{\hat{1}}v_{\hat{1}}(\mathcal{D}_{\mathcal{T},\hat{1}}b)+x_{\hat{2}}v_{\hat{2}}(\mathcal{D}_{\mathcal{T},\hat{2}}b)\big)\big|\le C(b)|\upsilon|^{1+\frac{1}{p}}.$$

Subtracting $3(x_{\hat{1}}+x_{\hat{2}})$ times the analog of (5.26) from and adding $6x_{\hat{1}}x_{\hat{2}}$ times the analog of (5.25) to twice the analog of (5.27), we obtain

$$\Big|\big(2(\mathcal{D}^{(3)}\tilde{\gamma}_{\mathcal{T}}(\upsilon))-3(x_{\hat{1}}+x_{\hat{2}})(\mathcal{D}^{(2)}\tilde{\gamma}_{\mathcal{T}}(\upsilon))\big)-3\big(x_{\hat{1}}-x_{\hat{2}}\big)\big(v_{\hat{1}}^2(\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}b)-v_{\hat{2}}^2(\mathcal{D}^{(2)}_{\mathcal{T},\hat{2}}b)\big)\Big|\le C(b)|\upsilon|^{2+\frac{1}{p}}.$$

The estimates of Case 3 follow from the last two equations and Lemma 5.8. The finer bound on $\varphi_{\mathcal{FS}^\perp}$ of Lemma 5.8 is essential here.

5.5 The Numbers $n_1^{(2)}(\mu)$ and $n_1^{(3)}(\mu)$ in the $n=3$ Case

In this subsection, we express the numbers $n_1^{(2)}(\mu)$ and $n_1^{(3)}(\mu)$ in the $n=3$ case in terms of intersection numbers on the spaces $\bar{\mathcal{V}}_1(\mu)$, $\bar{\mathcal{V}}_2(\mu)$, $\bar{\mathcal{V}}_3(\mu)$.

Lemma 5.11 *If $n=3$, $n_1^{(2)}(\mu)=4\langle 2a+c_1(\mathcal{L}^*),[\bar{\mathcal{S}}_1(\mu)]\rangle-2|\mathcal{S}_2(\mu)|$.*

Proof: (1) We continue with the notation of the previous subsection. The number $n_1^{(2)}(\mu)$ is the number of zeros of the affine map in (4.31). As in the proof of Lemma 5.3, we can replace $s_\Sigma^{(2,-)}$ by $\tilde{s}_\Sigma^{(2,-)}$. Since the linear part of the new affine map does not vanish on $\Sigma\times\mathcal{S}_1(\mu)$ (see Subsection 4.4), by Lemma 3.14,

$$\begin{aligned} n_1^{(2)}(\mu)&=\sum_{k=0}^{k=2}\big\langle c_1^{2-k}(\tilde{T}\Sigma^*\otimes L^{*\otimes 2})c_k(\mathcal{O}),\big[\Sigma\times\bar{\mathcal{S}}_1(\mu)\big]\big\rangle-\mathcal{C}_{\Sigma\times\partial\bar{\mathcal{S}}_1}(\alpha^\perp)\\ &=4\big\langle 2a+c_1(L^*),\big[\bar{\mathcal{S}}_1(\mu)\big]\big\rangle-\mathcal{C}_{\Sigma\times\partial\bar{\mathcal{S}}_1}(\alpha^\perp), \end{aligned} \tag{5.30}$$

where $\mathcal{O}=\mathcal{H}_{\Sigma}^{-}\otimes\mathrm{ev}^*T\mathbb{P}^3$, $\partial\bar{\mathcal{S}}_1=\bar{\mathcal{S}}_1(\mu)-\mathcal{S}_1(\mu)$, and α is the linear part of the affine map in (4.31), with $s_{\Sigma}^{(2,-)}$ replaced by $\tilde{s}_{\Sigma}^{(2,-)}$.
(2) If $\mathcal{T}=(S^2,[N],I;j,\underline{d})<\mathcal{T}^*$ and $\mathcal{S}_{\mathcal{T}}(\mu)\neq\emptyset$, $\mathcal{T}$ must have one of the three forms given by Lemma 5.10. Since $\mathcal{D}^{(2)}$ does not vanish on $\mathcal{S}_{\mathcal{T}}(\mu)$ in Case (1) of Lemma 5.10, $\mathcal{C}_{\Sigma\times\mathcal{S}_{\mathcal{T}}(\mu)}(\alpha^{\perp})=0$ in this case. In Case (2), i.e. $\mathcal{T}=\mathcal{T}^*(l)$ for some $l\in[N]$, $\mathcal{D}_{\mathcal{T},\hat{1}}^{(2)}$ does not vanish over $\mathcal{S}_{\mathcal{T}}(\mu)$; see Subsection 4.5. Thus, by Corollaries 3.13, 3.6, and the first estimate of Lemma 5.10, $\mathcal{C}_{\Sigma\times\mathcal{S}_{\mathcal{T}}(\mu)}(\alpha^{\perp})$ is *twice* the number of Lemma 3.14 corresponding to

$$\bar{\mathcal{M}}=\Sigma\times\mathcal{S}_{\mathcal{T}}(\mu),\quad E_2=\mathcal{F}\mathcal{T}^{\otimes 2}\approx\mathbb{C},\quad \mathcal{O}_2=\tilde{T}\Sigma^*\otimes L^*\otimes\mathcal{O}^{\perp}\approx\tilde{T}\Sigma^*\otimes\mathcal{O}^{\perp},$$

and $\alpha_2\in\Gamma(\bar{\mathcal{M}};E_2^*\otimes\mathcal{O}_2)$ that has full rank on every fiber. It follows that

$$\mathcal{C}_{\Sigma\times\mathcal{S}_{\mathcal{T}}(\mu)}(\alpha^{\perp})=2\big\langle c_1(\mathcal{O}_2)-c_1(E_2),\big[\Sigma\times\mathcal{S}_{\mathcal{T}}(\mu)\big]\big\rangle=4|\mathcal{S}_{\mathcal{T}}(\mu)|. \tag{5.31}$$

(3) Suppose $\mathcal{T}$ is as in Case (3) of Lemma 5.10. Since $x_{\hat{1}}\neq x_{\hat{2}}$, $\mathcal{D}_{\mathcal{T},\hat{1}}$ and $\mathcal{D}_{\mathcal{T},\hat{2}}$ do not vanish on $\mathcal{S}_{\mathcal{T}}(\mu)=\mathcal{U}_{\mathcal{T}}(\mu)\cap\mathcal{S}_2(\mu)$ (see Subsection 4.6), and $\mathcal{D}_{\mathcal{T},\hat{1}}+\mathcal{D}_{\mathcal{T},\hat{2}}$ vanishes on $\mathcal{F}\mathcal{S}$, $x_{\hat{1}}\mathcal{D}_{\mathcal{T},\hat{1}}+x_{\hat{2}}\mathcal{D}_{\mathcal{T},\hat{2}}$ does not vanish on $\mathcal{F}\mathcal{S}$. Thus, the third estimate of Lemma 5.10, Corollary 3.13, and Lemma 3.14,

$$\mathcal{C}_{\Sigma\times\mathcal{S}_{\mathcal{T}}(\mu)}(\alpha^{\perp})=\big\langle c_1(\mathcal{O}_2)-c_1(E),\big[\Sigma\times\mathcal{S}_{\mathcal{T}}(\mu)\big]\big\rangle=2|\mathcal{S}_{\mathcal{T}}(\mu)|. \tag{5.32}$$

Summing up equations (5.31) and (5.32) over all appropriate bubble types $\mathcal{T}<\mathcal{T}^*$ and substituting the result into (5.30), we obtain the claim.

Lemma 5.12 *If $n=3$, $n_1^{(3)}(\mu)=\langle 4a+5c_1(\mathcal{L}^*),[\bar{\mathcal{S}}_1(\mu)]\rangle-3|\mathcal{S}_2(\mu)|$.*

Proof: (1) We continue with the notation of Lemma 5.11. The number $n_1^{(3)}(\mu)$ is the number of zeros of the affine map in (4.33). Let

$$E=L^{\otimes 2}\oplus L^{\otimes 3}\longrightarrow\bar{\mathcal{S}}_1(\mu).$$

Since the linear part α of the affine map has full rank on $\mathcal{S}_1(\mu)$ (see Subsection 4.5),

$$\begin{aligned} n_1^{(3)}(\mu)&=\sum_{k=0}^{k=2}\big\langle\lambda_E^{2-k}c_k(\mathcal{O}),\big[\mathbb{P}E\big]\big\rangle-\mathcal{C}_{\mathbb{P}E|\partial\bar{\mathcal{S}}_1}(\alpha_E^{\perp})\\ &=\big\langle 4a+5c_1(L^*),\big[\bar{\mathcal{S}}_1(\mu)\big]\big\rangle-\mathcal{C}_{\mathbb{P}E|\partial\bar{\mathcal{S}}_1}(\alpha_E^{\perp}), \end{aligned} \tag{5.33}$$

where $\mathcal{O}=\mathrm{ev}^*T\mathbb{P}^3$.
(2) As in the proof of Lemma 5.11, $\mathcal{C}_{\mathbb{P}E|\mathcal{S}_{\mathcal{T}}(\mu)}(\alpha_E^{\perp})=0$ for bubble types $\mathcal{T}$ of Case (1) of Lemma 5.10. Suppose $\mathcal{T}=\mathcal{T}^*(l)$ for some $l\in[N]$, i.e. we are in Case (2) of Lemma 5.10. The normal bundle of $\mathbb{P}E|_{\mathcal{S}_{\mathcal{T}}(\mu)}$ in $\mathbb{P}E$ is $\pi_E^*\mathcal{F}\mathcal{T}\approx\mathbb{C}$. By the first two estimates of Lemma 5.10, with appropriate identifications,

$$\big|\alpha_E^{\perp}(\gamma_{\mathcal{S}}(b,v_{\hat{1}}))-\tilde{\alpha}_{\mathcal{T}}(b,v_{\hat{1}})\big|\leq C|v_{\hat{1}}|^{2+\frac{1}{p}}\qquad\forall(b,v_{\hat{1}})\in\mathcal{F}\mathcal{T}_{\delta},$$

for some $\tilde{\alpha}_{\mathcal{T}}\in\Gamma(\mathbb{P}E|_{\mathcal{S}_{\mathcal{T}}(\mu)};\mathcal{F}\mathcal{T}^{*\otimes 2}\otimes\gamma_E^*\otimes\mathcal{O}^{\perp})$ which vanishes only on

$$\mathcal{Z}_{\mathcal{T}}\equiv\big\{(b,[v,w])\in\mathbb{P}E|_{\mathcal{S}_{\mathcal{T}}(\mu)}:v-3x_{\hat{1}}w=0\big\}.$$

Thus, by Corollaries 3.13 and 3.6 and Lemma 3.14,

$$\begin{aligned}\mathcal{C}_{\mathbb{P}E|(\mathcal{S}_\mathcal{T}(\mu)-\mathcal{Z}_\mathcal{T})}(\alpha_E^\perp) &= 2\big(\big\langle c_1(\gamma_E^*\otimes\mathcal{O}^\perp)-c_1(\mathcal{F}\mathcal{T}),\big[\mathbb{P}E|_{\mathcal{S}_\mathcal{T}(\mu)}\big]\big\rangle - \mathcal{C}_{\mathcal{Z}_\mathcal{T}}(\tilde{\alpha}_\mathcal{T}^\perp)\big)\\ &= 4|\mathcal{S}_\mathcal{T}(\mu)| - 2\mathcal{C}_{\mathcal{Z}_\mathcal{T}}(\tilde{\alpha}_\mathcal{T}^\perp).\end{aligned}$$

By the first two estimates of Lemma 5.10, $\mathcal{C}_{\mathcal{Z}_\mathcal{T}}(\tilde{\alpha}_\mathcal{T}^\perp)=|\mathcal{Z}_\mathcal{T}|=|\mathcal{S}_\mathcal{T}(\mu)|$. Since the images of $\mathcal{D}^{(2)}_{\mathcal{T},\hat{1}}$ and $\mathcal{D}^{(3)}_{\mathcal{T},\hat{1}}$ are linearly independent in every fiber of $\mathrm{ev}^*T\mathbb{P}^3$ over $\mathcal{S}_\mathcal{T}(\mu)$, by the first two estimates of Lemma 5.10 and Corollary 3.13, $\mathcal{C}_{\mathcal{Z}_\mathcal{T}}(\alpha_E^\perp)=3|\mathcal{Z}_\mathcal{T}|$. Thus,

$$\mathcal{C}_{\mathbb{P}E|\mathcal{S}_\mathcal{T}(\mu)}(\alpha_E^\perp) = \big(4|\mathcal{S}_\mathcal{T}(\mu)| - 2|\mathcal{S}_\mathcal{T}(\mu)|\big) + 3|\mathcal{S}_\mathcal{T}(\mu)| = 5|\mathcal{S}_\mathcal{T}(\mu)|. \tag{5.34}$$

(3) Suppose $\mathcal{T}$ is as in Case (3). By the last two estimates of Lemma 5.10, with appropriate identifications,

$$\big|\alpha_E^\perp(\gamma_\mathcal{S}(b,v)) - \tilde{\alpha}_\mathcal{T}(b,v)\big| \le C|v_{\hat{1}}|^{1+\frac{1}{p}} \qquad \forall (b,v)\in F\mathcal{S}_\delta,$$

for some $\tilde{\alpha}_\mathcal{T}\in\Gamma(\mathbb{P}E|_{\mathcal{S}_\mathcal{T}(\mu)};F\mathcal{S}^*\otimes\gamma_E^*\otimes\mathcal{O}^\perp)$ which vanishes only on

$$\mathcal{Z}_\mathcal{T} \equiv \{(b,[v,w])\in\mathbb{P}E|_{\mathcal{S}_\mathcal{T}(\mu)}: 2v-3(x_{\hat{1}}+x_{\hat{2}})w=0\}.$$

Thus, by Corollary 3.13 and Lemma 3.14,

$$\begin{aligned}\mathcal{C}_{\mathbb{P}E|(\mathcal{S}_\mathcal{T}(\mu)-\mathcal{Z}_\mathcal{T})}(\alpha_E^\perp) &= \big\langle c_1(\gamma_E^*\otimes\mathcal{O}^\perp)-c_1(F\mathcal{S}),\big[\mathbb{P}E|_{\mathcal{S}_\mathcal{T}(\mu)}\big]\big\rangle - \mathcal{C}_{\mathcal{Z}_\mathcal{T}}(\tilde{\alpha}_\mathcal{T}^\perp)\\ &= 2|\mathcal{S}_\mathcal{T}(\mu)| - \mathcal{C}_{\mathcal{Z}_\mathcal{T}}(\tilde{\alpha}_\mathcal{T}^\perp).\end{aligned}$$

By the last two estimates of Lemma 5.10, $\mathcal{C}_{\mathcal{Z}_\mathcal{T}}(\tilde{\alpha}_\mathcal{T}^\perp)=|\mathcal{Z}_\mathcal{T}|$. Finally, by Lemma 5.10 and Corollary 3.13, $\mathcal{C}_{\mathcal{Z}_\mathcal{T}}(\alpha_E^\perp)=2|\mathcal{Z}_\mathcal{T}|$. Thus,

$$\mathcal{C}_{\mathbb{P}E|\mathcal{S}_\mathcal{T}(\mu)}(\alpha^\perp) = \big(2|\mathcal{S}_\mathcal{T}(\mu)| - |\mathcal{S}_\mathcal{T}(\mu)|\big) + 2|\mathcal{S}_\mathcal{T}(\mu)| = 3|\mathcal{S}_\mathcal{T}(\mu)|. \tag{5.35}$$

The claim follows by summing up equations (5.34) and (5.35) over the appropriate equivalence classes of bubble types $\mathcal{T}<\mathcal{T}^*$ and plugging the result back into (5.33).

The next step is to relate $\langle a,[\bar{\mathcal{S}}_1(\mu)]\rangle$ and $\langle c_1(\mathcal{L}^*),[\bar{\mathcal{S}}_1(\mu)]\rangle$ to intersection numbers on the spaces $\bar{\mathcal{V}}_1(\mu)$, $\bar{\mathcal{V}}_2(\mu)$, and $\bar{\mathcal{V}}_3(\mu)$. The approach is similar to the proof of Lemma 5.4, but first we need to interpret $\langle a,[\bar{\mathcal{S}}_1(\mu)]\rangle$ and $\langle c_1(\mathcal{L}^*),[\bar{\mathcal{S}}_1(\mu)]\rangle$ as the zero sets of some bundle sections. In our case, the spaces $\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$ and $\bar{\mathcal{U}}_{\mathcal{T}^*(l)}(\mu)$ for all $l\in[N]$ are topological manifolds (not just orbifolds). Thus, $c_1(\mathcal{L}^*)$ represents the first chern class of some line bundle $\mathcal{L}^*\longrightarrow\bar{\mathcal{U}}_\mathcal{T}(\mu)$. It is well-known in algebraic geometry that a slightly weaker statement is in fact true for any choice of constraints, and

$$\mathcal{L}^* = L^*\otimes\mathcal{O}\Big(-\sum_{l\in[N]}\bar{\mathcal{U}}_{\mathcal{T}^*(l)}\Big).$$

Let $V_1=\mathrm{ev}^*\mathcal{O}(1)\longrightarrow\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$, $V_2=\mathcal{L}^*\longrightarrow\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$, and $\eta_i=c_1(V_i)$. Choose sections $s_i\in\Gamma(\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu);V_i)$ such that s_i is smooth and transversal to the zero set on all smooth strata $\mathcal{U}_\mathcal{T}(\mu)\subset\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$ and on $\mathcal{S}_\mathcal{T}(\mu)\subset\bar{\mathcal{S}}_1(\mu)$. The second condition implies that s_i does not vanish on the finite set $\partial\bar{\mathcal{S}}_1$.

Lemma 5.13 *If $d\ge1$, μ is a tuple of p points and q lines in general position in $\mathbb{P}^3$ with $2p+q=4d-3$,*

$$\begin{aligned}\langle a,[\bar{\mathcal{S}}_1(\mu)]\rangle &= \big\langle 6a^3c_1(\mathcal{L}^*)+4a^2c_1^2(\mathcal{L}^*)+ac_1^3(\mathcal{L}^*),[\bar{\mathcal{V}}_1(\mu)]\big\rangle - \big\langle 4a^2+a(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)),[\bar{\mathcal{V}}_2(\mu)]\big\rangle;\\ \langle c_1(\mathcal{L}^*),[\bar{\mathcal{S}}_1(\mu)]\rangle &= \big\langle 4a^3c_1(\mathcal{L}^*)+6a^2c_1^2(\mathcal{L}^*)+4ac_1^3(\mathcal{L}^*)+c_1^4(\mathcal{L}^*),[\bar{\mathcal{V}}_1(\mu)]\big\rangle - \tau_3(\mu).\end{aligned}$$

Proof: (1) Similarly to the proof of Lemma 5.4,

$$\langle \eta_i, [\bar{\mathcal{S}}_1(\mu)] \rangle = \big\langle \eta_i c_3\big(L^* \otimes \mathrm{ev}^* T\mathbb{P}^3\big), [\bar{\mathcal{V}}_1(\mu)] \big\rangle - \mathcal{C}_{\partial \bar{\mathcal{V}}_1(\mu)}(\mathcal{D} \oplus s_i). \tag{5.36}$$

Suppose $\mathcal{T} = \big(S^2, [N], I; j, \underline{d}\big) < \mathcal{T}^*$ is a bubble type such that $\mathcal{U}_{\mathcal{T}}(\mu) \neq \emptyset$. If $d_{\hat{0}} \neq 0$, by our assumptions on s_i, $\mathcal{D} \oplus s_i$ does not vanish on $\mathcal{U}_{\mathcal{T}}(\mu)$. Thus, for the purposes of computing $\mathcal{C}_{\partial \bar{\mathcal{V}}_1(\mu)}(\mathcal{D} \oplus s_i)$, we can assume $d_{\hat{0}} = 0$.
(2) In order to compute the numbers $\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)}(\mathcal{D} \oplus s_i)$, we slightly modify the approach of Subsection 3.2, since we have a great amount of flexibility in choosing the section s_i. We consider a family $\psi_t = (t\nu + \mathcal{D}, s_i)$ of sections of $L^* \otimes \mathrm{ev}^* T\mathbb{P}^3 \oplus V_i$, with ν generic with respect to $\mathcal{D}$. Let $\pi\colon \mathcal{F}\mathcal{T} \longrightarrow \mathcal{U}_{\mathcal{T}}(\mu)$ be the bundle projection map and fix an identification of $\gamma_{\mathcal{T}}^{\mu *} V_i \longrightarrow \mathcal{F}\mathcal{T}_\delta$ with $\pi^* V_i$. It can be assumed that the section s_i has been chosen so that $\gamma_{\mathcal{T}}^{\mu *} s_i \in \Gamma(\mathcal{F}\mathcal{T}_\delta; \pi^* V_i)$ is constant on the fibers of $\mathcal{F}\mathcal{T}_\delta$ over an open subset $K_{\mathcal{T}}$ of $\mathcal{U}_{\mathcal{T}}(\mu)$ that contains all of the finitely many zeros of the affine map

$$\mathcal{F}\mathcal{T} \longrightarrow L^* \otimes \mathrm{ev}^* T\mathbb{P}^3 \oplus V_i, \quad (b, \upsilon) \longrightarrow \big(\bar{\nu}_b + \alpha_{\mathcal{T}}(\rho_{\mathcal{T}}(\upsilon)), s_i(b)\big),$$

over $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$, where $\alpha_{\mathcal{T}}$ and $\rho_{\mathcal{T}}$ are as in (2.21). Note that by our assumptions on s_i, the images of $\{\mathcal{D}_{\mathcal{T},h} : h \in \chi(\mathcal{T})\}$ are linearly independent in every fiber of $\mathrm{ev}^* T\mathbb{P}^n$ over $s_i^{-1}(0)$. Thus by Theorem 2.8, Corollary 3.13, and Lemma 3.2, $\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)}(\mathcal{D} \oplus s_i) = 0$ if $H_{\hat{0}}\mathcal{T} \neq \hat{I}$. Furthermore, if $H_{\hat{0}}\mathcal{T} = \hat{I}$, $\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)}(\mathcal{D} \oplus s_i)$ is the number of zeros of the affine map

$$\mathcal{F}\mathcal{T} \longrightarrow L^* \otimes \mathrm{ev}^* T\mathbb{P}^3, \quad \upsilon = (b, \upsilon) \longrightarrow \bar{\nu}_{\mathcal{T},b} + \alpha_{\mathcal{T}}(\upsilon), \tag{5.37}$$

over $s_i^{-1}(0) \cap \bar{\mathcal{U}}_{\mathcal{T}}(\mu)$, where $\bar{\nu}_{\mathcal{T}} \in \Gamma(\bar{\mathcal{U}}_{\mathcal{T}}(\mu); L^* \otimes \mathrm{ev}^* T\mathbb{P}^3)$ is a generic section. Thus, by Lemma 3.14,

$$\begin{aligned}
\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)}(\mathcal{D} \oplus s_i) &= \sum_{k=0}^{k=2} \big\langle \lambda_{\mathcal{F}\mathcal{T}}^{2-k} c_k(L^* \otimes \mathrm{ev}^* T\mathbb{P}^3), \big[\mathbb{P}\mathcal{F}\mathcal{T}\big|_{s_i^{-1}(0) \cap \bar{\mathcal{U}}_{\mathcal{T}}(\mu)}\big] \big\rangle - \mathcal{C}_{\mathbb{P}\mathcal{F}\mathcal{T}|s_i^{-1}(0) \cap \partial \bar{\mathcal{U}}_{\mathcal{T}}}(\alpha_{\mathcal{F}\mathcal{T}}^{\perp}), \\
&= \sum_{k=0}^{k=2} \big\langle \lambda_{\mathcal{F}\mathcal{T}}^{2-k} c_k(L^* \otimes \mathrm{ev}^* T\mathbb{P}^3) \eta_i, \big[\mathbb{P}\mathcal{F}\mathcal{T}\big] \big\rangle - \mathcal{C}_{\mathbb{P}\mathcal{F}\mathcal{T}|s_i^{-1}(0) \cap \partial \bar{\mathcal{U}}_{\mathcal{T}}}(\alpha_{\mathcal{F}\mathcal{T}}^{\perp}),
\end{aligned} \tag{5.38}$$

where $\partial \bar{\mathcal{U}}_{\mathcal{T}} = \bar{\mathcal{U}}_{\mathcal{T}}(\mu) - \mathcal{U}_{\mathcal{T}}(\mu)$ and $\alpha_{\mathcal{F}\mathcal{T}} \in \Gamma(\mathbb{P}\mathcal{F}\mathcal{T}; \gamma_{\mathcal{F}\mathcal{T}}^* \otimes L^* \otimes \mathrm{ev}^* T\mathbb{P}^3)$ is the section induced by $\alpha_{\mathcal{T}}$.
(3) Suppose $i = 1$, i.e. $\eta_i = a$. If $\mathcal{T} = \mathcal{T}^*(l)$ and $\mathcal{T}' = \big(S^2, [N], I'; j, \underline{d}'\big) < \mathcal{T}$ is a bubble type such that $s_1^{-1}(0) \cap \mathcal{U}_{\mathcal{T}'}(\mu) \cap \alpha_{\mathcal{T}}^{-1}(0) \neq \emptyset$, $\mathcal{T}'$ must have the form

$$|I' - I| = 2, \quad H_{\hat{1}}\mathcal{T}' = \{\hat{2}, \hat{3}\}, \quad d'_{\hat{1}} = 0,\ d'_{\hat{2}} \neq 0,\ d'_{\hat{3}} \neq 0.$$

By Theorem 2.8 applied to $\bar{\mathcal{T}}' < \bar{\mathcal{T}}$, and Corollary 3.13,

$$\mathcal{C}_{s_1^{-1}(0) \cap \mathcal{U}_{\mathcal{T}'}(\mu)}(\alpha_{\mathcal{F}\mathcal{T}}^{\perp}) = \big|\mathcal{U}_{\mathcal{T}'}(\mu) \cap s_1^{-1}(0)\big| = \langle a, [\bar{\mathcal{U}}_{\mathcal{T}'}(\mu)] \rangle.$$

Thus, summing up equation (5.38) over $\mathcal{T} = \mathcal{T}^*(l)$ with $l \in [N]$, we obtain

$$\sum_{l \in [N]} \mathcal{C}_{\mathcal{U}_{\mathcal{T}^*(l)}(\mu)}(\mathcal{D} \oplus s_1) = \sum_{l \in [N]} \big\langle 6a^3 + 4a^2 c_1(L_{\hat{1}}^* \mathcal{T}^*(l)) + a c_1^2(L_{\hat{1}}^* \mathcal{T}^*(l)), \big[\bar{\mathcal{U}}_{\mathcal{T}^*(l)}(\mu)\big] \big\rangle - \tau_2^{(1)}(\mu), \tag{5.39}$$

where $\tau_2^{(1)}(\mu)$ is the number of two-component connected degree-d curves passing through the constraints with the node at the intersection of one of the constraints with a generic plane in $\mathbb{P}^3$. If $|H_{\hat{0}}\mathcal{T}| = |\hat{I}| = 2$, $|M_{\hat{0}}\mathcal{T}| = 0$, and $\mathcal{T}'$ is as above, up to equivalence of bubble types,

$$|I' - I| = 1, \quad \iota'_{\hat{3}} = \hat{1}, \quad d'_{\hat{1}} = 0,\ d'_{\hat{2}} \neq 0,\ d'_{\hat{3}} \neq 0,$$

i.e. $\bar{\mathcal{T}}'=\bar{\mathcal{T}}(l)$ for some $l\in[N]$. By Theorem 2.8 applied to $\bar{\mathcal{T}}'<\bar{\mathcal{T}}$ and Corollary 3.13,

$$\mathcal{C}_{\mathbb{P}\mathcal{F}\mathcal{T}|s_1^{-1}(0)\cap\mathcal{U}_{\mathcal{T}'}(\mu)}(\alpha^{\perp}_{\mathcal{F}\mathcal{T}}) = \big|\mathcal{U}_{\mathcal{T}'}(\mu)\cap s_1^{-1}(0)\big| = \langle a,[\bar{\mathcal{U}}_{\mathcal{T}'}(\mu)]\rangle.$$

Thus, summing up equation (5.38) over $\mathcal{T}$ with $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2$ and $|M_{\hat{0}}\mathcal{T}|=0$, we obtain

$$\begin{split}\sum_{|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2,|M_{\hat{0}}\mathcal{T}|=0}\big\langle a\big(4a+c_1(L_{\hat{1}}^*\mathcal{T})+c_1(L_{\hat{2}}^*\mathcal{T})\big),\big[\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\big]\big\rangle-2\tau_2^{(1)}(\mu)\\ =\big\langle 4a^2+a(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)),\big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle.\end{split}\tag{5.40}$$

If $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2$ and $|M_{\hat{0}}\mathcal{T}|=1$, $\alpha_{\mathcal{T}}$ has full rank on all of $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$. Thus, by Corollary 3.13,

$$\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)}(\mathcal{D}\oplus s_i) = \big\langle c_1(L^*\otimes\mathrm{ev}^*T\mathbb{P}^3)-c_1(\mathcal{F}\mathcal{T}),[\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\cap s_1^{-1}(0)]\big\rangle = |\mathcal{U}_{\bar{\mathcal{T}}}(\mu)|.$$

Here we used $\mathcal{F}\mathcal{T}=L^*\otimes\big(L_{\hat{1}}\oplus L_{\hat{2}}\big)\approx L^*\oplus L^*$ and Corollary 5.22. Thus, summing up equation (5.38) over $\mathcal{T}$ with $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2$ and $|M_{\hat{0}}\mathcal{T}|=1$ gives

$$\sum_{|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2,|M_{\hat{0}}\mathcal{T}|=1}\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)}(\mathcal{D}\oplus s_i) = \tau_2^{(1)}(\mu).\tag{5.41}$$

Finally, if $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=3$, $\eta_1|_{\bar{\mathcal{U}}'_{\mathcal{T}}(\mu)}=0$. The first claim follows by plugging the sum of equations (5.39)-(5.41) into (5.36). See also equations (5.3) and (5.4).
(4) Suppose $\eta_i=c_1(\mathcal{L}^*)$. We continue as in (3) above. If $\mathcal{T}=\mathcal{T}^*(l)$, $\alpha_{\mathcal{T}}$ does not vanish anywhere on $s_2^{-1}(0)\cap\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$. Thus, by Corollary 3.13,

$$\begin{split}\sum_{l\in[N]}\mathcal{C}_{\mathcal{U}_{\mathcal{T}^*(l)}(\mu)}(\mathcal{D}\oplus s_2) &=\sum_{l\in[N]}\big\langle c(L^*\otimes\mathrm{ev}^*T\mathbb{P}^3)c(L_{\hat{1}}\mathcal{T})^{-1},\big[\bar{\mathcal{U}}_{\mathcal{T}^*(l)}(\mu)\cap s_2^{-1}(0)\big]\big\rangle\\ &=\sum_{l\in[N]}\big\langle c_1(\mathcal{L}^*)\big(6a^2+4ac_1(L_{\hat{1}}^*\mathcal{T}^*(l))+c_1^2(L_{\hat{1}}^*\mathcal{T}^*(l))\big),\big[\bar{\mathcal{U}}_{\mathcal{T}^*(l)}(\mu)\big]\big\rangle.\end{split}\tag{5.42}$$

If $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=3$, $\alpha_{\mathcal{T}}$ again does not vanish anywhere on $s_2^{-1}(0)\cap\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$, and thus

$$\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)}(\mathcal{D}\oplus s_2) = \big|\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\cap s_2^{-1}(0)\big| = \big\langle c_1(\mathcal{L}^*),[\bar{\mathcal{U}}_{\mathcal{T}}(\mu)]\big\rangle = \big|\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)\big|.\tag{5.43}$$

Here we used Corollary 5.22 again. Note that if $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2$, $\eta_2|_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)}=0$. This is immediate in the case $|M_{\hat{0}}\mathcal{T}|=0$ and follows from Corollary 5.22 and (5.3) in the case $|M_{\hat{0}}\mathcal{T}|=1$. The second claim of the lemma is obtained by summing (5.43) over all equivalence classes of bubble types $\mathcal{T}<\mathcal{T}^*$ with $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=3$, and plugging the result along with (5.42) into (5.36). Note that

$$a^3\big|_{\bar{\mathcal{U}}_{\mathcal{T}^*(l)}(\mu)}=0\quad\forall\, l\in[N]\Longrightarrow\big\langle 4a^3c_1(L^*),[\bar{\mathcal{V}}_1(\mu)]\big\rangle=\big\langle 4a^3c_1(\mathcal{L}^*),[\bar{\mathcal{V}}_1(\mu)]\big\rangle.$$

5.6 The Number $n_1^{(1)}(\mu)$ in the $n=3$ Case

We finally compute the remaining number $n_1^{(1)}(\mu)$. The computation parallels the proof of Lemma 5.6.

Lemma 5.14 *If $n=3$,*

$$n_1^{(1)}(\mu)=4\big\langle 10a^3c_1(\mathcal{L}^*)+3a^2c_1^2(\mathcal{L}^*),\big[\bar{\mathcal{V}}_1(\mu)\big]\big\rangle-12\tau_2^{(2)}(\mu),$$

where $\tau_2^{(2)}(\mu)$ denotes the number of two-component connected degree-d curves that pass through the constraints and with the node on a generic line in $\mathbb{P}^3$.

Proof: (1) We use the same notation as in the proof of Lemma 5.6. By equation (4.30) and Lemma 3.14,

$$\begin{aligned} n_1^{(1)}(\mu)&=\sum_{k=0}^{k=5}\big\langle c_k(\mathcal{O})c_1^{5-k}(T^*\Sigma\otimes L^*),\big[\Sigma\times\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)\big]\big\rangle-\mathcal{C}_{\Sigma\times\mathcal{D}^{-1}(0)}(\alpha^\perp),\\ &=2\big\langle 112a^3c_1(L^*)+84a^2c_1^2(L^*)+32ac_1^3(L^*)+5c_1^4(L^*),\big[\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)\big]\big\rangle-\mathcal{C}_{\Sigma\times\mathcal{D}^{-1}(0)}(\alpha^\perp), \end{aligned}\tag{5.44}$$

where $\mathcal{O}=\mathcal{H}_\Sigma^{0,1}\otimes\mathrm{ev}^*T\mathbb{P}^3$ and $\alpha\in\Gamma(\Sigma\times\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu);T^*\Sigma\otimes L^*\otimes\mathcal{O})$ is the linear part of the affine map $\psi_1^{(1)}$ of (4.30). Let $\mathcal{O}_2=T^*\Sigma\otimes L^*\otimes\mathcal{O}^\perp$.
(2) Similarly to (2) of the proof of Lemma 5.4,

$$\begin{aligned} \mathcal{C}_{\Sigma\times\mathcal{S}_1(\mu)}(\alpha^\perp)&=\big\langle e\big(T^*\Sigma\otimes L^*\otimes(\mathcal{H}_\Sigma^-\otimes\mathrm{ev}^*T\mathbb{P}^3/\mathbb{C})\big),\big[\Sigma\times\bar{\mathcal{S}}_1(\mu)\big]\big\rangle\\ &=2\big\langle 12a+5c_1(L^*),[\bar{\mathcal{S}}_1(\mu)]\big\rangle. \end{aligned}\tag{5.45}$$

Suppose $\mathcal{T}=(S^2,[N],I;j,\underline{d})<\mathcal{T}^*$ is a bubble type such that $\mathcal{S}_\mathcal{T}(\mu)\neq\emptyset$. By Lemma 5.10, there are three possibilities for the structure of $\mathcal{T}$, but $\mathcal{C}_{\Sigma\times\mathcal{D}^{-1}(0)}(\alpha^\perp)=0$ in all three cases. This claim follows from Corollaries 5.9 and 3.13 and Lemma 3.2.
(3) As before, if $\mathcal{T}<\mathcal{T}^*$ and $\mathcal{C}_{\Sigma\times(\mathcal{U}_\mathcal{T}(\mu)-\mathcal{S}_\mathcal{T}(\mu))}(\alpha^\perp)\neq0$, $d_{\hat{0}}=0$ and $H_{\hat{0}}\mathcal{T}=\hat{I}$. In such a case,

$$\mathcal{C}_{\Sigma\times(\mathcal{U}_\mathcal{T}(\mu)-\mathcal{S}_\mathcal{T}(\mu))}(\alpha^\perp)=\sum_{k=0}^{k=4}\big\langle\lambda_{\mathcal{F}\mathcal{T}}^{4-k}c_k(\mathcal{O}_2),\big[\mathbb{P}\mathcal{F}\mathcal{T}\big]\big\rangle-\mathcal{C}_{\tilde{\alpha}_{\mathcal{F}\mathcal{T}}^{-1}(0)}(\tilde{\alpha}_{\mathcal{F}\mathcal{T}}^\perp),\tag{5.46}$$

where $\tilde{\alpha}_{\mathcal{F}\mathcal{T}}\in\Gamma(\mathbb{P}\mathcal{F}\mathcal{T};\gamma_{\mathcal{F}\mathcal{T}}^*\otimes\mathcal{O}_2)$ is the section induced by the section

$$\pi^\perp\circ(\alpha_\mathcal{T}\circ s_\Sigma)\in\Gamma\big(\Sigma\times\bar{\mathcal{U}}_\mathcal{T}(\mu);\mathcal{F}\mathcal{T}^*\otimes\mathcal{O}_2\big).$$

(4) If $\mathcal{T}=\mathcal{T}^*(l)$ for some $l\in[N]$, $F\mathcal{T}\approx L_{\hat{1}}\mathcal{T}$ over $\bar{\mathcal{U}}_\mathcal{T}(\mu)$, and $\alpha_\mathcal{T}^{-1}(0)=\Sigma\times\mathcal{D}_{\mathcal{T},\hat{1}}^{-1}(0)$. Thus, by (5.46),

$$\begin{aligned} \mathcal{C}_{\Sigma\times(\mathcal{U}_\mathcal{T}(\mu)-\mathcal{S}_\mathcal{T}(\mu))}(\alpha^\perp)=2\big\langle 112a^3+84a^2c_1(L_{\hat{1}}^*\mathcal{T})+32ac_1^2(L_{\hat{1}}^*\mathcal{T})+5c_1^3(L_{\hat{1}}^*\mathcal{T}),\big[\bar{\mathcal{U}}_\mathcal{T}(\mu)\big]\big\rangle\\ -\mathcal{C}_{\Sigma\times\mathcal{D}_{\mathcal{T},\hat{1}}^{-1}(0)}(\tilde{\alpha}_{\mathcal{F}\mathcal{T}}^\perp). \end{aligned}$$

By Corollaries 6.3 and 3.13,

$$\mathcal{C}_{\Sigma\times\mathcal{S}_\mathcal{T}(\mu)}(\tilde{\alpha}_{\mathcal{F}\mathcal{T}}^\perp)=\big\langle c(\mathcal{F}\mathcal{T}^*\otimes\mathcal{O}_2^\perp)c(L_{\mathcal{T},\hat{1}}^*\otimes\mathrm{ev}^*T\mathbb{P}^3)^{-1},\big[\Sigma\times\mathcal{S}_\mathcal{T}(\mu)\big]\big\rangle=10\big|\mathcal{S}_\mathcal{T}(\mu)\big|.$$

On the other hand, if $\mathcal{T}'=(S^2,[N],I';j',\underline{d}')<\mathcal{T}$, we apply Theorem 2.8 to $\bar{\mathcal{T}}'<\bar{\mathcal{T}}$. Then for the same reason as before, $\mathcal{C}_{\Sigma\times\mathcal{U}_{\mathcal{T}'}}(\tilde{\alpha}_{\mathcal{F}\mathcal{T}}^\perp)=0$ unless $d_{\hat{1}}=0$ and $H_{\hat{1}}\mathcal{T}'=I'-I$, i.e.

$$\hat{I}'=\{\hat{1},\hat{2},\hat{3}\},\quad \iota'_{\hat{2}}=\hat{1},\ \iota'_{\hat{3}}=\hat{1},\quad d'_{\hat{1}}=0,\ d'_{\hat{2}}\neq0,\ d'_{\hat{3}}\neq0.$$

In such a case, with $E = L_{\hat{2}}\mathcal{T}' \oplus L_{\hat{3}}\mathcal{T}'$, by Corollary 3.13,

$$\begin{aligned}\mathcal{C}_{\Sigma\times\mathcal{U}_{\mathcal{T}'}(\mu)}(\tilde{\alpha}^{\perp}_{\mathcal{F}\mathcal{T}}) &= \big\langle c(F\mathcal{T}^*\otimes\mathcal{O}_2)c(E)^{-1}, \big[\Sigma\times\bar{\mathcal{U}}_{\mathcal{T}'}(\mu)\big]\big\rangle \\ &= \big\langle 32a + 5(c_1(L^*_{\hat{2}}\mathcal{T}')+c_1(L^*_{\hat{3}}\mathcal{T}')), \big[\bar{\mathcal{U}}_{\mathcal{T}'}(\mu)\big]\big\rangle.\end{aligned}$$

Summing equation (5.46) over $\mathcal{T}=\mathcal{T}^*(l)$, we thus obtain

$$\begin{aligned}\sum_{l\in[N]}\mathcal{C}_{\Sigma\times(\mathcal{U}_{\mathcal{T}^*(l)}(\mu)-\mathcal{S}_{\mathcal{T}^*(l)}(\mu))}(\alpha^{\perp}) &= -64\tau_2^{(1)}(\mu) - 10\big\langle c_1(L_1^*)+c_1(L_2^*), \big[\bar{\mathcal{V}}_{2,1}(\mu)\big]\big\rangle \\ &+ 2\sum_{l\in[N]}\Big(\big\langle 112a^3+84a^2c_1(L^*_{\hat{1}})+32ac_1^2(L^*_{\hat{1}})+5c_1^3(L^*_{\hat{1}}), \big[\bar{\mathcal{U}}_{\mathcal{T}^*(l)}(\mu)\big]\big\rangle - 5\big|\bar{\mathcal{U}}_{\mathcal{T}^*(l)}\cap\bar{\mathcal{S}}_1(\mu)\big|\Big),\end{aligned} \tag{5.47}$$

where $L_{\hat{1}} = L_{\hat{1}}\mathcal{T}^*(l)$, $\bar{\mathcal{V}}_{2,1}(\mu) = \bigcup_{l\in[N]}\bar{\mathcal{V}}_{2,1;l}(\mu)$, and $\bar{\mathcal{V}}_{2,1;l}(\mu)$ denotes the union of the spaces $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ taken over all equivalence classes of basic bubble types $\mathcal{T}=(S^2,[N]-\{l\},\{\hat{1},\hat{2}\};j,\underline{d})$ with $d_{\hat{1}},d_{\hat{2}}>0$ and $d_{\hat{1}}+d_{\hat{2}}=d$.

(5) If $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2$ and $|M_{\hat{0}}\mathcal{T}|=0$, $\mathcal{F}\mathcal{T}\approx L_{\hat{1}}\mathcal{T}\oplus L_{\hat{2}}\mathcal{T}$ over $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ and $\tilde{\alpha}^{-1}_{\mathcal{F}\mathcal{T}}(0)$ consists of a section $\mathcal{Z}_{\mathcal{T}}$ of $\mathbb{P}F\mathcal{T}$ over $\Sigma\times\mathcal{S}_{\mathcal{T}}(\mu)$ and the spaces $\Sigma\times\bar{\mathcal{U}}_{\mathcal{T}'}(\mu)$, with $\bar{\mathcal{T}}'$ corresponding to the bubble types $\mathcal{T}$ described in (2) in the proof of Lemma 5.5. By Corollaries 6.3 and 3.13,

$$\mathcal{C}_{\Sigma\times\mathcal{S}_{\mathcal{T}}(\mu)}(\tilde{\alpha}^{\perp}_{\mathcal{F}\mathcal{T}}) = \big\langle c(\gamma^*_{\mathcal{F}\mathcal{T}}\otimes\mathcal{O}_2^{\perp})c(\mathbb{C}^3)^{-1}, \big[\Sigma\times\mathcal{S}_{\mathcal{T}}(\mu)\big]\big\rangle = 10\big|\mathcal{S}_{\mathcal{T}}(\mu)\big|.$$

On the other hand, if $\mathcal{T}'=(S^2,[N],I';j',\underline{d}')<\mathcal{T}$ and $\mathcal{C}_{\mathbb{P}\mathcal{F}\mathcal{T}|\Sigma\times\mathcal{U}'_{\mathcal{T}}(\mu)}(\tilde{\alpha}^{\perp}_{\mathcal{F}\mathcal{T}})\neq 0$,

$$|I'-I|\in\{1,2\}, \quad H_{\hat{2}}\mathcal{T} = I'-I, \quad d'_{\hat{2}}=0,\ d'_h\neq 0 \text{ if } h\in\hat{I}'-\{\hat{2}\}.$$

If $|I'-I|=1$, by Corollary 3.13,

$$\begin{aligned}\mathcal{C}_{\mathbb{P}\mathcal{F}\mathcal{T}|\Sigma\times\mathcal{U}'_{\mathcal{T}}(\mu)}(\tilde{\alpha}^{\perp}_{\mathcal{F}\mathcal{T}}) &= \big\langle c(\mathcal{O}_2)c\big(L_{\hat{1}}\mathcal{T}'\oplus L_{\hat{3}}\mathcal{T}'\big)^{-1}, \big[\Sigma\times\bar{\mathcal{U}}_{\mathcal{T}'}(\mu)\big]\big\rangle \\ &= 2\big\langle 32a + 5(c_1(L^*_{\hat{1}}\mathcal{T}')+c_1(L^*_{\hat{3}}\mathcal{T}')), \big[\bar{\mathcal{U}}_{\mathcal{T}'}(\mu)\big]\big\rangle;\end{aligned}$$

see the proof of Lemma 5.13 for more details. If $|I'-I|=2$, by Corollary 3.13,

$$\mathcal{C}_{\mathbb{P}\mathcal{F}\mathcal{T}|\Sigma\times\mathcal{U}'_{\mathcal{T}}(\mu)}(\tilde{\alpha}^{\perp}_{\mathcal{F}\mathcal{T}}) = \big\langle c_1(\mathcal{O}_2)-c_1(\mathbb{C}^3), \big[\Sigma\times\bar{\mathcal{U}}_{\mathcal{T}'}(\mu)\big]\big\rangle = 10\big|\bar{\mathcal{U}}_{\mathcal{T}'}(\mu)\big|.$$

Thus, summing equation (5.46) over $\mathcal{T}<\mathcal{T}^*$ with $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2$ and $|M_{\hat{0}}\mathcal{T}|=0$, we obtain

$$\begin{aligned}\sum_{[\mathcal{T}]}\mathcal{C}_{\Sigma\times(\mathcal{U}_{\mathcal{T}}(\mu)-\mathcal{S}_{\mathcal{T}}(\mu))}(\alpha^{\perp}) &= -30\tau_3(\mu) - 10|\mathcal{S}_2(\mu)| \\ &+ 2\big\langle 84a^2 + 32a(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*)) + 5(c_1^2(\mathcal{L}_1^*)+c_1^2(\mathcal{L}_2^*)) + 5c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*), \big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle\end{aligned} \tag{5.48}$$

(6) If $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2$ and $|M_{\hat{0}}\mathcal{T}|=1$, $\tilde{\alpha}_{\mathcal{F}\mathcal{T}}$ does not vanish on $\Sigma\times\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)$. Thus, by Corollary 3.13,

$$\begin{aligned}\mathcal{C}_{\Sigma\times\mathcal{U}_{\mathcal{T}}(\mu)}(\alpha^{\perp}) &= \big\langle c(\mathcal{O}_2)c\big(L^*\otimes(L_{\hat{1}}\mathcal{T}\oplus L_{\hat{2}}\mathcal{T})\big)^{-1}, \big[\Sigma\times\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\big]\big\rangle \\ &= 2\big\langle 32a + 5(c_1(L^*_{\hat{1}}\mathcal{T})+c_1(L^*_{\hat{2}}\mathcal{T})), \big[\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)\big]\big\rangle.\end{aligned}$$

Here we used the decomposition (5.2) and Corollary 5.22. Summing equation (5.46) over $\mathcal{T}<\mathcal{T}^*$ with $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=2$ and $|M_{\hat{0}}\mathcal{T}|=1$, we obtain

$$\sum_{[\mathcal{T}]}\mathcal{C}_{\Sigma\times\mathcal{U}_{\mathcal{T}}(\mu)}(\alpha^{\perp}) = 64\tau_2^{(1)}(\mu) + 10\big\langle c_1(L_1^*)+c_1(L_2^*), [\bar{\mathcal{V}}_{2,1}(\mu)]\big\rangle. \tag{5.49}$$

(7) Finally, if $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=3$, $\mathcal{F}\mathcal{T}\approx L^*\oplus L^*\oplus L^*$ over $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$, and $\tilde{\alpha}_{\mathcal{F}\mathcal{T}}$ again does not vanish over $\Sigma\times\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)$. Then by Corollary 3.13,

$$\mathcal{C}_{\Sigma\times\mathcal{U}_{\mathcal{T}}(\mu)}(\alpha^{\perp}) = \big\langle c(\mathcal{O}_2)c(L^*\oplus L^*\oplus L^*)^{-1}, \big[\Sigma\times\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\big]\big\rangle = 10|\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)|.$$

Thus, summing equation (5.46) over $\mathcal{T}<\mathcal{T}^*$ with $|H_{\hat{0}}\mathcal{T}|=|\hat{I}|=3$, we obtain

$$\sum_{|H_{\hat{0}}\mathcal{T}|=3}\mathcal{C}_{\Sigma\times\mathcal{U}_{\mathcal{T}}(\mu)}(\alpha^{\perp}) = 10\tau_3(\mu). \tag{5.50}$$

From equations (5.44), (5.45), and (5.47)-(5.50), we conclude that

$$\begin{aligned} n_1^{(1)}(\mu) =& 2\big\langle 112a^3c_1(\mathcal{L}^*)+84a^2c_1^2(\mathcal{L}^*)+32ac_1^3(\mathcal{L}^*)+5c_1^4(\mathcal{L}^*), \big[\bar{\mathcal{V}}_1(\mu)\big]\big\rangle \\ &- 2\big\langle 84a^2+32a(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*))+5(c_1^2(\mathcal{L}_1^*)+c_1^2(\mathcal{L}_2^*))+5c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*), \big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle \\ &- 2\big\langle 12a+5c_1(\mathcal{L}^*), [\bar{\mathcal{S}}_1(\mu)]\big\rangle+10\big|\mathcal{S}_2(\mu)\big| + 20\tau_3(\mu). \end{aligned} \tag{5.51}$$

The claim follows by using Lemma 5.5 and 5.13.

5.7 Computation of Chern Classes

In this subsection, we show that all intersection numbers of the spaces $\bar{\mathcal{V}}_k(\mu)$ involving powers of a and powers of $c_1(\mathcal{L}_i^*)$ are computable. We can then conclude that the numbers $n_m^{(k)}(\mu)$ are computable. The computability of intersection numbers of tautological classes of $\bar{\mathcal{V}}_k(\mu)$, which include a and $c_1(\mathcal{L}_i^*)$, has been shown in [P2]. For the sake of completeness, a slightly different approach is presented below.

If $d_{\hat{0}}$ and $d_{\hat{1}}$ are nonnegative integers and μ is an N-tuple of any generic constraints in $\mathbb{P}^n$, let $\bar{\mathcal{M}}_{(d_{\hat{0}},d_{\hat{1}})}(\mu)$ denote the union of the spaces $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$, where $\mathcal{T}$ is a simple bubble type of the form

$$\mathcal{T} = \big(S^2, [N], \{\hat{0},\hat{1}\}; j, \{d_{\hat{0}}, d_{\hat{1}}\}\big).$$

Then $\bar{\mathcal{M}}_{\mathcal{T},(d_{\hat{0}},d_{\hat{1}})}(\mu)$ is a complex codimension-one homology class in the space $\bar{\mathcal{V}}_1(\mu)$ with $d=d_{\hat{0}}+d_{\hat{1}}$. If $d>0$, let

$$\sum_{d_{\hat{0}}+d_{\hat{1}}=d}^{\geq} f(d_{\hat{0}},d_{\hat{1}}) = \sum_{\substack{d_{\hat{0}}+d_{\hat{1}}=d\\ d_{\hat{0}},d_{\hat{1}}\geq 0}} f(d_{\hat{0}},d_{\hat{1}}), \qquad \sum_{d_{\hat{0}}+d_{\hat{1}}=d}^{>} f(d_{\hat{0}},d_{\hat{1}}) = \sum_{\substack{d_{\hat{0}}+d_{\hat{1}}=d\\ d_{\hat{0}},d_{\hat{1}}>0}} f(d_{\hat{0}},d_{\hat{1}}),$$

whenever f is any function defined on the appropriate subset of $\mathbb{Z}\times\mathbb{Z}$.

Lemma 5.15 *Let $\mathcal{T}^*=(S^2,[N],\{\hat{0}\};\hat{0},d)$ be a bubble type with $d>0$. Then in $H^*(\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu))$,*

$$c_1(L^*)=\frac{1}{d^2}\Big(\mathcal{H}-2da+\sum_{d_{\hat{0}}+d_{\hat{1}}=d}^{\geq} d_{\hat{1}}^2\bar{\mathcal{M}}_{(d_{\hat{0}},d_{\hat{1}})}(\mu)\Big),$$

where $\mathcal{H}$ denotes the subset of elements in $\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$ that pass through a generic codimension-two linear subspace of $\mathbb{P}^n$.

Proof: (1) We restate the proof of [I] in terms of the line bundle $L^{*\otimes d^2}\longrightarrow\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$, instead of passing to a cover of $\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$. Define a section $\psi\in\Gamma(\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu);L^{*\otimes d^2})$ as follows. Let H_0 and H_1 be two fixed hyperplanes in $\mathbb{P}^n$, generic with respect to the constraints $\mu_1,\ldots\mu_N$. Suppose

$$[b]=\big[(S^2,[N],\hat{0};,(\hat{0},y),u_{\hat{0}})\big]\in\mathcal{U}_{\mathcal{T}^*}(\mu)$$

is such that $u_{\hat{0}}$ is transversal to H_0 and H_1. Then,

$$u_{\hat{0}}^{-1}(H_i)=\big\{[x_1^{(i)},y_1^{[i]}],\ldots,[x_d^{(i)},y_d^{[i]}]\big\},\quad i=0,1,$$

for some $[x_k^{(i)},y_k^{[i]}]\in\mathbb{P}^1$. Define $\psi([b])$ by

$$\psi([b,c])=c^{d^2}\prod_{k,l\in[d]}\bigg(\frac{x_k^{(0)}}{y_k^{(0)}}-\frac{x_l^{(1)}}{y_l^{(1)}}\bigg). \tag{5.52}$$

While this section could be infinite, it is well-defined, i.e. independent of the choice of a representative $b\in\mathcal{B}_{\mathcal{T}^*}$ for $[b]$. With an appropriate coordinate change on $\mathbb{C}^{n+1}$, it can be assumed that $H_i=\{X_i=0\}$. The map $u_{\hat{0}}$ corresponds to $(n+1)$ homogeneous polynomials of degree d: $p_0,\ldots,p_n$. Since the right-hand side of (5.52) is symmetric in the roots of p_0 and separately in the roots of p_1, ψ is a rational function in the coefficients of p_0 and p_1. Thus, ψ extends over all of $\mathcal{U}_{\mathcal{T}^*}(\mu)$. Furthermore, this section extends by zero over $\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)-\mathcal{U}_{\mathcal{T}^*}(\mu)$.
(2) We now identify the zero set of the section ψ. From equation (5.52), it is clear that ψ vanishes with multiplicity one if p_0 and p_1 have a common root, i.e. if $u_{\hat{0}}$ passes through $H_0\cap H_1$. The section ψ also has a pole of order d along the sets of maps

$$X_0=\{b\colon y_k^{(0)}(b)=0 \text{ for a unique } k\in[d],\ p_1(1,0)\neq 0\},$$
$$X_1=\{b\colon y_k^{(1)}(b)=0 \text{ for a unique } k\in[d],\ p_0(1,0)\neq 0\}.$$

Note that $\bar{X}_i=\mathrm{ev}^{-1}(H_i)$. Finally, while ψ vanishes outside of $\mathcal{U}_{\mathcal{T}^*}(\mu)$, $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ has (complex) codimension one in $\bar{\mathcal{U}}_{\mathcal{T}^*}(\mu)$ if and only if $\mathcal{T}<\mathcal{T}^*$ is a two-bubble strata, i.e. as described just before the statement of the lemma. Let $d_{\hat{0}}$ and $d_{\hat{1}}$ be the corresponding degrees. It follows from equation (5.52) that ψ has a zero of order $d_{\hat{1}}^2$ along an open subset of $\mathcal{U}_{\mathcal{T}}(\mu)$. Thus, we obtain

$$c_1\big(L^{*\otimes d^2}\big)=\mathcal{H}-2da+\sum_{d_{\hat{0}}+d_{\hat{1}}=d}^{\geq} d_{\hat{1}}^2\bar{\mathcal{M}}_{(d_{\hat{0}},d_{\hat{1}})}(\mu).$$

Corollary 5.16 *With notation as in Lemma 5.15,*

$$c_1(\mathcal{L}^*)=\frac{1}{d^2}\Big(\mathcal{H}-2da+\sum_{d_{\hat{0}}+d_{\hat{1}}=d}^{>} d_{\hat{1}}^2\bar{\mathcal{M}}_{(d_{\hat{0}},d_{\hat{1}})}(\mu)\Big).$$

Proof: This is immediate from Lemma 5.15 and (5.3).

If $\mathcal{T}=(S^2,[N],I;j,\underline{d})$ is any bubble type, let $\mathcal{T}_{\hat{0}}=(S^2,M_{\hat{0}}\mathcal{T}\sqcup H_{\hat{0}}\mathcal{T},\{\hat{0}\};\hat{0},d_{\hat{0}})$. Denote by $\mathcal{T}_k$ for $k\in H_{\hat{0}}\mathcal{T}$ the simple bubble types corresponding to $\bar{\mathcal{T}}$. Then,

$$\begin{aligned}\bar{\mathcal{U}}_{\mathcal{T}}(\mu) &= \bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}}(\mu)\times_{\prod_{k\in H_{\hat{0}}\mathcal{T}}(\mathrm{ev}_k\times\mathrm{ev})}\prod_{k\in H_{\hat{0}}\mathcal{T}}\bar{\mathcal{U}}_{\mathcal{T}_k}(\mu) \\ &\equiv \Big\{\big(b_{\hat{0}},(b_k)_{k\in H_{\hat{0}}\mathcal{T}}\big)\in\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}}(\mu)\times\prod_{k\in H_{\hat{0}}\mathcal{T}}\bar{\mathcal{U}}_{\mathcal{T}_k}(\mu):\mathrm{ev}_k(b_{\hat{0}})=\mathrm{ev}(b_h)\ \forall k\in H_{\hat{0}}\mathcal{T}\Big\}.\end{aligned} \tag{5.53}$$

Lemma 5.17 *With notation as above, if $\mathcal{T}<\mathcal{T}^*$ and $d_{\hat{0}}\neq 0$,*

$$c_1(\mathcal{L}^*\mathcal{T}^*)\big|_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)}=\Big\{c_1(\mathcal{L}^*\mathcal{T}_{\hat{0}})\big|_{\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}}(\mu)}+\sum_{\emptyset\neq M_0\subset M_{\hat{0}}\mathcal{T}_{\hat{0}},M_0\cap H_{\hat{0}}\mathcal{T}\neq\emptyset}\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0)}(\mu)\Big\}\times_{\prod_{k\in H_{\hat{0}}\mathcal{T}}(ev_k\times ev)}\prod_{k\in H_{\hat{0}}\mathcal{T}}\bar{\mathcal{U}}_{\mathcal{T}_k}(\mu).$$

Proof: Since $L_{\hat{0}}\mathcal{T}^*|_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)}=L_{\hat{0}}\mathcal{T}$ and $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\cap\bar{\mathcal{U}}_{\mathcal{T}^*(M_0)}(\mu)=\emptyset$ unless $M_0\subset M_{\hat{0}}\mathcal{T}$, by (5.3)

$$\begin{aligned}c_1(\mathcal{L}^*\mathcal{T}^*)\big|_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)} &= c_1(L^*\mathcal{T})\big|_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)}-\sum_{\emptyset\neq M_0\subset M_{\hat{0}}\mathcal{T}}\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\cdot\bar{\mathcal{U}}_{\mathcal{T}^*(M_0)}(\mu)\\ &= c_1(L^*_{\hat{0}}\mathcal{T}_{\hat{0}})\big|_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)}-\sum_{\emptyset\neq M_0\subset M_{\hat{0}}\mathcal{T}}\bar{\mathcal{U}}_{\mathcal{T}(M_0)}(\mu).\end{aligned}$$

The claim follows by using equation (5.3) again.

Corollary 5.18 *All intersection numbers on $\bar{\mathcal{V}}_k(\mu)$ involving only the powers of a and $c_1(\mathcal{L}_k^*)$ are computable.*

Proof: Corollary 5.16 and Lemma 5.17 reduce the computation of such numbers to understanding the restrictions $c_1(\mathcal{L}^*\mathcal{T}_{\hat{0}})|_{\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0)}}$, where M_0 is a subset of $M_{\hat{0}}\mathcal{T}_{\hat{0}}$ intersecting $H_{\hat{0}}\mathcal{T}$. By (5.2),

$$\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0)}\approx\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}\sqcup M_0}\times\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}/M_0}.$$

We express $c_1(\mathcal{L}^*\mathcal{T}_{\hat{0}})|_{\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0)}}$ in terms of cohomology classes on $\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}\sqcup M_0}$. By definition, $L\mathcal{T}_{\hat{0}}|_{\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0)}}$ comes from a line bundle over $\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}\sqcup M_0}$. In fact,

$$c_1(L^*\mathcal{T}_{\hat{0}})\big|_{\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}\sqcup M_0}\times\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}/M_0}}=\psi_{\hat{0}}\times 1,$$

where $\psi_{\hat{0}}$ is the ψ-class of $\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}\sqcup M_0}$ corresponding to the marked point $\hat{0}$. Since $L^*\mathcal{T}_{\hat{0}}|_{\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0)}}$ is $L^*\mathcal{T}_{\hat{0}}(M_0)$,

$$\begin{aligned}c_1(\mathcal{L}^*\mathcal{T}_{\hat{0}})\big|_{\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0)}} &= c_1(L^*\mathcal{T}_{\hat{0}})\big|_{\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0)}}-\sum_{\emptyset\neq M_0'\subset M_{\hat{0}}\mathcal{T}}\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0')}\cdot\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0)}\\ &= \psi_{\hat{0}}\times 1-\sum_{\emptyset\neq M_0'\subset(M_0-H_{\hat{0}}\mathcal{T})}\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0';M_0-M_0')}=\tilde{\psi}_{M_0-H_{\hat{0}}\mathcal{T}}\times 1\big|_{\bar{\mathcal{M}}_{\{\hat{0},\hat{1}\}\sqcup M_0}\times\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}/M_0}},\end{aligned}$$

where $\mathcal{T}_{\hat{0}}(M_0';M_0-M_0')\equiv\{\mathcal{T}_{\hat{0}}(M_0)\}(M_0')$ and for any proper subset $\tilde{J}$ of J we define the cohomology class $\tilde{\psi}_{\tilde{J}}$ on $\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}\sqcup J}$ by

$$\tilde{\psi}_{\tilde{J}}=\psi_{\hat{0}}-\sum_{\emptyset\neq J'\subset\tilde{J}}\bar{\mathcal{M}}_{0,(\{\hat{0}\}\sqcup J',\{\hat{1}\}\sqcup(J-J'))}.$$

Here $\bar{\mathcal{M}}_{0,(\{\hat{0}\}\sqcup J',\{\hat{1}\}\sqcup(J-J'))}$ is the closure in $\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}\sqcup J}$ of the two-component strata such that the marked points on one of the components are $\{\hat{0}\}\sqcup J'$. The numbers

$$\chi\big(|J|,|\tilde{J}|\big)\equiv\big\langle\tilde{\psi}_{\tilde{J}}^{|J|-1},\big[\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}\sqcup J}\big]\big\rangle$$

are given in Corollary 5.20, which is a consequence of the following well-known lemma; see [P2] for example.

Lemma 5.19 *(1) For any $j^*\in J$, $\tilde{\psi}_{J-\{j^*\}}=0$ in $H^*(\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}\sqcup J})$.*
(2) If $\mathcal{N}\bar{\mathcal{M}}_{0,(\{\hat{0}\}\sqcup J',\{\hat{1}\}\sqcup(J-J'))}$ is the normal bundle of

$$\bar{\mathcal{M}}_{0,(\{\hat{0}\}\sqcup J',\{\hat{1}\}\sqcup(J-J'))}\approx\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}\sqcup J'}\times\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}\sqcup(J-J')}$$

in $\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}\sqcup J}$,

$$c_1\big(\mathcal{N}\bar{\mathcal{M}}_{0,(\{\hat{0}\}\sqcup J',\{\hat{1}\}\sqcup(J-J'))}\big)=-\psi_{\hat{1}}\times 1-1\times\psi_{\hat{0}}.$$

Corollary 5.20 *If $m>0$, $\chi(m,0)=1$. If $m>k>0$, $\chi(m,k)=0$.*

For our purposes, we can assume that the constraints $\mu_1,\ldots,\mu_N$ are disjoint. In the case of $\mathbb{P}^2$, the dimension of the space $\bar{\mathcal{V}}_k(\mu)$ is at most 2. Thus, by a dimension count, if $\bar{\mathcal{U}}_{\mathcal{T}_{\hat{0}}(M_0)}(\mu)$ is nonempty and appears in the computation of the intersection numbers of Corollary 5.18 via Lemma 5.17, then $H_{\hat{0}}\mathcal{T}$ consists of a single element and $M_0=H_{\hat{0}}\mathcal{T}$. The corresponding moduli space $\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}\sqcup M_0}$ is a single point and thus

$$\big\langle\tilde{\psi}_{M_0-H_{\hat{0}}\mathcal{T}}^{|M_0|-1},\big[\bar{\mathcal{M}}_{\{\hat{0},\hat{1}\}\sqcup M_0}\big]\big\rangle=1.$$

In the case of $\mathbb{P}^3$, $\bar{\mathcal{V}}_1(\mu)$ is four-dimensional, and we encounter two cases when $\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}\sqcup M_0}$ is positive-dimensional. One possibility is that $H_{\hat{0}}\mathcal{T}$ is still a single-element set, but M_0 contains one of the N marked points. In this case, by Corollary 5.20 or simply by the first statement of Lemma 5.19,

$$\big\langle\tilde{\psi}_{M_0-H_{\hat{0}}\mathcal{T}}^{|M_0|-1},\big[\bar{\mathcal{M}}_{\{\hat{0},\hat{1}\}\sqcup M_0}\big]\big\rangle=\chi(2,1)=0.$$

In fact, we can replace the first statement of Lemma 5.19 with the direct computation of the degree $\psi_{\hat{0}}$ on $\bar{\mathcal{M}}_{0,4}$ given by Lemma 5.21 below. The other case when $\bar{\mathcal{M}}_{0,\{\hat{0},\hat{1}\}\sqcup M_0}$ is positive-dimensional is $M_0=H_{\hat{0}}\mathcal{T}$ is a two-element set. Then

$$\big\langle\tilde{\psi}_{M_0-H_{\hat{0}}\mathcal{T}}^{|M_0|-1},\big[\bar{\mathcal{M}}_{\{\hat{0},\hat{1}\}\sqcup M_0}\big]\big\rangle=\chi(2,0)=1.$$

Lemma 5.21 *Let $\bar{\mathcal{M}}_{0,4}^{(0)}=\big\{(y_1,y_2,y_3)\in\mathbb{C}^3: y_1+y_2+y_3=0,\ \beta(|y_1|)+\beta(|y_2|)+\beta(|y_3|)=\frac{1}{2}\big\}$. Then the action of S^1 on $\bar{\mathcal{M}}_{0,4}^{(0)}$ induced from the standard action on $\mathbb{C}$ is free,*

$$\bar{\mathcal{M}}_{0,4}=\bar{\mathcal{M}}_{0,4}^{(0)}/S^1\approx\mathbb{P}^1,$$

and the line bundle associated to this quotient is the tautological line bundle over $\mathbb{P}^1$.

Proof: Identify $\bar{\mathcal{M}}_{0,4}^{(0)}$ with $S^3 \subset \mathbb{C}^2$ S^1-equivariantly by the map

$$(y_1, y_2, y_3) \longrightarrow \frac{(y_1, y_2)}{|y_1| + |y_2|}.$$

Our assumptions on β imply that this map is a diffeomorphism; see Subsection 1.3.

Corollary 5.22 *If* $\mathcal{T} = (S^2, [3], \{\hat{0}\}; \hat{0}, 0)$, $\langle c_1(L^*), [\bar{\mathcal{U}}_{\mathcal{T}}] \rangle = 1$.

Remark: In [Z1], we extend the definition of $\bar{\mathcal{M}}_{0,4}^{(0)}$ of Corollary 5.21 to construct spaces $\bar{\mathcal{M}}_{\mathcal{T}}^{(0)}$ for all bubble types $\mathcal{T}$.

5.8 The Final Formulas

We finally put everything together to arrive at formulas for the numbers $n_{2,d}(\mu)$ for $\mathbb{P}^2$ and $\mathbb{P}^3$. It can be assumed that μ is a tuple of $(3d-2)$ points in the case of $\mathbb{P}^2$ and of p points and q lines, with $2p+q = 4d-3$, in the case of $\mathbb{P}^3$. In the former case, we write $n_{2,d}$ for $n_{2,d}(\mu)$ and n_d for the number of rational plane degree d curves passing through $3d-1$ points.

If $\nu \in \Gamma(\Sigma \times \mathbb{P}^n; \Lambda^{0,1}\pi_\Sigma^* T^*\Sigma \otimes \pi_{\mathbb{P}^n}^* T^*\mathbb{P}^n)$ is generic, for all $t \in (0,1)$, the signed cardinality of the set $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$ is the symplectic invariant $RT_{2,d}(;\mu)$. If $t > 0$ is sufficiently small, every element of $\mathcal{M}_{\Sigma,t\nu,d}(\mu)$ lies either in a small neighborhood U of the set $\mathcal{H}_{\Sigma,d}(\mu)$ or in a small neighborhood W of the space of all bubble map with singular domains. Furthermore,

$$^{\pm}\big|\mathcal{M}_{\Sigma,t\nu,d}(\mu) \cap U\big| = \big|\mathcal{H}_{\Sigma,d}(\mu)\big| = 2n_{2,d}(\mu).$$

On the other hand, by Subsection 4.9,

$$\big|\mathcal{M}_{\Sigma,t\nu,d}(\mu) \cap W\big| = \begin{cases} n_1^{(1)}(\mu) + 2n_1^{(2)}(\mu) + 18n_1^{(3)}(\mu) + n_2^{(1)}(\mu), & \text{if } n=2; \\ n_1^{(1)}(\mu) + 2n_1^{(2)}(\mu) + 18n_1^{(3)}(\mu) & \\ \qquad + n_2^{(1)}(\mu) + 2n_2^{(2)}(\mu) + n_3^{(1)}(\mu), & \text{if } n=3. \end{cases} \tag{5.54}$$

Thus, $n_{2,d}(\mu)$ is one-half of the difference between $RT_{2,d}(;\mu)$ and the number in (5.54). We write $CR(\mu)$ for the number given by (5.54).

We first consider to the $n=2$ case. We abbreviate $\mathcal{M}_{(d_1,d_2)}(\mu)$ as $\mathcal{M}_{d_1,d_2}$. Let

$$\mathcal{Z}_{2;d} = \Big(\bigcup_{\substack{d_1,d_2>0 \\ d_1+d_2=0}} \mathcal{Z}_{d_1,d_2} \Big) / \mathbb{Z}_2, \quad \text{where} \quad \mathcal{Z}_{d_1,d_2} = \bigcup_{j_l=1,2} \bar{\mathcal{U}}_{(S^2,[N],I;j,\{0,d_1,d_2\})}(\mu),$$

and the partial ordering on $I = \{\hat{0}, 1, 2\}$ is $0 < 1, 2$. The set $\mathcal{Z}_{2;d}$ is the zero-dimensional space of three-bubble maps passing through the (3d-2) points μ, such that the map is trivial on the principal component. Note that

$$\big|\mathcal{Z}_{d;2}\big| = \tau_2(\mu) = \frac{1}{2} \sum_{d_1+d_2=d} \binom{3d-2}{3d_1-1} d_1 d_2 n_{d_1} n_{d_2}. \tag{5.55}$$

The binomial coefficient counts the number of possible ways of distributing the constraints between the two nontrivial bubbles. Without the factor $d_1 d_2$, the above number would have been precisely the number of two-component rational curves passing through $(3d–2)$ generic points in $\mathbb{P}^2$. However, we have to account for the image of the evaluation map at $\hat{0}$, which must be one of the $d_1 d_2$ points of intersection of two rational curves of degrees d_1 and d_2.

Lemma 5.23 *In the $n=2$ case, the total correction is given by*

$$CR(\mu) = \big\langle 78a^2 + 72ac_1(\mathcal{L}^*) + 22c_1^2(\mathcal{L}^*), \big[\bar{\mathcal{V}}_1(\mu)\big]\big\rangle - 18\tau_2(\mu).$$

Proof: The four numbers of (5.54) are given by Lemmas 5.6 and 5.3 and by Corollary 5.2. The cardinality of $\mathcal{S}_1(\mu)$ is given by Lemma 5.4.

Lemma 5.24 *With notation as above,*

$$\big\langle ac_1(\mathcal{L}^*), [\bar{\mathcal{V}}_1(\mu)]\big\rangle = \frac{1}{d}\Big(- n_d + \frac{1}{2}\sum_{d_1+d_2=d} d_1^2 d_2^2 \binom{3d-2}{3d_1-1} n_{d_1} n_{d_2}\Big).$$

Proof: By Corollary 5.16,

$$ac_1(\mathcal{L}^*) = \frac{1}{d^2} a\Big(\mathcal{H} - 2da + \sum_{d_1+d_2=d}^{>} d_2^2 \mathcal{M}_{d_1,d_2}\Big). \tag{5.56}$$

Note that

$$\begin{aligned} \sum_{d_1+d_2=d} d_2^2 \big\langle a, [\mathcal{M}_{d_1,d_2}]\big\rangle &= \sum_{d_1+d_2=d} d_1(d_1 d_2) d_2^2 \binom{3d-2}{3d_1-1} n_{d_1} n_{d_2} \\ &= \frac{1}{2} d \sum_{d_1+d_2=d} d_1^2 d_2^2 \binom{3d-2}{3d_1-1} n_{d_1} n_{d_2}. \end{aligned} \tag{5.57}$$

The reason for the appearance of the factor $d_1 d_2$ in (5.57) is the same one as in (5.55). On the other hand, the factor d_1 appears because we need to count the number of times the first rational component intersects a line in $\mathbb{P}^2$. Since

$$\big\langle a\mathcal{H}, [\bar{\mathcal{V}}_1(\mu)]\big\rangle = dn_d \quad \text{and} \quad \big\langle a^2, [\bar{\mathcal{V}}_1(\mu)]\big\rangle = n_d,$$

the claim follows by plugging (5.57) into (5.56).

Lemma 5.25 *With notation as above,*

$$\big\langle c_1^2(\mathcal{L}^*), [\bar{\mathcal{V}}_1(\mu)]\big\rangle = -\frac{1}{2}\sum_{d_1+d_2=d} \binom{3d-2}{3d_1-1} d_1 d_2 n_{d_1} n_{d_2}.$$

Proof: By Corollary 5.16,

$$c_1^2(\mathcal{L}^*) = \frac{1}{d^2} c_1(\mathcal{L}^*)\Big(\mathcal{H} - 2da + \sum_{d_1+d_2=d} d_2^2 \mathcal{M}_{d_1,d_2}\Big). \tag{5.58}$$

Since there are no two-component rational curves of total degree d passing through $(3d-1)$ generic points in $\mathbb{P}^2$ and there are no three-component rational curves of total degree d passing through $(3d-2)$ generic points in $\mathbb{P}^2$, by Corollary 5.16

$$\big\langle \mathcal{H}c_1(\mathcal{L}^*), [\bar{\mathcal{V}}_1(\mu)]\big\rangle = \frac{1}{d^2}\big\langle -2da\mathcal{H}, [\bar{\mathcal{V}}_1(\mu)]\big\rangle = -2n_d. \tag{5.59}$$

Similarly by Corollary 5.16 and Lemma 5.17,

$$\begin{aligned}\big\langle c_1(\mathcal{L}^*), [\mathcal{M}_{d_1,d_2}]\big\rangle &= \frac{1}{d_1^2}\big\langle -2d_1a\mathcal{H}, [\mathcal{M}_{d_1,d_2}]\big\rangle + \big|\mathcal{Z}_{d_1,d_2}\big| = -\big|\mathcal{Z}_{d_1,d_2}\big| \\ &= -d_1d_2\binom{3d-2}{3d_1-1}n_{d_1}n_{d_2}.\end{aligned} \tag{5.60}$$

Note that by symmetry

$$\sum_{d_1+d_2=d} d_1d_2^3\binom{3d-2}{3d_1-1}n_{d_1}n_{d_2} = \frac{1}{2}\sum_{d_1+d_2=d} d_1d_2\left(d^2-2d_1d_2\right)\binom{3d-2}{3d_1-1}n_{d_1}n_{d_2}. \tag{5.61}$$

The claim now follows from equations (5.58)-(5.61) and Lemma 5.24.

Corollary 5.26 *The total correction term is given by*

$$CR(\mu) = 78n_d + 72\frac{1}{d}\Big(-n_d + \frac{1}{2}\sum_{d_1+d_2=d} d_1^2d_2^2\binom{3d-2}{3d_1-1}n_{d_1}n_{d_2}\Big) - 20\sum_{d_1+d_2=d} d_1d_2\binom{3d-2}{3d_1-1}n_{d_1}n_{d_2}.$$

Proof: This claim is immediate from Lemmas 5.23-5.25 and equation (5.55).

Lemma 5.27 *The genus-two degree-d RT-invariant of* $\mathbb{P}^2$ *is given by*

$$RT_{2,d}(;\mu) \equiv RT_{2,d}(;p_{[3d-2]}) = 6d^2n_d + \sum_{d_1+d_2=d} d_1^3d_2^3\binom{3d-2}{3d_1-1}n_{d_1}n_{d_2}.$$

Proof: Applying the genus-reducing composition law of [RT] twice, we obtain

$$\begin{aligned}RT_{2,d}(;p_{[3d-2]}) &= 2RT_{1,d}(p,\mathbb{P}^2;p_{[3d-2]}) + RT_{1,d}(\ell,\ell;p_{[3d-2]}) \\ &= 4RT_{0,d}(p,\mathbb{P}^2,p,\mathbb{P}^2;p_{[3d-2]}) + 4RT_{0,d}(p,\mathbb{P}^2,\ell,\ell;p_{[3d-2]}) + RT_{0,d}(\ell,\ell,\ell,\ell;p_{[3d-2]}) \\ &= 0 + 4RT_{0,d}(p,\ell,\ell;p_{[3d-2]}) + RT_{0,d}(\ell,\ell,\ell,\ell;p_{[3d-2]}).\end{aligned} \tag{5.62}$$

Since the genus-zero three-point RT-invariant is the usual enumerative invariant, the middle term above is simply $4d^2n_d$. On the other hand, by the component-splitting composition law of [RT],

$$\begin{aligned}RT_{0,d}(\ell,\ell,\ell,\ell;p_{[3d-2]}) &= 2RT_{0,0}(\ell,\ell,\mathbb{P}^2;)RT_{0,d}(\ell,\ell,p;p_{[3d-2]}) \\ &\quad + \sum_{d_1+d_2=d}\ \sum_{J_1+J_2=[3d-2]} RT_{0,d_1}(\ell,\ell,\ell;p_{J_1})RT_{0,d_2}(\ell,\ell,\ell;p_{J_2}) \\ &= 2d^2n_d + \sum_{d_1+d_2=d} d_1^3d_2^3\binom{3d-2}{3d_1-1}n_{d_1}n_{d_2}.\end{aligned} \tag{5.63}$$

The lemma follows from equations (5.62) and (5.63).

Theorem 1.1 is nearly proved. We can simplify the expression in Corollary 5.26 by using a recursive relation for the numbers n_d; see [RT, p363]. The expression of Theorem 1.1 is half of the difference between the quantity of Lemma 5.27 and Corollary 5.26. Note that the numbers n_d with $d=1,2,3$ have long been known to be zero; see [ACGH]. Strictly speaking, our computation does not apply to the cases $d=1,2$. However, these two cases do provide a consistency check.

The case of $\mathbb{P}^3$ is significantly harder than the $n=2$ case. An explicit recursive formula as in Theorem 1.1 would be rather long, so we do not provide one. Instead we express $n_{2,d}(\mu)$ in terms of the corresponding symplectic invariant and intersection numbers of the spaces $\bar{\mathcal{V}}_1(\mu)$, $\bar{\mathcal{V}}_2(\mu)$, and $\bar{\mathcal{V}}_3(\mu)$.

Theorem 5.28 *If d is a positive integer and μ is a tuple of p points and q lines in general position in $\mathbb{P}^3$ with $2p+q=4d-3$,*

$$2n_{2,d}(\mu) = RT_{2,d}(\cdot;\mu) - CR(\mu), \qquad \textit{where}$$

$$\begin{aligned}\frac{1}{2}CR(\mu) =&\big\langle 480a^3c_1(\mathcal{L}^*)+476a^2c_1^2(\mathcal{L}^*)+240ac_1^3(\mathcal{L}^*)+49c_1^4(\mathcal{L}^*), \big[\bar{\mathcal{V}}_1(\mu)\big]\big\rangle + 36\tau_3(\mu)\\ &-\big\langle 324a^2+144a(c_1(\mathcal{L}_1^*)+c_1(\mathcal{L}_2^*))+27(c_1^2(\mathcal{L}_1^*)+c_1^2(\mathcal{L}_2^*))+25c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*), \big[\bar{\mathcal{V}}_2(\mu)\big]\big\rangle.\end{aligned}$$

Furthermore, $RT(\cdot;\mu)$ and all intersection numbers above are computable.

Proof: The six numbers of (5.54) in the $n=3$ case are given by Lemmas 5.14, 5.11, 5.12, 5.7, 5.3, and Corollary 5.2, respectively. The numbers $\langle a, [\bar{\mathcal{S}}_1(\mu)]\rangle$, $\langle c_1(\mathcal{L}^*), [\bar{\mathcal{S}}_1(\mu)]\rangle$, and $\big|\bar{\mathcal{S}}_2(\mu)\big|$ are given by Lemmas 5.5 and 5.13. The symplectic invariant $RT_{2,d}(\cdot;\mu)$ is well-known to be computable; see [RT]. The above intersection numbers are computable by Corollary 5.18.

As in the case of $\mathbb{P}^2$, we recover the well-known fact that all degree-one, -two, and -three numbers are zero. The only degree-one number, the number of genus-two degree-one curves through a line, is zero because there are no holomorphic degree-one maps from a positive-genus curve into $\mathbb{P}^n$; see [ACGH]. The eight degree-two and -three numbers are zero because the image of any holomorphic map of degree two or three from a genus-two curve into $\mathbb{P}^n$ is a line, see [ACGH], while no line passes through the required constraints. The first three degree-four numbers given below have also been known to be zero, since the image of any holomorphic map of degree four from a genus-two curve into $\mathbb{P}^n$ must lie in a plane. Finally, observe that the fourth degree-four number is the number $n_{2,4}$ given by Theorem 1.1, as should be the case.

degree	4					5
(p,q)	(6,1)	(5,3)	(4,5)	(3,7)	(0,13)	(5,7)
$\mathrm{RT}_{2,d}(\cdot;\mu)$	7,872	64,960	548,608	4,906,304	5,130,826,752	290,439,680
$CR(\mu)$	7,872	64,960	548,608	4,877,504	4,998,465,792	258,287,360
$n_{2,d}(\mu)$	0	0	0	14,400	66,180,480	16,076,160

6 Appendix

6.1 A Short Exact Sequence on $\mathbb{P}^n$

If M is a Kahler manifold and $E \longrightarrow M$ is a holomorphic vector bundle, let $\mathcal{O}(E)$ denote the sheaf of holomorphic sections of E. If $E \longrightarrow M$ is the trivial holomorphic line bundle, we write $\mathcal{O}$ for $\mathcal{O}(E)$. Let $H \longrightarrow \mathbb{P}^n$ be the hyperplane bundle.

Lemma 6.1 *There exists an exact sequence of sheaves over* $\mathbb{P}^n$*:*

$$0 \longrightarrow \mathcal{O} \longrightarrow (n+1)\mathcal{O}(H) \longrightarrow \mathcal{O}(T\mathbb{P}^n) \longrightarrow 0.$$

Proof: (1) Let $[X_0 : \ldots : X_n]$ denote the homogeneous coordinates on $\mathbb{P}^n$. Denote by $\bar{X}_i$ the section of the hyperplane bundle given by

$$\bar{X}_i|_{[X_0:\ldots:X_n]}(X_0, \ldots, X_n) = X_i \in \mathbb{C}.$$

Then we define a sheaf map $\mathcal{O} \longrightarrow (n+1)\mathcal{O}(H)$ by

$$f \longrightarrow (f\bar{X}_0, \ldots, f\bar{X}_n).$$

Let $U_i = \{[X_0 : \ldots : X_n] : X_i \neq 0\}$. On U_i, we can use the complex coordinates

$$z_{i,k} = \frac{X_k}{X_i}, \qquad k \in \{0, \ldots, n\} - \{i\}.$$

Using these coordinates, we define a sheaf map $(n+1)\mathcal{O}(H) \longrightarrow \mathcal{O}(T\mathbb{P}^n)$ by

$$(p_0, \ldots, p_n) \longrightarrow \sum_{k\neq i} \big(p_k(z_{i,0}, \ldots, z_{i,n}) - z_{i,k}p_i(z_{i,0}, \ldots, z_{i,n})\big)\frac{\partial}{\partial z_{i,k}}, \tag{6.1}$$

where $z_{i,i} = 1$. We need to see that this map is well-defined. Suppose $j \neq i$. Then,

$$z_{j,l} = z_{i,j}^{-1} z_{i,l} \Longrightarrow \frac{\partial}{\partial z_{i,k}} = \sum_{l\neq j} \frac{\partial z_{j,l}}{\partial z_{i,k}} \frac{\partial}{\partial z_{j,l}} = \begin{cases} z_{i,j}^{-1} \frac{\partial}{\partial z_{j,k}}, & \text{if } k \neq j; \\ -z_{i,j}^{-2}\Big(\frac{\partial}{\partial z_{j,i}} + \sum\limits_{l\neq i,j} z_{i,l} \frac{\partial}{\partial z_{j,l}}\Big), & \text{if } k = j. \end{cases} \tag{6.2}$$

Since each p_l is a linear functional, if $k \neq i, j$, we can write the kth summand in (6.1) as

$$\begin{aligned} &\big(z_{j,i}^{-1} p_k(z_{j,0}, \ldots, z_{j,n}) - z_{j,i}^{-2} z_{j,k} p_i(z_{j,0}, \ldots, z_{j,n})\big) z_{i,j}^{-1} \frac{\partial}{\partial z_{j,k}} \\ &\qquad = \big(p_k(z_{j,0}, \ldots, z_{j,n}) - z_{j,i}^{-1} z_{j,k} p_i(z_{j,0}, \ldots, z_{j,n})\big) \frac{\partial}{\partial z_{j,k}}. \end{aligned} \tag{6.3}$$

The remaining, $k = j$, summand in (6.1) is equal to

$$\begin{aligned} &\big(z_{j,i}^{-1} p_j(z_{j,0}, \ldots, z_{j,n}) - z_{j,i}^{-2} p_i(z_{j,0}, \ldots, z_{j,n})\big)(-z_{i,j}^{-2})\Big(\frac{\partial}{\partial z_{j,i}} + \sum_{k\neq i,j} z_{i,k} \frac{\partial}{\partial z_{j,k}}\Big) \\ &\qquad = \big(p_i(z_{j,0}, \ldots, z_{j,n}) - z_{j,i} p_j(z_{j,0}, \ldots, z_{j,n})\big)\Big(\frac{\partial}{\partial z_{j,i}} + \sum_{k\neq i,j} z_{i,k} \frac{\partial}{\partial z_{j,k}}\Big). \end{aligned} \tag{6.4}$$

Since $z_{j,i}z_{i,k} = z_{j,k}$, collecting similar terms in (6.3) and (6.4), we obtain equation (6.1) with i replaced by j.
(2) It is clear that the first map is injective, the second is surjective, and the composite is zero. Finally, if $(p_0, \ldots, p_n)$ is mapped to zero by the second map, then (6.1) implies that $\bar{X}_j p_i = \bar{X}_i p_j$ for all i and j. Thus, the function f, given by

$$f([X_0 : \ldots : X_n]) = \frac{p_i(X_0, \ldots, X_n)}{X_i},$$

is well-defined and holomorphic wherever $(p_0, \ldots, p_n)$ is.

6.2 On Regularity of Kernel of D_b

Lemma 6.2 *If $u\colon S^2 \longrightarrow \mathbb{P}^n$ is a holomorphic map, there is a surjection*

$$(n+1)H^1\big(S^2; \mathcal{O}(u^*H \otimes (-(k+1)p))\big) \longrightarrow H^1\big(S^2; \mathcal{O}(u^*T\mathbb{P}^n \otimes (-(k+1)p))\big),$$

where p denotes the divisor corresponding to a point $p \in S^2$. If the degree of u is at least k, then both cohomology groups are trivial.

Proof: Pulling back the short exact sequence of sheaves of Lemma 6.1 by u, tensoring it with $-(k+1)p$, and taking the corresponding long exact sequence, we obtain:

$$\begin{aligned} &\longrightarrow (n+1)H^1\big(S^2; \mathcal{O}(u^*H \otimes (-(k+1)p))\big) \longrightarrow H^1\big(S^2; \mathcal{O}(u^*T\mathbb{P}^n \otimes (-(k+1)p))\big) \\ &\longrightarrow H^2\big(S^2; \mathcal{O}(-(k+1)p)\big) \longrightarrow \ldots \end{aligned} \tag{6.5}$$

Since S^2 is a one-dimensional complex manifold, the last cohomology group in (6.5) must vanish, and the first statement of the lemma follows. On the other hand, by Kodaira-Serre duality,

$$\begin{aligned} H^1\big(S^2; \mathcal{O}(u^*H \otimes (-(k+1)p))\big) &= H^1\big(S^2; \Omega^1(u^*H \otimes (-(k-1)p)\big) \\ &\approx H^0\big(S^2; \mathcal{O}((u^*H \otimes (-(k-1)p))^*)\big)^*. \end{aligned} \tag{6.6}$$

The last group in (6.6) is trivial if $\mathcal{O}(u^*H \otimes (-(k-1)p))$ is positive, i.e. if

$$\big\langle c_1\big(u^*H \otimes (-(k-1)p\big), [S^2]\big\rangle = d - (k-1) > 0,$$

where d is the degree of u.

Corollary 6.3 *If $u : S^2 \longrightarrow \mathbb{P}^n$ is holomorphic map of degree d, for any $p \in S^2$ and nonzero $v \in T_pS^2$, the map*

$$\phi^{(k)}_{p,v}\colon \ker D_u \longrightarrow \bigoplus_{m\in\langle k\rangle} T_{u(p)}\mathbb{P}^n, \quad \phi^{(k)}_{p,v}\xi = \big(\xi_p, D\xi|_{p,v}, \ldots, D^{(k)}\xi|_{p,v}\big),$$

where $D\xi|_{p,v}$ denotes the covariant derivative of ξ along u in the direction of v, is surjective provided $d \geq k$.

Remark: If one defines $D^{(k)}\xi$ with respect to the metric $g_{\mathbb{P}^n,u(p)}$ on $\mathbb{P}^n$, $D^{(k)} \in T_{u(p)}\mathbb{P}^n \otimes T^*S^{2\otimes k}$, where T^*S^2 is viewed as a complex line bundle. However, the statement is independent of the choice of metric on $\mathbb{P}^n$.

Proof: Since ξ is holomorphic, if $\phi_{p,v}^{(k)}\xi$ is zero, ξ has a zero of order $k+1$ at p. Thus, $\phi_{p,v}^{(k)}$ induces a short exact sequence of sheaves on S^2:

$$0 \longrightarrow \mathcal{O}\big(u^*T\mathbb{P}^n \otimes (-(k+1)p)\big) \longrightarrow \mathcal{O}(u^*T\mathbb{P}^n) \xrightarrow{\phi_{p,v}^{(k)}} (k+1)\mathcal{O}\big((u^*T\mathbb{P}^n)_p\big) \longrightarrow 0,$$

where we view $\mathcal{O}((u^*T\mathbb{P}^n)_p)$ as a sheaf on S^2 via extension by 0; see [GH, p38]. Taking the corresponding long exact sequence in cohomology, we obtain

$$\begin{aligned} \ldots \longrightarrow H^0\big(S^2;\mathcal{O}(u^*T\mathbb{P}^n)\big) \xrightarrow{\phi_{p,v}^{(k)}} (k+1)H^0\big(S^2;\mathcal{O}((u^*T\mathbb{P}^n)_p)\big) \\ \longrightarrow H^1\big(S^2;\mathcal{O}(u^*T\mathbb{P}^n \otimes (-(k+1)p))\big) \ldots \end{aligned} \tag{6.7}$$

By Lemma 6.2, the last cohomology group in (6.7) is zero if $d \geq k$. It follows that the map $\phi_{p,v}^{(k)}$ is surjective.

6.3 Dimension Counts

Lemma 6.4 *Let Σ be a compact Riemann surface. If $u\colon \Sigma \longrightarrow \mathbb{P}^n$ is a holomorphic map, there exists a surjection*

$$(n+1)H^1\big(\Sigma;\mathcal{O}(u^*H)\big) \longrightarrow H^1\big(\Sigma;\mathcal{O}(u^*T\mathbb{P}^n)\big).$$

Proof: Pulling back the short exact sequence of Lemma 6.1 by u gives a long exact sequence in sheaf cohomology:

$$\ldots (n+1)H^1\big(\Sigma;\mathcal{O}(u^*H)\big) \longrightarrow H^1\big(\Sigma;\mathcal{O}(u^*T\mathbb{P}^n)\big) \longrightarrow H^2\big(\Sigma;\mathcal{O}\big) \ldots \tag{6.8}$$

Since the complex dimension of Σ is one, the last group vanishes, and the claim follows.

Corollary 6.5 *Let Σ be a compact Riemann surface. If $u\colon \Sigma \longrightarrow \mathbb{P}^n$ is a holomorphic map, the $\bar{\partial}$-operator for the bundle $u^*T\mathbb{P}^n$,*

$$D_u\colon \Gamma(\Sigma;u^*T\mathbb{P}^n) \longrightarrow \Gamma(\Sigma;\Lambda^{0,1}T^*\Sigma\otimes u^*T\mathbb{P}^n)$$

is surjective, provided $d+\chi(\Sigma)>0$, where d is the degree of u.

Proof: The cokernel of D_u is $H^1_{\bar{\partial}}(\Sigma;u^*T\mathbb{P}^n)$. By Dolbeault Theorem,

$$H^1_{\bar{\partial}}(\Sigma;u^*T\mathbb{P}^n) = H^1\big(\Sigma;\mathcal{O}(u^*T\mathbb{P}^n)\big). \tag{6.9}$$

On the other hand, by Kodaira-Serre duality (see [GH, p153]),

$$\begin{aligned} H^1\big(\Sigma;\mathcal{O}(u^*H)\big) &= H^1\big(\Sigma;\Omega^1(T\Sigma\otimes u^*H)\big) \\ &= H^0\big(\Sigma;\mathcal{O}((T\Sigma\otimes u^*H)^*)\big)^* = H^0_{\bar{\partial}}\big(\Sigma;(T\Sigma\otimes u^*H)^*\big)^*. \end{aligned} \tag{6.10}$$

The bundle $(T\Sigma\otimes u^*H)^*$ does not admit any holomorphic section if it is negative, i.e. if

$$\big\langle c_1((T\Sigma\otimes u^*H)),[\Sigma]\big\rangle = \big\langle c_1(T\Sigma)+c_1(u^*H),[\Sigma]\big\rangle = \chi(\Sigma)+d > 0.$$

Thus, the claim follows from equations (6.9) and (6.10) and Lemma 6.4,

Proposition 6.6 *Let Σ be a Riemann surface of genus 2 and let d and n be positive integers with $n\le 4$. If $n=4$, assume that $d\neq 2$. Suppose $\mu=(\mu_1,\ldots,\mu_N)$ is an N-tuple of proper complex submanifolds of $\mathbb{P}^n$ of total complex codimension $d(n+1)-n+N$ in general position. If*

$$\mathcal{T}=\big(\Sigma,[N],I;j,\underline{d}'\big)<\mathcal{T}^*=\big(\Sigma,[N],\{\hat{0}\};\hat{0},d\big)$$

is a bubble type such that $d'_{\hat{0}}>0$, then $\mathcal{H}_{\mathcal{T}}(\mu)=\emptyset$. Furthermore, if

$$b=\big(\Sigma,[N],\{\hat{0}\};,(\hat{0},y),u\big)\in\mathcal{H}_{\mathcal{T}^*}(\mu),$$

then the map u is not multiply-covered.

Proof: (1) If $d'_{\hat{0}}\ge 3$, by Corollaries 6.3 and 6.5 and standard arguments such as in [MS], the space $\mathcal{H}_{\mathcal{T}}$ is a smooth manifold and the maps ev_l are smooth. If $b\in\mathcal{H}_{\mathcal{T}}$, a neighborhood of b in $\mathcal{H}_{\mathcal{T}}$ can be modeled on $\ker D_b\oplus\bigoplus_{l=1}^{l=n}T_{y_l}\Sigma_{b,j_l}$. In particular, by the Index Theorem,

$$\dim_{\mathbb{C}}\mathcal{H}_{\mathcal{T}}=\sum_{i\in I}\big(d'_i(n+1)+n(1-g(\Sigma_{b,i}))\big)-(n-1)|\hat{I}|+N=d(n+1)-n+|\hat{I}|+N.$$

Thus, if the map

$$\mathrm{ev}_{[N]}\equiv\mathrm{ev}_1\times\ldots\times\mathrm{ev}_N\colon\mathcal{H}_{\mathcal{T}}\longrightarrow\mathbb{P}^n\times\ldots\times\mathbb{P}^n,$$

is smooth and transversal to $\mu_1\times\ldots\times\mu_N$, $\mathcal{H}_{\mathcal{T}}(\mu)$ is a smooth manifold of (complex) dimension $|\hat{I}|$. Since the map $\mathrm{ev}_{[N]}$ is invariant under the action of $2|\hat{I}|$-dimensional group

$$\mathcal{G}_{\mathcal{T}}\equiv\{g\in PSL_2\colon g(\infty)=\infty\}^{\hat{I}},$$

$\mathcal{G}_{\mathcal{T}}$ acts smoothly on $\mathcal{H}_{\mathcal{T}}(\mu)$. Furthermore, the stabilizer at each point is finite. Thus, $\mathcal{H}_{\mathcal{T}}(\mu)=\emptyset$.
(2) Suppose $d'_{\hat{0}}=2$. If $b=\big(\Sigma,[N],I;x,(j,y),u\big)\in\mathcal{H}_{\mathcal{T}}$, the map $u_{\hat{0}}$ must factor through a degree-one map $\tilde{u}_{\hat{0}}\colon S^2\longrightarrow\mathbb{P}^n$; see [ACGH, p116]. Thus, it is enough to show that the space $\mathcal{H}_{\mathcal{T}'}(\mu)$ is empty, where $\mathcal{T}'=(S^2,[N],I;j,\underline{d}'')$, $d''_h=d'_h$ if $h\in\hat{I}$ and $d''_{\hat{0}}=1$. By Corollary 6.3, the space $\mathcal{H}_{\mathcal{T}'}$ is a smooth manifold of dimension

$$\dim_{\mathbb{C}}\mathcal{H}_{\mathcal{T}'}=(d-1)(n+1)+n+|\hat{I}|+N.$$

Similarly to (1) above, it follows that $\mathcal{H}_{\mathcal{T}'}(\mu)$ is a smooth manifold of dimension $n-1+|\hat{I}|$ on which the $(2|\hat{I}|+3)$-dimensional group $PSL_2\times\mathcal{G}_{\mathcal{T}}$ acts with only finite stabilizers. It follows that $\mathcal{H}_{\mathcal{T}'}(\mu)=\emptyset$ if $n<|\hat{I}|+4$. Note that the case $\hat{I}=\emptyset$ can occur only if $d=d_{\hat{0}}=2$. Finally, if $d_{\hat{0}}=1$, the entire space $\mathcal{H}_{\mathcal{T}}$ is empty, since there are no holomorphic degree-one maps from Σ into $\mathbb{P}^n$.
(3) Suppose $b=(\Sigma,[N],\{\hat{0}\};,(\hat{0},y),u)\in\mathcal{H}_{\mathcal{T}^*}(\mu)$ and $u\colon\Sigma\longrightarrow\mathbb{P}^n$ factors through a k-fold cover of S^2, where $k\ge 2$ and k divides d. Then b arises from the space $\mathcal{H}_{\mathcal{T}'}(\mu)$, where

$$\mathcal{T}'=\big(S^2,[N],\{\hat{0}\};\hat{0},d/k\big).$$

Similarly to the above, this space is a smooth manifold of dimension

$$\big((d/k)(n+1)+n+N\big)-\big(d(n+1)-n+N\big)=-\frac{k-1}{k}d(n+1)+2n.$$

Thus, $\mathcal{H}_{\mathcal{T}'}(\mu)=\emptyset$, provided $d\geq 3$. In fact, since $\mathcal{H}_{\mathcal{T}'}(\mu)$ has a three-dimensional group of symmetry, $\mathcal{H}_{\mathcal{T}'}(\mu)=\emptyset$ unless $d=2$ and $n\geq 4$.
(4) Suppose b is as in (3) and u factors through a k-fold cover of a torus T, where $k\geq 2$ and k divides d. Then b arises from the space

$$\tilde{\mathcal{H}}_{1,d/k}(\mu)\equiv\big\{(\mathcal{E},y_{[N]},u)\colon \mathcal{E} \text{ is smooth elliptic curve, } u\colon\mathbb{C}\longrightarrow\mathbb{P}^n,\\ \bar{\partial}u=0,\ u_*[\mathcal{E}]=\frac{d}{k}\lambda;\ u(y_l)\in\mu_l\ \forall l\in[N]\big\}.$$

Similarly to the above, Corollary 6.5 implies that $\tilde{\mathcal{H}}_{1,d/k}(\mu)$ is a smooth space of dimension

$$\big((d/k)(n{+}1)+1+N\big)-\big(d(n{+}1)-n+N\big)=-\Big(\frac{k-1}{k}d-1\Big)(n{+}1)<1.$$

Since $\tilde{\mathcal{H}}_{1,d/k}(\mu)$ has a one-dimensional group of symmetries, $\tilde{\mathcal{H}}_{1,d/k}(\mu)=\emptyset$.

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